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BIBLIOTHECA
REGIA
MONACENSIS.

LIGHT.

PART I.

Of unpolarized Light.

§ I. INTRODUCTION.

Light. Part I.

1. Subject stated. In this article we propose to give an account of the properties of light; of the physico-mathematical laws which regulate the direction, intensity, state of polarization, colours, and interferences of its rays; to state the theories which have been advanced for explaining the complicated and splendid phenomena of optics; to explain the laws of vision, and their application, by the combined ingenuity of the philosopher and the artist, to the improvement of our sight; and the examination and measurement of those objects and appearances which, from their remoteness, minuteness, or refinement, would otherwise elude our senses.

2. The sight is the most perfect of our senses; the most various and accurate in the information it affords us; and the most delightful in its exercise. Apart from all considerations of utility, the mere perception of light is in itself a source of enjoyment. Instances are not wanting of individuals debarred from infancy by a natural defect from the use of their eyes, whose highest enjoyment still consisted in that feeble glimmering a strong sunshine could excite in their obstructed organs; but when to this we join the exact perception of form and motion, the wonderous richness and variety of colour, and the ubiquity conferred upon us by just impressions of situation and distance, we are lost in amazement and gratitude.

3. What are the means and mechanism by which we receive this inestimable benefit? Curiosity may well prompt the inquiry, but a more direct interest urges us to pursue it. Knowledge is power; and a careful examination of the means by which we see, not only may, but actually has led us to the discovery of artificial aids by which this particular sense may be strengthened and improved to a most extraordinary degree; giving to man at once the glance of the eagle, and the scrutiny of the insect—by which the infirmities of age may be deferred or remedied—nay, by which the sight itself when lost may be restored, and its blessings conferred after long years of privation and darkness, or on those who from infancy have never seen. But as we proceed in the inquiry we shall find inducements enough to pursue it from purely intellectual motives. A train of minute adaptation and wonderful contrivance is disclosed to us, in which are blended the utmost extremes of grandeur and delicacy; the one overpowering, the other eluding, our conceptions. In consequence of those peculiar and singular properties which are found to belong to light in its various states of polarization, it affords to the philosopher information respecting the intimate constitution of bodies, and the nature of the material world, totally distinct from the more general impressions of form, colour, distance, &c. which it conveys to the vulgar. Its notices, it is true, in this respect, are addressed rather to the intellect than the sense; but they are not on that account less real, or less to be depended on. Polarized light is, in the hands of the natural philosopher, not merely a medium of vision; it is an instrument by which he may be almost said to feel the ultimate molecules of natural bodies, to detect the existence and investigate the nature of powers and properties ascertainable only by this test, and connected with the most important and intricate inquiries in the study of nature.

4. Bodies regarded as self-luminous and opaque. The ancients imagined vision to be performed by a kind of emanation proceeding from the eye to the object seen. Were this the case, no good reason could be shown why objects should not be seen equally well in the dark. Something more, however, is necessary for seeing than the mere presence of the object. It must be in a certain state, which we express by saying that it is *luminous.* Among natural bodies some possess in themselves the property of exciting in our eyes the sensation of *brightness*, or light; as the sun, the stars, a lamp, red-hot iron, &c. Such bodies are called *self-luminous;* but by far the greater part possess no such property. Such bodies in the dark remain invisible, though our eyes are turned directly towards them; and are therefore termed dark, non-luminous, or *opaque*, though this word is also used occasionally to express want of transparency. All bodies, however, though not luminous of themselves, nor capable of exciting any sensation in our eyes, become so on being placed in the presence of a self-luminous body. When a lamp is brought into a dark room, we see, not only the lamp, but all the other bodies in the room. They are all, so long as the lamp remains, rendered luminous, and are in their turn capable of illuminating others. Thus a sunbeam passing into a darkened room renders luminous, and therefore visible, a sheet of paper on which it falls; and this, in its turn, will in like manner illuminate the whole apartment, and render visible every object it contains, so long as it continues to receive the sunbeam. The moon and planets are opaque bodies; but those parts of them on which the sun shines become for the time luminous, and perform all the offices of self-luminous bodies. Thus we see, that the communication which we call light, subsists not only between luminous bodies and our eyes, but between luminous and non-luminous bodies, or between luminous bodies and each other.

Opaque bodies become luminous in the presence of a luminous body.

Light. Part I.

5. Opaque bodies intercept light.

Many bodies possess the property of intercepting this peculiar intercourse between luminous bodies and our eyes, or other bodies. A screen of metal interposed between the sun and our eyes prevents our seeing it; interposed between the sun and a sheet of white paper, or other object, it *casts a shadow* on such object: *i. e.* renders it non-luminous. By this power of bodies to intercept light, we learn that the communication which constitutes it takes place in straight lines. We cannot see through a bent metallic tube, nor perceive the least glimpse of light through three small holes in as many plates of metal placed one behind the other at a distance, unless the holes be situated exactly in one straight line. Moreover, the shadows of bodies, when fairly received on smooth surfaces perpendicular to the line in which the luminous body lies, are similar in figure to the section of the body which produces them, which could not be, except the light were communicated in straight lines from their edges to the borders of the shadow. We express this property by saying that light *emanates*, or *radiates*, or is *propagated* from luminous bodies in straight lines; by which expressions nothing more is to be understood than the mere fact, without in any way prejudging the question as to the intimate nature of this emanation. Moreover, *it emanates from them in all directions*, for we see them in all situations of the eye, provided nothing intervene to intercept the light. This is the essential distinction between luminous bodies and optical images; from which, as we shall see, light emanates only in certain directions. Whether it emanates *equally* in all directions will be considered farther on.

Light emanates in straight lines,

in all directions,

6. and from every physical point in a luminous surface.

Light also radiates from every point (at least from every *physical* point) of a luminous body. This may, perhaps, be regarded as a truism; for those points of a luminous body from which (as from the spots in the sun) no light emanates, are, in fact, non-luminous, and the body is only partially so; the figure of the spots is only seen, because it is also necessarily that of the luminous surface which surrounds them. Still it should be borne in mind, for reasons which will appear when we come to speak of the formation of images. It is possible (nay, probable) that a luminous surface, such as that of the flame of a candle, may consist only of an immense but finite number of luminous points, surrounded by non-luminous spaces; but it is not ocular demonstration this idea admits of; and it is sufficient for our purpose that, so far as our senses inform us, every physical point of a luminous surface is a *separate* and *independent* source of light. We may magnify in a telescope the sun's disc to any extent, and intercept all but the very smallest portions of it, (spots excepted,) yet the *visibility* of one part is no way impaired by the exclusion of the rest. In this sense the proposition is no truism, but an important fact, of which we shall hereafter trace the consequences.

7. Fig. 6.

When the sun shines through a small hole, and is received on a white screen behind at a considerable distance, we see a round luminous spot, which enlarges as the screen recedes from the hole. If we measure the diameter of this image at different distances from the hole, it will be found that (laying out of the question certain small causes of difference not now in contemplation) the angle subtended by the spot at the centre of the hole is constant, and equal to the apparent angular diameter of the sun. The reason of this is obvious; the light from every point in the sun's disc passes through the hole, and continues its course in a right line beyond it till it reaches the screen. Thus every point in the sun's disc has a point corresponding to it in the screen; and the whole circular spot on the screen is, in fact, an *image* or representation of the face of the sun. That this is really the case, is evidently seen by making the experiment in the time of a solar eclipse, when the image on the screen, instead of appearing round, appears horned, like the sun itself.* In like manner, if a pin-hole in a card be held between a candle and a piece of white paper in a dark room, an exact representation of the flame, but inverted, will be seen depicted on the paper, which enlarges as the paper recedes from the hole; and if in a dark room a white screen be extended at a few feet from a small round hole, an exact picture of all external objects, of their natural colours and forms, will be seen traced upon the screen; moving objects being represented in motion, and quiescent ones at rest. (See fig. 6.) To understand this, we must recollect that all objects exposed to light are luminous; that from every physical point of them light radiates in all directions, so that every point in the screen is receiving light at once from every point in the object. The same may be said of the hole; but the light that falls on the hole passes through it, and continues its course in straight lines behind. Thus the hole becomes the vertex of a conoidal solid prolonged both ways, having the object for its base at one end, and the screen at the other. The section of this solid by the screen is the picture we see projected on it, which must manifestly be exactly similar to the object, and inverted, according to the simplest rules of Geometry.

8. Fig. 7.

Now if in our screen receiving (suppose) the image of the sun we make another small hole, and behind it place another screen, the light falling on the space occupied by this hole will pass beyond it, and reach the other screen; but it is clear that it will no longer dilate itself, after passing through the second hole, and form another image of the whole sun, but only an image of that very minute portion of the sun which corresponds to the space occupied in his image on the first screen by the hole made there. The lines bounding the conoidal surface will in this case have much less divergency, and, if the holes be small enough, and very distant from each other, will approach to physical lines, and that the nearer, as the holes are smaller and their distance greater. (See fig. 7.) If we conceive the holes reduced to mere physical points, these lines form what we call rays of light. Mathematically speaking, a ray of light is an infinitesimal pyramid, having for its vertex a luminous point, and for its base an infinitely small portion of any surface illuminated by it, and supposed to be filled with the luminous emanation, whatever that may be. This pyramid, in homogeneous media, and when the course of the ray is not interrupted, has, as we have seen,

* In the eclipse of September 7, 1820, this horned appearance was very striking in the luminous interstices between the shadows of small irregular objects, as the leaves of trees, &c. It was noticed by those who had no idea of its cause.

its sides straight lines. If cases should occur (as they will) when the course of the light is curved, or suddenly broken, we may still conceive such a pyramid having curved or broken sides to correspond; or we may (for brevity's sake) substitute for it a mere mathematical line, straight, curved, or broken, as the case may be.

9. Velocity of light. Light requires time for its propagation. Two spectators at different distances from a luminous object suddenly disclosed, will not begin to see it at the same mathematical instant of time. The nearer will see it sooner than the more remote; in the same way as two persons at unequal distances from a gun hear the report at different moments. In like manner, if a luminous object be suddenly extinguished, a spectator will continue to see it for a certain time afterwards, as if it still continued luminous, and this time will be greater the farther he is from it. The interval in question is, however, so excessively small in such distances as occur on the earth's surface, as to be absolutely insensible; but in the immense expanse of the celestial regions the case is different. The eclipses and emersions of Jupiter's satellites become visible much sooner (nearly a quarter of an hour) when the earth is at its least distance from Jupiter than when at its greatest. *Light then takes time to travel over space.* It has a finite, though immense velocity, viz. 192500 miles per second; and this important conclusion, deduced by calculation from the phenomenon just mentioned, and which, if it stood unsupported, might startle us with its vastness, and incline us to look out for some other mode of explanation, receives full confirmation from another astronomical phenomenon, the *aberration* of light, which (without entering into any close examination of the mode in which vision is produced) may be explained as follows:

Aberration of light.

10. Fig. 1. Let a ray of light from a star S, at such a distance that all rays from it may be regarded as parallel, be received on a small screen A, having an extremely minute opening A in its centre; and let that ray which passes through the opening be received at any distance A B, on a screen B perpendicular to its direction; and let B be the point on which it falls, the whole apparatus being supposed at rest. If then we join the points A, B by an imaginary line, that line will be the direction in which the ray has really travelled, and will indicate to us the direction of the star; and the angle between that line and any fixed direction (that of the plumb-line, for instance) will determine the star's place as referred to that fixed direction. For simplicity, we will suppose this angle nothing, or the star directly vertical; then the point B on which the ray falls will be precisely that marked by a plumb-line let fall from A; and the direction in which we judge the star to lie will coincide precisely with the direction of gravity. Such will be the case, supposing the earth, the spectator, and the whole apparatus at rest; but now suppose them carried along in a horizontal direction A C, B D, with a uniform and equal velocity, of whose existence they will therefore be perfectly insensible, and the pumb-line will hang steadily as before, and coincide with the same point of the screen. At the moment when the ray S A from the star passes through the orifice A, let A, B be the respective places of the orifice, and the point on the screen vertically below it. When the ray has passed the orifice, it will pursue its course in the same straight line S A B as before, independent of the motion of the apparatus, and in some certain time $\left(= \frac{\text{distance A B}}{\text{velocity of light}} = t\right)$ will reach the lower screen. But in this time the aperture, screens, and plumb-line will have moved away through a space

$$\text{A } a = \text{B } b \left(= t \times \text{velocity of motion} = \text{A B} \times \frac{\text{earth's velocity}}{\text{velocity of light}}\right)$$

At the instant, then, that the ray impinges on the lower screen, the plumb-line will hang, not from A on B, but from *a* on *b*; and *a* being the *real orifice*, and B the *real point of incidence* of the light on the screen, the spectator, judging only from these facts, will necessarily be led to regard the ray as having deviated from its vertical direction, and as inclining from the vertical, in the direction of the earth's motion through an angle whose tangent is $\frac{\text{A } a}{\text{A B}}$ or $\frac{\text{earth's velocity}}{\text{velocity of light}}$.

11. The eye is such an apparatus. Its retina is the screen on which the light of the star or luminary falls, and we judge of its place only by the actual point on this screen where the impression is made. The pupil is the orifice. If, the eye preserving a fixed direction, the whole body be carried to one side with a velocity commensurate to that of light, before the rays can traverse the space which separates the pupil from the retina, the latter will have shifted its place; and the point which receives the impression is no longer the same which would have received it had the eye and spectator remained at rest; and this deviation is the *aberration of light.*

12. Every spectator on the earth participates in the general motion of the whole earth, which in its annual orbit about the sun is very rapid, and though far from equal to that of light, is by no means insensible, compared to it. Hence the stars, the sun, and planets, all appear removed from their true places in the direction in which the earth is moving.

13. This direction is varying every instant, as the earth describes an orbit round the sun. The direction therefore of the apparent displacement of any star from its true situation continually changes, *i. e.* the apparent place describes a small orbit about the true. This phenomenon is that alluded to. It was noticed as a fact by Bradley, while ignorant of its cause, that the stars appear to describe annually small ellipses in the heavens of about 40″ in diameter. The discovery of the velocity of light by the eclipses of Jupiter's satellites, then recently made by Roemer, however, soon furnished its explanation. Later observations, especially those of Brinkley and Struve, have enabled us to assign, with great precision, the numerical amount of this inequality, and thence to deduce the velocity of light, which by this method comes out 191515 miles per second, differing

Light. from the former only by a two hundredth part of its whole quantity. This determination is certainly to be preferred. Part I.

14. Light uniform in its motion. But this is not the only information respecting light which astronomical observations furnish. We learn from them also, "That the light of the sun, the planets, and all the fixed stars, travels with one and the same velocity." Now as we know these bodies to be at different and variable distances from us, we hence conclude that the velocity of light is independent of the particular source from which it emanates, and the distance over which it has travelled before reaching our eye.

15. The velocity of light, therefore, in that free and perhaps void space which intervenes between us and the planets and fixed stars, cannot be supposed other than uniform; and the calculations of the eclipses of Jupiter's satellites, and the places of the distant planets made on this supposition agreeing with observation, prove it to be so. In entering such media as it traverses, when arrived within the limits of the atmospheres of the earth and other planets, we shall find reason hereafter to conclude that its velocity undergoes a change; but, at all events, we have no reason to suppose it to differ in different parts of one and the same homogeneous medium.

16. Velocity of light illustrated by comparisons. The enormous velocity here assigned to light, surprising as it may seem, is among those conclusions which rest on the best evidence that science can afford, and may serve to prepare us for other yet more amazing numerical estimates. It is when we attempt to measure the vastness of the phenomena of nature with our feeble scale of units, such as we are conversant with on this our planet, that we become sensible of its insignificance in the system of the universe. Demonstrably true as are the results, they fail to give us distinct conceptions; we are lost in the immensity of our numbers, and must have recourse to other ways of rendering them sensible. A cannon ball would require seventeen years at least to reach the sun, supposing its velocity to continue uniform from the moment of its discharge. Yet light travels over the same space in $7\frac{1}{2}$ minutes. The swiftest bird, at its utmost speed, would require nearly three weeks to make the tour of the earth. Light performs the same distance in much less time than is required for a single stroke of his wing; yet its rapidity is but commensurate to the distances it has to travel. It is demonstrable that light cannot possibly arrive at our system from the nearest of the fixed stars in less than five years, and telescopes disclose to us objects probably many thousand times more remote.

But these are considerations which belong rather to astronomy than to the present subject; and we will, therefore, return to the consideration of the phenomena of emitted light.

§ II. *Of Photometry.*

17. Light diminishes as the distance of its source increases. Of these, one of the most striking is certainly the diminution of the illuminating power of any source of light, arising from an increase of its distance. We see very well to read by the light of a candle at a certain distance: remove the candle twice, or ten times as far, and we can see to read no longer.

The numerical estimation of the degrees of intensity of light constitutes that branch of optics which is termed *Photometry.* (*φως, μετρω.*)

18. Its intensity inversely as the square of the distance. If light be a material emanation, a something scattered in minute particles in all directions, it is obvious that the same quantity which is diffused over the surface of a sphere concentric with the luminous points, if it continue its course, will successively be diffused over larger and larger concentric spherical surfaces; and that its intensity, or the number of rays which fall on a given space, in each will be inversely as the whole surfaces over which it is diffused; that is, inversely as the squares of their radii, or of their distances from the source of light. Without assuming this hypothesis, the same thing may be rendered evident as follows. Let a candle be placed behind an opaque screen full of small equal and similar holes; the light will shine through these, and be intercepted in all other parts, forming a pyramidal bundle of rays, having the candle in the common vertex. If a sheet of white paper be placed behind this, it will be seen dotted over with small luminous specks, disposed exactly as the holes in the screen. Suppose the holes so small, their number so great, and the eye so distant from the paper that it cannot distinguish the individual specks, it will still receive a general impression of brightness; the paper will appear illuminated, and present a mottled appearance, which, however, will grow more uniform as the holes are smaller, and closer, and the eye more distant; and if extremely so, the paper will appear uniformly bright. Now, if every alternate hole be stopped, the paper will manifestly receive only half the light, and will therefore be only half as much illuminated, and *cæteris paribus*, the degree of illumination is proportional to the number of the holes in the screen, or to the number of equally illuminated specks on its surface, *i. e.* if the specks be infinitely diminished in size, and infinitely increased in number, to the number of rays which fall on it from the original source of light.

19. Let a screen, so pierced with innumerable equal and very small holes in the manner described, be placed at a given distance (1 yard) from a candle; and in the diverging pyramid of rays behind it place a small piece of white paper of a given area, (1 square inch, for instance,) so as to be entirely included in the pyramid. It is manifest that the number of rays which fall on it will be fewer as it is placed farther from the screen, because the whole number which pass the screen are scattered continually over a larger and larger space. Thus were it close to the screen it would receive a number equal to that of the holes in a square inch of the screen, but at twice the distance (2 yards) from the candle this number will be spread over four square inches by their divergence, and the paper can therefore receive only a fourth part of that number. If, therefore, its illumination when close to the screen be represented by I, it will at twice the distance be only $\frac{I}{4}$, and

at D times the assumed unit of distance, its illumination will be $\frac{I}{D^2}$, the areas of sections of a pyramid being as the squares of their distances from the vertex.

20. As this reasoning is independent of the number and size of the holes, and therefore of the ratio of the sum of their areas to that of the unperforated part of the screen, we may conceive this ratio increased *ad infinitum*. The screen then disappears, and the paper is freely illuminated. Hence we conclude that when a small plane object of given area is freely and perpendicularly exposed to a luminary at different distances, the quantity of light it receives, or the degree of its illumination, is inversely as the squares of its distance from the luminary, *cæteris paribus*.

21. Illumination proportional to the number and intensity of the rays; If a single candle be placed before a system of holes in a screen, as before, and the rays received on a screen at a given distance, (1,) the degree of illumination may be represented by a given quantity, I. Now if a second candle be placed immediately behind the other, and close to it, so as to shine through the same holes, the illumination of the screen is perceived to be increased, though the number and size of the illuminated points on it be the same. Each point is then said to be more intensely illuminated. Now, (the eye being all along supposed so distant, and the illuminated points so small as to produce only a general sense of brightness, without distinguishing the individual points,) if the one candle be shifted a little sideways, without altering its distance, the illumination of the paper will not be altered. In this case the number of illuminated points is doubled, but each is illuminated with only half the light it had before. The same holds for any number of candles. Hence we conclude that the illumination of a surface is constant when the number of rays it receives is inversely as the intensity of each, and that consequently the degree of illumination is proportional to the number and intensity of the rays jointly.

22. and to the area of the illuminating surface. Now if for any number of candles placed side by side we substitute mere physical luminous points, each of these will be the vertex of a pyramid of rays, and the number of equally illuminated points in the paper, and therefore illuminations will be proportional to the number of such points. If we conceive the number of these increased, and their size diminished *ad infinitum*, so as to form a continuous luminous surface, their number will be represented by its area. Hence the illumination of the paper will be, *cæteris paribus*, as the area of the illuminating surface, (supposed of uniform brightness.)

23. General expression. Uniting all these circumstances, we see that when a given object is enlightened by a luminous surface of small but sensible size, the degree of its illumination is proportional to the

$$\frac{\text{area of the luminous surface} \times \text{intensity of its illuminating power}}{\text{square of the distance of the surface illuminated.}}$$

24. Oblique illumination. The foregoing reasoning applies only to the case when the luminous disc is a small portion of a spherical surface concentric with the illuminated object, in which case all its points are equidistant from it, and all the light falls perpendicularly on the object. When the object is exposed obliquely, conceive its surface divided into equal infinitely small portions, and regard each of them as the base of an oblique pyramid, having its vertex at any one point of the luminary; then will the perpendicular section of this pyramid at the same distance be equal to the base $\times$ sine of inclination of the base to the axis, or the element of the illuminated surface $\times$ by the sine of the inclination of the ray. But the number of rays which falls on the base is evidently equal to those which fall on the section, and being spread over a larger area their effect will be to illuminate it less intensely in the proportion of the area of the section to that of the base, *i. e.* in the proportion of the sine of inclination to radius. But the illumination of the section is equal to the

$$\frac{\text{area of the luminary} \times \text{intrinsic brightness}}{(\text{distance})^2} \quad (23)$$

therefore that of the elementary surface equals this fraction multiplied by the sine of the rays' inclination; or, calling A the area of the luminary, I its intrinsic brightness, D its distance, and θ the inclination of the ray to the illuminated surface $\frac{A \,.\, I \sin \theta}{D^2}$ will represent the intensity of illumination.

25. If L represent the absolute quantity of light emitted by the luminary *in a given direction*, which may be called its *absolute light*, we have $L = A \times I$, provided the surface of the luminary be perpendicular to the given direction. If not, A must represent the area of the section of a cylindroidal surface bounded by the outline of the luminary, and having its axis parallel to the given direction; consequently $\frac{L \,.\, \sin \theta}{D^2}$ represents in this case the intensity of illumination of the elementary surface.

To illustrate the application of these principles we will resolve the following

Problem.

26. A small white surface is laid horizontally on a table, and illuminated by a candle placed at a given (horizontal) distance: What ought to be the height of the flame, so as to give the greatest possible illumination to the surface?

Fig. 2. Let A be the surface, BC the candle. Put $AB = a$. $AC = D$; $BC = \sqrt{D^2 - a^2}$. Then, since the illumination of A is, *cæteris paribus*, as $\frac{\sin CAB}{AC^2}$, or as $\frac{CB}{AC^3} = \frac{\sqrt{D^2 - a^2}}{D^3}$ (= F) we have to make this

Light. quantity a maximum; consequently $d\,F = o$, or $d\,.\,F^2 = o$, that is, Part I.

$$d\left\{\frac{1}{D^4} - \frac{a^2}{D^6}\right\} = o, \text{ or } -\frac{4}{D^5} + \frac{6\,a^2}{D^7} = o,\ 2\,D^2 - 3\,a^2 = o,$$

$$\text{or } D = a\,.\,\sqrt{\frac{3}{2}} \text{ and } BC = \sqrt{D^2 - a^2} = \frac{a}{\sqrt{2}} = 0.707 \times AB.$$

27. Apparent magnitude defined. *Definition.* The *apparent superficial magnitude*, or the apparent magnitude of any object, is a portion of a spherical surface described about the eye as a centre, with a radius equal to 1, and bounded by an outline being the intersection of this spherical surface with a conoidal surface, having the object for its base and the eye for its vertex.

28. Hence the apparent superficial magnitude of a small object is directly as the area of a section (perpendicular to the line of sight) of this conoidal surface, at the place of the object, and inversely as the square of its distance. If the object be a surface perpendicular to the line of sight, this ratio reduces itself to the area of the object divided by the square of its distance.

29. Real intrinsic brightness defined. *Definition.* The *real intrinsic brightness* of a luminous object is the intensity of the light of each physical point in its surface, or the numerical measure of the degree in which such a point (of given magnitude) would illuminate a given object at a given distance, referred to some standard degree of illumination as a unit. When we speak simply of intrinsic brightness, *real intrinsic brightness* is meant.

30. *Corol.* 1. Consequently the degree of illumination of an object exposed perpendicularly to a luminary is as the apparent magnitude of the luminary and its intrinsic brightness jointly.

31. *Corol.* 2. Conversely, if these remain the same, the degree of illumination remains the same. For example, the illumination of direct sunshine is the same as would be produced by a circular portion of the surface of the sun of one inch in diameter, placed at about 10 feet from the illuminated object, and the rest of the sun annihilated; for such a circular portion would have the same apparent superficial magnitude as the sun itself. This will serve to give some idea of the intense brightness of the sun's disc.

32. Apparent intrinsic brightness. *Definition.* The *apparent intrinsic brightness* of any object, or luminary, is the degree of illumination of its image or picture at the bottom of the eye. It is this illumination only by which we judge of brightness. A luminary may in reality be ever so much brighter than another; but if by any cause the illumination of its image in the eye be enfeebled, it will appear no brighter than in proportion to its diminished intensity. Thus we can gaze steadily at the sun through a dark glass, or the vapours of the horizon.

33. Absolute light defined. *Definition.* The *absolute light* of a luminary is the sum of the areas of its elementary portions, each multiplied by its own intrinsic brightness; or, if every part of the surface be equally bright, simply the area multiplied by the intrinsic brightness. It is, therefore, the same quantity as that above represented by L.

34. Apparent light defined. *Definition.* The *apparent light* of an object is the total quantity of light which enters our eyes from it, however distributed on the retina.

In common language, when we speak of the brightness of an object of considerable size, we often mean its *apparent intrinsic brightness.* When, however, the object has no sensible size, as a star, we always mean its

35. *apparent light*, (or, as it might be termed, its *apparent absolute brightness*,) because, as we cannot distinguish such an object into parts, we can only be affected by its total light indiscriminately. The same holds good with all small objects which require attention to distinguish them into parts. Optical writers have occasionally fallen into much confusion for want of attending to these distinctions.

36. Diminution of apparent light by distance. As we recede from a luminary, its *apparent light* diminishes, from two causes; first, our eyes, being of a given size, present a given area to its light, and therefore receive from it a quantity of light inversely as the square of the distance; secondly, in passing through the air, a portion of the light is stopped, and lost from its want of perfect transparency. This, however, we will not now consider. In virtue of the first cause only, then, the apparent light of a luminary is inversely as the square of its distance, and directly as its *absolute light.*

37. Objects appear equally bright at all distances. The apparent intrinsic brightness is equal to the apparent light divided by the area of the picture on the retina of our eye. But this area is as the apparent superficial magnitude of the luminary, that is, as its real area A divided by the square of its distance D, or as $\frac{A}{D^2}$. Moreover, the apparent light, as we have just seen, is as $\frac{A\,I}{D^2}$ where I is the real intrinsic brightness. Consequently the apparent intrinsic brightness is proportional to $\frac{A\,I}{D^2} \div \frac{A}{D^2}$, or simply to I, and is independent on A or D. The apparent intrinsic brightness is, therefore, the same at all distances, and is simply proportional to the real intrinsic brightness of the object. This conclusion is usually announced by optical writers by saying, that *objects appear equally bright at all distances,* which must be understood only of *apparent intrinsic brightness*, and the truth of which supposes also that no loss of light takes place in the media traversed.

In what sense to be understood.

38. Angle of emanation defined. *The angle of emanation* of a ray of light from a luminous surface is the inclination of the ray to the surface at the point from which it emanates.

39. A question has been agitated among optical philosophers, whether the intensity of the light of luminous bodies be the same in all directions; or whether, on the other hand, it be not dependent on the angle of emanation. Euler, in his *Réflexions sur les divers degrés de la lumière du Soliel, &c.* Berlin, Mém. 1750, p. 280, has adopted

Question among opticians respecting the dependence of the emission of light on the angle of emanation.

the former principle. Lambert, on the other hand, *Photometria*, p. 41, contends that the intensity of the light, or density of the rays, issuing from a luminous surface in any direction is proportional to the line of the angle of emanation. If we knew the intimate nature of light, and the real mechanism by which bodies emit and reflect it, it might be possible to decide this question *à priori*. If, for instance, we were assured that light emanated strictly and solely from the molecules situated on the external surface of bodies, and that the emanation from each physical point of the surface were totally uninfluenced by the rest of the molecules of which the body consists, and dispersed itself equally in all directions, then, since every point of a plane surface is visible to an eye wherever situated above it, and each is supposed to send the same number of rays to the eye in an oblique as in a perpendicular situation, the *total light* received from a given area of the surface in the eye ought to be the same at all angles of emanation. But as the *apparent magnitude* of this area is as the sine of its inclination to the line of sight, *i. e.* of its angle of emanation, this light is distributed over a less *apparent* area; and therefore its intensity, or the apparent brightness of the surface, should be increased in the inverse ratio of the sine of the angle of emanation. On the other hand, if, as there is every reason to suppose, light emanates, not strictly from the surfaces of bodies, but from sensible depths within their substance; if the surfaces themselves be not true mathematical planes, but consist of a series of physical points retained in their places by attractive and repulsive forces, and if the intensity of emanation of each of these points depend in any way on its relation to the points adjacent, there is no reason, *à priori*, to suppose the equal emanation of light in all directions; and to find what its law really is, we must have recourse to direct observation.

Astronomy teaches us that the sun is a sphere. Hence the several parts of its visible disc appear to us under every possible angle of inclination. Now if we examine the surface of the sun with a telescope, the circumference certainly does not appear brighter than the centre. But if the hypothesis of equal emanation were correct, the brightness ought to increase from the centre outwards, and should become infinite at the edges, so that the disc ought to appear surrounded by an annulus of infinitely greater splendour than the central parts. To this it may, however, be justly objected, that as the surface of the sun is obviously though *generally* spherical, yet full of local irregularities, every minute portion of it may be regarded as presenting every possible variety of inclination to our eye; and the brightness of every part being thus an average of all the gradations of which it is susceptible, should be alike throughout.

40. Bouguer, in his *Traité d'Optique*, Paris, 1760, p. 90, states himself to have found, by direct comparison, that the central portions of the disc of the sun are really *much more* luminous than the borders. A result so extraordinary, however, and so apparently incompatible with all we know of the constitution of the sun and the mode of emission of light from its surface, would require to be verified by very careful and delicate reexamination. If found correct, the only way of accounting for it would be to suppose a dense and imperfectly transparent atmosphere of great extent floating above the luminous clouds which form its visible surface. This is certainly possible, but our ignorance on the subject renders it unphilosophical to resort to a body so little within our reach for the establishment of any fundamental law of emanation. The objection above advanced, it will be observed, applies with nearly the same force to all surfaces. If we examine a piece of white paper with a magnifier, we shall find its texture to be in the last degree rough and coarse, presenting no approach to a plane; and so of all surfaces rough enough to reflect light in all directions.

41. Surfaces appear equally bright at all angles.

However, as it is only with such luminous surfaces as occur in nature that we have any concern, we must take their properties as we find them; and, waiving all consideration of what would be the law of emanation from a mathematical surface, it may be stated as a result of observation, that *luminous surfaces appear equally bright at all angles of inclination to the line of sight.*

This may be tried with a surface of red-hot iron; its apparent *intrinsic brightness* is not sensibly increased by inclining it obliquely to the eye.

42. Experimental proof of the law of emanation.

If we take a smooth square bar of iron, or better, of silver, or a polished cylinder of either metal, heated to redness, into a dark room, the cylinder will appear equally bright in the middle of its convexity next the eye, and at the edges, and cannot be distinguished at all from a flat bar; and the square bar, when so presented as to have two of its sides at very different angles to the line of sight, will still appear of perfectly equable brightness, nor can the angle separating the adjacent sides be at all discerned; and if the whole bar be turned round on its axis, the motion can only be recognised by an alternating increase and decrease of its apparent diameter, according as it is seen alternately diagonally and laterally, its appearance being always that of a flat plate perpendicularly exposed to the eye. These and similar experiments with surfaces artificially illuminated, which the reader will have no difficulty in imagining and making, as well as those recorded by Mr. Ritchie in the *Edinburgh Philosophical Journal*, are sufficient to establish the principle announced in Article 42, to which (for the reasons already mentioned) the observation of Bouguer on the unequal brightness of the sun's disc offers no conclusive objection.

43. Law of the oblique emanation of light.

Hence it follows, that the surfaces of luminous bodies, at least their ultimate molecules, do not emit light with equal copiousness in all directions; but that, on the contrary, *the copiousness of emission, in any direction, is as the sine of the angle of emanation from the surface.*

Problem.

44. To determine the intensity of illumination of a small plane surface any how exposed to the rays from a luminary of any given size, figure, and distance; the luminary being supposed uniformly bright in every part. Conceive the surface of the luminary divided into infinitesimal elementary portions, of which let each be regarded as an oblique section of a pyramid, having for its vertex the centre of the infinitely small illuminated

Light. Illumination of a plane by any luminary investigated. Fig. 3.

plane B, fig. 3. Let P Q be any such portion, and let the pyramid B P be continued till it meets the surface of the heavens in p, there projecting the surface P Q into the areola $p\,q$, and let the whole luminary C D E F be in like manner projected into the disc $c\,d\,e\,f$. Let π Q be a section of the pyramid A P Q, perpendicular to its axis. Then, first, the plane B will be illuminated by the element P Q, just as it would be by a surface π Q equally bright, in virtue of the principle just established. Hence P Q is equivalent to an equally bright surface π Q. Again, since the apparent magnitude of π Q seen from B is the same with that of $p\,q$, the area π Q is equivalent to an equally bright area $p\,q$ placed at $p\,q$, (Art. 29, 30, 31, Cor. 1, 2.) P Q is, therefore, equivalent to $p\,q$. And since the same holds good of every other elementary portion of the surface, and the total light received by B is the sum of the lights it receives from all the elements of the luminary, the whole surface C D E F must be equivalent to its projection $c\,d\,e\,f$.

45. Hence the illumination of B depends, not at all on the real, but only on the apparent figure and magnitude of the luminary; and whatever the luminary be, we may always substitute for it a portion of the visible heavens, supposed of equal intrinsic brightness, and bounded by the same outline.

46. Thus, instead of the sun, we may suppose a small circle equal in apparent diameter to the sun, and equally bright; instead of a luminous rectangle perpendicular to the illuminated plane B, and of infinite height, as A G H I, fig. 3, we may substitute the spherical sector Z A G, bounded by the two vertical circles Z A, Z G, and so on.

47. Let then $p\,q$, any elementary rectangle infinitely small in both dimensions of the spherical surface, be represented by d^2 A, so that $\iint d^2$ A shall represent the surface $c\,d\,e\,f$ itself; then if we put $z =$ the zenith distance Z p of this portion, its illuminating power on A will be d^2 A . cos z, and the total illuminating power of the whole surface A will be

$$\mathrm{L} = \iint d^2\,\mathrm{A} \,.\, \cos z.$$

48. General formula for illumination of a small plane.

Example 1. To find the illuminating power of the sector Z A G confined between any two vertical circles and the horizon. (fig. 3.) Here, putting θ for the azimuth of the element d^2 A, if we consider it as terminated by two contiguous verticals and two contiguous parallels of altitude, we have $d^2\,\mathrm{A} = dz \times d\theta \,.\, \sin z$. Hence we have

$$\mathrm{L} = \iint d\theta\, dz \,.\, \sin z \,.\, \cos z = \tfrac{1}{2} \iint d\theta\, dz \,.\, \sin 2z = \tfrac{1}{2} \int (\theta + \mathrm{C})\, dz \,.\, \sin 2z\,;$$

and extending the integral from $\theta = o$ to $\theta =$ A G, the amplitude of the sector, (which we will call a,) we get

$$\mathrm{L} = \frac{a}{2} \int dz \,.\, \sin 2z = \frac{a}{2} (\mathrm{C} - \tfrac{1}{2} \cos 2z)$$

which extended from Z $= o$ to $z = 90°$ becomes simply $\mathrm{L} = \frac{a}{2}$.

49. *Corol.* 1. This is a measure of the illuminating power of the sector, on *the same scale* that that of an infinitely small area (A) placed at the zenith would be represented by A itself. Because in this case

$$\cos z = o, \text{ and } \iint d^2\,\mathrm{A} \,.\, \cos z = \mathrm{A}.$$

50. *Corol.* 2. On the same scale the illuminating power of the whole hemisphere is π where $\pi = 3.14159535$

51. *Example* 2. Required the illuminating power of a circular portion of the heavens whose centre is the zenith. Calling z the zenith distance of any element, and θ its azimuth, we shall have, as before,

$d^2\,\mathrm{A} = d\theta\, dz \,.\, \sin z$, and therefore $\mathrm{L} = \iint d\theta\, dz \,.\, \sin z \,.\, \cos z = \int \theta \,.\, \frac{dz \,.\, \sin 2z}{2} = \pi \int dz \,.\, \sin 2z$

extending the integral from $\theta = o$ to $\theta = 2\pi$. That is $\mathrm{L} = \pi \,.\, (\text{const} - \frac{1}{2} \cos 2z)$ which being made to vanish when $z = o$ becomes $\mathrm{L} = \frac{\pi}{2} (1 - \cos 2z) = \pi \,.\, (\sin z)^2$

52. *Corol.* 3. The illuminating power of a circular luminary, whose centre is in the zenith, is as the square of the sine of its apparent semidiameter.

53. Illuminating power of any circular portion of an equally bright heaven.

Example 3. Required the illuminating power of any circular portion of the heavens whatever.

Let T K L M be the illuminating circle; conceive it composed of annuli concentric with itself, and of one of them, X Y Z, (fig. 4,) let X x be an infinitesimal parallelogram terminated by contiguous radii S X and S x, S being the centre.

Put Z S $= a$; S X $= x$, Z X $= z$,

Angle Z S X $= \phi$, S T $= r$.

Area d^2 A $=$ X x, $= dx \times d\phi \,.\, \sin x$

$$\therefore \mathrm{L} = \iint d\phi\, dx \,.\, \sin x \,.\, \cos z.$$

but, by spherical trigonometry, $\cos z = \cos a \,.\, \cos x \,.\, + \sin a \,.\, \sin x \cos \phi$.

Light. Therefore $$L = \iint dx \,.\, d\phi \,.\, \sin x \left\{ \cos a \,.\, \cos x + \sin a \,.\, \sin x \,.\, \cos \phi. \right\}$$ Part I.

The first integration performed relative to ϕ, and extended from $\phi = o$ to $\phi = 360^0$, or 2π, gives

$$L = \int dx \,.\, \sin x \times 2\pi \,.\, \cos a \,.\, \cos x.$$

After which integrating, with respect to x, and extending the integral from $x = o$ to $x = ST = r$, we find

$$L = \frac{\pi \,.\, \cos a}{2} (1 - \cos 2r) = \pi \,.\, \cos a \,(\sin r)^2.$$

This result is particularly elegant and remarkable. It shows, that to obtain the illuminating effect of a circular luminary (of any apparent diameter) at any altitude, on a horizontal plane, we have only to reduce its illuminating effect when in the zenith, in the ratio of radius to the cosine of the zenith distance, or sine of the altitude. For other examples, the reader may consult Lambert's *Photometria*, cap. ii. from which this is taken.

54. General expression for illumination when the luminary is not equally bright throughout.

If the illuminating surface be not equally intrinsically bright in every part, if we call I the intrinsic brightness of the element d^2 A, we shall have

$$L = \iint I \, d^2 A \,.\, \cos z$$

for the general formula expressing the illuminating power of the surface A. The moon, Venus and Mercury in their phases, the sky during twilight, a white sphere illuminated by the sun, &c. afford examples of this when themselves regarded as luminaries.

PROBLEM.

55. To compare the illumination of a horizontal plane by the sun in the zenith with the illumination it would have were the whole surface of the heavens of equal brightness with the sun.

By Art. 53 we have $L = \pi \,.\, \cos a \,.\, (\sin r)^2$. If, therefore, we call L and L′ the two illuminations in question, we shall have

$$L : L' :: \pi \,.\, \cos o^\circ \,.\, (\sin \odot\text{'s semidiam.})^2 : \pi \,.\, \cos o^\circ \,.\, (\sin 90^\circ)^2$$
$$:: (\sin 16')^2 : 1 :: 1 : 46166.$$

56. Illumination at the sun's surface.

The illumination of a plane in contact with the sun's surface is the same as that of a plane on the earth's surface illuminated by a whole hemisphere of equal brightness with the sun in the zenith. Hence we see that the illumination of such a plane as the sun's surface would be nearly 50,000 times greater than that of the earth's surface at noon under the equator. Such would be the effect (in point of light alone) of bringing the earth's surface in contact with the sun's!

57. Photometers

For measuring the intensity of any given light, various instruments called Photometers have been contrived, many of which have little to recommend them on the score of exactness, and some are essentially defective in principle, being adapted to measure—not the illuminating—but the heating power of the rays of light; and, therefore, must be regarded as undeserving the name of photometers.

58. The eye an imperfect judge of degrees of illumination

We know of no instrument, no contrivance, as yet, by which light alone (as such) can be made to produce mechanical motion, so as to mark a point upon a scale, or in any way to give a direct reading off of its intensity, or quantity, at any moment. This obliges us to refer all our estimations of the degrees of brightness at once to our organs of vision, and to judge of their amount by the impression they produce immediately on our sense of sight. But the eye, though sensible to an astonishing range of different degrees of illumination, is (partly on that very account) but little capable of judging of their relative strength, or even of recognising their identity when presented at intervals of time, especially at distant intervals. In this manner the judgment of the eye is as little to be depended on for a measure of light, as that of the hand would be for the weight of a body casually presented. This uncertainty, too, is increased by the nature of the organ itself, which is in a constant state of fluctuation; the opening of the pupil, which admits the light, being continually expanding and contracting by the stimulus of the light itself, and the sensibility of the nerves which feel the impression varying at every instant. Let any one call to mind the blinding and overpowering effect of a flash of lightning in a dark night, compared with the sensation an equally vivid flash produces in full daylight. In the one case the eye is painfully affected, and the violent agitation into which the nerves of the retina are thrown is sensible for many seconds afterwards, in a series of imaginary alternations of light and darkness. By day no such effect is produced, and we trace the course of the flash, and the zig-zags of its motion with perfect distinctness and tranquillity, and without any of those ideas of overpowering intensity which previous and subsequent total darkness attach to it.

59. But yet more. When two unequally illuminated objects (surfaces of white paper, for instance) are presented at once to the sight, though we pronounce immediately on the existence of a difference, and see that one is brighter than the other, we are quite unable to say what is the proportion between them. Illuminate half a sheet of paper by the light of one candle, and the other half by that of several; the difference will be evident. But if ten different persons are desired, from their appearance only, to guess at the number of candles shining on each, the probability is that no two will agree. Nay, even the same person at different times will form different judgments. This throws additional difficulty in the way of photometrical estimations, and would seem to render this one of the most delicate and difficult departments of optics.

However, the eye, under favourable circumstances, is a tolerably exact judge of the *equality* of two degrees of illumination seen at once; and availing ourselves of this, we may by proper management obtain correct information as to the relative intensities of all lights. What these favourable circumstances are, we come now to consider.

60. The eye capable of judging of the equality of two degrees of illumination, and under what circumstances.

1st. The degrees of illumination compared must be of moderate intensity. If so bright as to dazzle, or so faint as to strain the eye, no correct judgment can be formed.

Hence, it is rarely adviseable to compare two luminaries directly with each other. It is generally better to let them shine on a smooth white surface, and judge of the degree in which they illuminate it; for it is an obvious axiom, *That two luminaries are equal in absolute light when, being placed at equal distances from, and in similar situations with respect to, a given smooth white surface, or two equal and precisely similar white surfaces, they illuminate it or them equally.*

Axiom in photometry.

63. 2nd. The luminaries, or illuminated surfaces compared, must be of equal apparent magnitude, and similar figure, and of such small dimensions as to allow of the illumination in every part of each being regarded as the same.

64. 3rd. They must be brought close together, in apparent contact; the boundary of one cutting upon that of the other by a well-defined straight line.

65. 4th. They should be viewed at once by the same eye, and not one by one eye, and the other by the other.

66. 5th. All other light but that of the two objects whose illumination is compared should be most carefully excluded.

67. 6th. The lights which illuminate both surfaces must be of the same colour. Between very differently coloured illuminations no exact equalization can ever be obtained, and in proportion as they differ our judgment is uncertain.

68. When all these conditions obtain, we can pronounce very certainly on the equality or inequality of two illuminations. When the limit between them cannot be perceived, on passing the eye backwards and forwards across it, we may be sure that their lights are equal.

69. Bouguer's principle of comparative photometry.

Bouguer, in his *Traité d'Optique*, 1760, p. 35, has applied these principles to the measure or rather the comparison of different degrees of illumination. Two surfaces of white paper, of exactly equal size and reflective power, (cut from the same piece in contact,) are illuminated, the one by the light whose illuminating power is to be measured, the other by a light whose intensity can be varied at pleasure by an increase of distance, and can therefore be exactly estimated. The variable light is to be removed, or approached, till the two surfaces are judged to be equally bright, when, the distances of the luminaries being measured, or otherwise allowed for, the measure required is ascertained.

70. Ritchie's photometer:

Mr. Ritchie has lately made a very elegant and simple application of this principle. His photometer consists of a rectangular box, about an inch and a half or two inches square, open at both ends, of which A B C D (fig. 5) is a section. It is blackened within, to absorb extraneous light. Within, inclined at angles of 45° to its axis, are placed two rectangular pieces of plane looking-glass F C, F D, *cut from one and the same rectangular strip*, of twice the length of either, to ensure the exact equality of their reflecting powers, and fastened so as to meet at F, in the middle of a narrow slit E F G about an inch long, and an eighth of an inch broad, which is covered with a slip of fine tissue or oiled paper. The rectangular slit should have a slip of blackened card at F, to prevent the lights reflected from the looking-glasses mingling with each other.

71. its use.

Suppose we would compare the illuminating powers of two sources of light (two flames, for instance) P and Q. They must be placed at such a distance from each other, and from the instrument between them, that the light from every part of each shall fall on the reflector next it, and be reflected to the corresponding portion of the paper E F or F G. The instrument is then to be moved nearer to the one or the other, till the paper on either side of the division F appears equally illuminated. To judge of this, it should be viewed through a prismoidal box blackened within, one end resting on the upper part A B of the photometer; the other applied quite close to the eye. When the lights are thus exactly equalized, it is clear that the total illuminating powers of the luminaries are directly as the squares of their distances from the middle of the instrument.

72. Experimental proof of the diminution of light as the squares of the distances

By means of this instrument we are furnished with an easy experimental proof of the decrease of light as the inverse squares of the distances. For if we place four candles at P, and one at Q, (as nearly equal as possible, and burning with equal flames,) it is found that the portions E F, G F of the paper will be equally illuminated when the distances P F, Q F are as 2 : 1, and so for any number of candles at each side.

To render the comparison of the lights more exact, the equalization of the lights should be performed several times, turning the instrument end for end each time. The mean of the several determinations will then be very near the truth.

73.

74. Another construction.

In some cases the looking-glasses are better dispensed with, and a slip of paper pasted over them, so as to present two oblique surfaces of white paper inclined at equal angles to the incident light. In this case the paper stretched over the slit E F G is taken away, and the white surfaces below examined and compared. One advantage of this disposition is the avoiding of a black interval between the two halves of the slit, which renders the exact comparison of their illuminations somewhat precarious.

75. Comparison of lights of different colours.

If the lights compared be of different colours (as daylight, or moonlight, and candlelight,) their precise equalization is impracticable, (art. 67.) The best way of employing the instrument, in this case, is to move it till one of the sides of the slit (in spite of the difference of colours) is judged to be decidedly the brighter, and then to move it the other way, till the other becomes decidedly the brighter. The position half way between these points is to be taken as the true point of equal illumination.

76. If we would compare the degrees of illumination, or the intrinsic brightnesses of two surfaces, a given portion of each must be insulated for examination; this may be best done by the adaptation of two blackened tubes to

the openings of the photometer, of equal length, and terminated by orifices of equal area, or subtending equal angles at the middle of the instrument. These, of course, cut off equal apparent magnitudes of the bright surfaces, the light of which is then to be equalized on the oiled papers of the slit E F, as in the case of candles, &c. Bouguer, *Traité*, p. 31.

Comparison of degrees of brightness of illuminated surfaces.

77. Another method of comparing the intensity of the light from two luminaries, which is also very ready and convenient, and possesses in some cases considerable advantages, has been proposed by Count Rumford. (See *Phil. Trans.*, vol. 84, p. 67.) It consists in the equalization of the *shadows* cast by them on a white surface illuminated by them both at once. Suppose, for instance, we would compare the illuminating power of two flames L and *l* of different sizes, or from different combustibles, as of wax and tallow. Before a screen C D of white paper, in a darkened room, place a blackened cylindrical stick S, and let the flames L *l* be so placed as to throw the shadows A B of the stick on the screen, side by side, and with an interval between them about equal in breadth to either shadow. Moreover the inclination of the rays L S A and *l* S B to the surface of the screen must be adjusted to exact equality. The brighter flame must then be removed, or the feebler brought nearer to the screen, till the two shadows appear of equal intensity, when their distances (or the distance of the screen) from the lights must be measured, and their total illuminating powers will be in the direct ratio of the squares of the distances. The rationale of this is obvious, the shadow thrown by each flame is illuminated by the light of the other. The screen by the sum of the lights. The eye in this case judges of the degrees of defalcation of brightness from this sum; and if these degrees be alike, it is clear that the remaining illuminations must be equal.

Fig. 8.

78. This method becomes uncertain when the lights are of considerable size and near the screen, as the penumbræ of the shadows prevent any fair comparison of the relative intensities of their central portions. It is still more so, and can hardly be used when the lights differ considerably in colour. Its convenience, however, as an extemporaneous method, requiring no apparatus but what is always at hand, (as the use of a *blackened* stick, though preferable, is not essential,) renders it often useful in the absence of more refined means.

When the lights to be compared are immovable.

79. It may happen that the lights to be compared are not movable, or not conveniently so. In this case the equalization of the shadows may be performed by inclining the screen at different angles to the directions in which it receives the light of each, and noting the angles of inclination of the rays. In this case the illuminating powers of the luminaries are as the squares of their distances directly, and the sines of the respective angles of inclination of their rays to the screen inversely.

Modifications of light enumerated.

80. When a ray of light proceeds in empty space, or in a perfectly homogeneous medium, its course, as we have seen, is rectilinear, and its velocity uniform; but when it encounters an obstacle, or a different medium, it undergoes changes or modifications which may be stated as follows:

Regular reflexion.

81. It is separated into several parts, which pursue different courses, or are otherwise differently modified. One of these parts is *regularly reflected*, and pursues, after reflexion, a course wholly exterior to the new medium, or obstacle.

Regular refraction. Single and double refraction.

82. A second and a third portion are *regularly refracted*, that is, they enter the medium, and there pursue their course according to the *laws of refraction*. In many media these portions follow the same course precisely, and *perhaps* are no way distinguishable from each other. In such media (comprehending most uncrystallized substances and liquids) the refraction is said to be *single*. In numerous others (for instance in most crystallized media) they follow different courses, and also retain different physical characters. In these the refraction is said to be *double*.

Scattering.

83. A fourth portion is scattered in all directions, one part being intromitted into the medium, and distributed over the hemisphere interior to it, while the other is in like manner scattered over the exterior hemisphere. These two portions are those which render visible the surfaces of bodies to eyes situated any how with respect to them, and are therefore of the utmost importance to vision.

Absorption.

84. Of those portions which enter the medium, a part more or less considerable is *absorbed*, stifled, or lost, without any further change of direction; and that not at once, but progressively, as they penetrate deeper and deeper into its substance. In perfectly opaque media, such as the metals, this *absorption* is total, and takes place within a space less than we can appreciate; yet even here we have good reasons for believing that it does not take place *per saltum*. In crystallized bodies, those at least which are coloured, this absorption takes place differently on the two portions into which the regularly refracted ray is divided, according to laws to be explained when we come to treat of the absorption of light.

Separation into colours, or dispersion

85. The regularly refracted portions of a ray of white or solar light are (except in peculiar circumstances) separated into a multitude of rays of different colours, and otherwise differing in their physical properties, each of which rays pursues its course afterwards, independently of all the rest, according to the laws of regular refraction or reflexion. The laws of this separation, or *dispersion*, of the coloured rays, and their physical and sensible properties, form the subject of *Chromatics*.

Polarization.

86. All those portions which are either regularly reflected, or regularly refracted, undergo, more or less, a modification termed *polarization*, in virtue of which they present, on their encountering another medium, different phenomena of reflexion and refraction from those presented by *unpolarized* light. Generally speaking, *polarized* light obeys the same laws of reflexion and refraction as *unpolarized*, as to the *directions* which the several portions, into which it is divided on encountering a new medium, take; but differs from it in the relative intensities of those portions, which vary according to the situation in which the surface of the medium and certain imaginary lines, or axes within it, are presented to the polarized ray.

Interference

87. The rays of light under certain circumstances exercise a mutual influence on each other, increasing, diminishing, or modifying each other's effects according to peculiar laws. This mutual influence is called the *interference* of the rays of light. We shall proceed to treat of these several modifications in order; and first of the regular reflexion of light.

§ 3. *Of the regular Reflexion of unpolarized Light from Plane Surfaces.*

88. When a ray of light is incident on a smooth-polished surface, a portion of it is regularly reflected, and pursues its course after reflexion in a right line wholly exterior to the reflecting medium. The direction and intensity of this portion are the objects of inquiry in this section; the physical properties acquired by the ray in the act of reflexion being reserved for examination at a more advanced period. And first, with regard to the direction of the reflected ray. This is determined by the following laws:

Laws of Reflexion.

89. Laws of reflexion. *Law* 1. When the reflecting surface is a plane. At the point on which the ray is incident raise a perpendicular. The reflected ray will lie in the same plane with this perpendicular, and with the incident ray. It will lie on the opposite side of the perpendicular, and will make an angle with it equal to that made by the incident ray.

90. The plane in which the perpendicular to any surface at the point of incidence, and the incident ray, both lie, is called the *plane of incidence.*

91. The angle included between the incident ray and the perpendicular is called the *angle of incidence.*

92. The plane in which the reflected ray and perpendicular both lie is called the *plane of reflexion;* and the angle included between the perpendicular and reflected ray is, in like manner, termed the *angle of reflexion.*

93. Adopting these definitions, the law of reflexion from a plane surface may be announced by saying, that the plane of reflexion is the same with that of incidence, and the angle of reflexion equal to that of incidence, but situated on the contrary side of the perpendicular.

94. *Corol.* The incident and reflected rays are equally inclined to the surface at the point of incidence.

Law 2. When the surface is a curved one, the course of a ray reflected from any point is the same as if it were reflected at the same point from a plane, a tangent to the curve surface at that point; *i. e.* if a perpendicular be raised to the curve surface at the point of incidence, the reflected ray will lie in the plane of incidence, and the angle of reflexion will equal that of incidence.

95. Demonstrated by experiment. The demonstration of these laws is a matter of experiment. If we admit a small sunbeam through a hole in the shutter of a darkened chamber, and receive it on a polished surface of glass, or metal, we may easily with proper instruments measure the inclinations of the incident and reflected rays to the surface, which will be found equal. But this method is rude and coarse. A much more delicate verification of this law is afforded by astronomical observations. It is the practice of astronomers to observe the altitudes of the stars above the horizon by direct vision; and, at the same instant, the apparent depression below the horizon of their images reflected at the surface of Mercury, (which is necessarily exactly horizontal,) and the depression so observed is always found precisely equal to the altitude, whatever the latter may be, whether great or small. Now as these observations, when made with large instruments, are susceptible of almost mathematical accuracy, we may regard the law of reflexion, or plane surfaces, as the best established in nature.

96. Reflexion at a curved surface may be considered as taking place at that infinitely small portion of the surface which is common to it, and to its tangent plane at the point of incidence; so that if a perpendicular to the surface be erected at the point of incidence, the incident and reflected rays will make equal angles with it on opposite sides.

97. *Proposition.* To find the direction of a ray of light after reflexion at any number of plane surfaces, given in position.

Construction. Since the *direction* of the ray after reflexion is the same whether it be reflected at the given surfaces, or at surfaces parallel to them, conceive surfaces parallel to the given ones to pass through any point C, (fig. 9,) and from C draw the straight lines C P, C P′, C P″, &c. respectively perpendicular to these respective surfaces, and lying wholly exterior to the reflecting media. Draw S C parallel to the ray when incident on the first surface, and in the plane S C P, and on the opposite side of C P, from the incident ray S C make the angle P C s' = P C S, then will C s' be the direction of the ray after reflexion at the first surface. Prolong s' C to S′, then S′ C will represent the ray at the moment of its incidence on the second surface, whose normal is C P′. Again, make the angle P′ C s'' in the plane S′ C P″, but on the other side of C P′, equal to the angle S′ C P′, then will C s'' represent the ray at the moment of its reflexion from the second surface, and, producing s'' C to S″, S″ C will represent it at the moment of its incidence on the third surface, whose normal is C P″. Similarly in the plane S″ C P″; but on the other side of C P″ make the angle P″ C s''' = P″ C S″, and C s''' will be the direction of the ray at the moment of its quitting the third surface, and so on.

Fig. 9.

98. Fig. 10. *Analysis.* About C as a centre conceive a spherical surface described, (fig. 10,) then will the plane P S s intersect it in a great circle P S S′ p; and the plane in which C P, C P′ lie, or the plane at right angles to the two first reflecting planes in another great circle P P′ p; and the planes S′ C s'' and S C s'' in other great circles S′ P′ s'' and S k s''.

Since C P and C P′ are given directions, the angle P C P′, or the arc P P′ (which is equal to *the inclination of the two first surfaces to each other*) is given. Call this I. Again, since the direction S C of the incident ray is given, *the angle of incidence*, or the *first surface* P C S (= a) and the angle S P P′, or *the inclination of the plane of the first reflection to the plane* P P′ *perpendicular to both surfaces* (= ψ) are given. Hence in

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the spherical triangle P P′ S′ we have P P′ = I; P S′ = 180° − α; and the angle P′ P S′ = ψ; consequently S′ P′, and therefore 2 S′ P′ = S′ s″ and the angle S S′ P′ are known, as also the angle P P′ S′, and therefore its supplement P P′ s″, which is the angle made by the second reflexion with the plane P P′. Again, in the spherical triangle S S′ s″ we have given S S′ = 180° − 2 α; S′ s″ = 2 S′ P′ and the included angle S S′ s″, whence the third side S s″ may be found, which is the angle between the incident and twice reflected rays.

Similarly, if a third reflexion be supposed, we have given P′ S″ = 180° − S′ P′; P′ P″ = I′, and the angle S″ P′ P″ = S′ P′ P″ = P P′ P″ − P P′ S′, whence we may compute S″ P″ and proceed as before, and so on to any extent.

99. Confining ourselves however to the case of two reflexions we have, by spherical trigonometry, putting P′ S′ = a' = *the angle of incidence on the second reflecting surface*, P S′ P′ = θ; P P′ S′ = ϕ, and 180° − S s'' = D, *the deviation of the ray after the second reflexion*, the following equations:

General equations of reflexion at two planes.

$$\left.\begin{aligned} -\cos a' &= \cos a \,.\, \cos \mathrm{I} - \sin a \,.\, \sin \mathrm{I} \,.\, \cos \psi \\ \sin \theta &= \frac{\sin \mathrm{I}}{\sin a'} \,.\, \sin \psi \\ \sin \phi &= \frac{\sin a}{\sin a'} \,.\, \sin \psi \\ \cos \mathrm{D} &= \cos 2a \,.\, \cos 2a' - \sin 2a \,.\, \sin 2a' \,.\, \cos \theta \end{aligned}\right\} \quad (\mathrm{A})$$

100. From these equations, any three of the seven quantities a, a', I, θ, ϕ, ψ, D being given, the other four may be found. It will be observed, that ϕ is *the angle between the plane of the second reflexion and the principal section of the two reflecting planes, and* θ *the angle between the planes of the first and second reflexion.* If ϕ and D only be sought, θ must be regarded as merely an auxiliary angle; but this may not be the case, and cases may occur in which θ alone may be sought, or in which it enters as a given quantity, &c. In short, the foregoing equations contain in themselves all the conditions which can arise in any proposed case of two reflexions.

Values of the symbols.

101. *Corol.* If $\psi = o$, or if the incident ray coincide with the principal section P C P, *i. e.* if the two reflexions both take place in the plane perpendicular to the reflecting surfaces, these formulæ take a very simple form, for we then have

$$\theta = o; \quad \phi = 180°; \quad \cos a' = -\cos (a + \mathrm{I})$$

that is $(a + a') = 180° - \mathrm{I}$; and consequently $\cos (2a + 2a') = \cos (360° - 2\,\mathrm{I}) = \cos 2\,\mathrm{I}$, or $2a + 2a' = 2\,\mathrm{I}$. But since $\theta = o$, we have by the last of the equations (A) $\cos \mathrm{D} = \cos 2\,(a + a')$; consequently $\mathrm{D} = 2a + 2a' = 2\,\mathrm{I}$. That is to say, the deviation in this case, after two reflexions, is equal to twice the inclination of the reflecting planes, whatever be the original direction of the ray. This elegant property is the foundation of the common sextant and of the reflecting circle, and is commonly regarded as having been first applied to the measurement of angles by Hadley, though Newton appears also to have proposed it for the same object. See the explanation of these instruments.

Case where both reflexions are in one plane.

102. In other cases, however, D, the deviation, is essentially a function of the angles expressing the position of the incident ray, and can only be obtained from the equations above stated.

103. *Proposition.* Given the angles of incidence on the two planes, and the angle made by the plane of the first reflexion with that of the second; required the positions of the incident and twice reflected rays, the deviation of the ray after both reflexions, and the angle included between the reflecting surfaces.

Retaining the same notation, we have given, a, a' θ, required I, D, and ϕ, ψ.

1st, D is given at once, by the last of the general equations, (A.)

2ndly, To find the rest, put $x = \sin \mathrm{I}$; $y = \sin \psi$; and $a = \sin a' \,.\, \sin \theta$; put also $\cos a = c$; $\sin a = s$; $\cos a' = c'$; $\sin a' = s'$. We have then $x\,y = a$, or $y = \frac{a}{x}$; and the first of the equations (A) then gives

$$-c' = c\sqrt{1 - x^2} - s\sqrt{x^2 - a^2}$$

which, cleared of radicals and reduced, gives

$$o = x^4 + x^2\{2c'^2(c^2 \quad s^2) - 2c^2 - 2s^2a^2\} + (c'^2 - c^2)^2 + 2a^2s^2(c'^2 + c^2) + a^4s^4$$

and this equation, which, though biquadratic, is of a quadratic form, contains the general solution of the problem.

104. *Corol.* 1. If $\theta = 90°$, or if the planes of the first and second reflexions be at right angles to each other, we have simply $\sin \mathrm{I} \,.\, \sin \psi = \sin a'$, and $a = \sin a' = s'$.

Case when the planes of the two reflexions are at right angles.

In this case our final equation becomes

$$o = x^4 - 2x^2(1 - c^2c'^2) + (1 - c^2c'^2)^2$$

which, being a complete square, gives $x^2 = 1 - c^2c'^2$.

Now $x = \sin \mathrm{I}$, therefore $x^2 = 1 - \cos \mathrm{I}^2$, consequently we have the following simple result,

$$\cos \mathrm{I}\ (= c\,c') = \cos a \,.\, \cos a'.$$

Light. Or the cosine of the inclination of the planes to each other is equal to the product of the cosines of the angles of incidence on each. And, *vice versâ*, if this relation holds good, the planes of the two reflexions will necessarily be at right angles to each other; for, this relation being supposed, we have of course $x^2 = 1 - c^2 c'^2$, and therefore $1 - c^2 c'^2$ being put for x^2 in the general equation, the whole must vanish; now this substitution gives a biquadratic of a quadratic form for determining a, which must evidently be satisfied by taking Part I.

$$a = \sin a', \text{ and consequently } \theta = 90°.$$

This elegant property will be useful when we come to treat of the polarization of light.

105. *Corol.* 2. In the same case if $\theta = 90$, the deviation D is given by the equation

$$\cos \mathrm{D} = \cos 2a \,.\, \cos 2a',$$

or, the cosine of the deviation is equal to the product of the cosines of the doubles of the angles of incidence.

106. *Problem.* A ray of light is reflected from each of two planes in such a manner that all the angles of incidence and reflexion are equal. Given the inclination of the planes, and the angles of incidence; required, first, the deviation; secondly, the inclination of the planes of the first and second reflexion to each other, and the angles made by each of these planes with the principal section of the reflecting planes.

Preserving the same notation we have $a = a'$, and therefore by the third of the equations (A) $\psi = \phi$, so that these equations become

$$\left.\begin{aligned} \cos a \;(1 + \cos \mathrm{I}) &= \sin a \,.\, \sin \mathrm{I} \,.\, \cos \psi \\ \sin a \,.\, \sin \theta &= \sin \mathrm{I} \,.\, \sin \psi \\ \cos \mathrm{D} &= (\cos 2a)^2 - (\sin 2a)^2 \,.\, \cos \theta \end{aligned}\right\} \qquad (a)$$

107. The first of these gives (putting for $1 + \cos \mathrm{I}$ its value $2\left(\cos \frac{\mathrm{I}}{2}\right)^2$ and for $\sin \mathrm{I}$ its equal $2 \,.\, \sin \frac{\mathrm{I}}{2} \,.\, \cos \frac{\mathrm{I}}{2}$)

$$\cos \psi = \operatorname{cotan} a \,.\, \operatorname{cotan} \frac{\mathrm{I}}{2}, \qquad (b)$$

whence ψ is immediately known. Hence ψ is had by the equation

$$\sin \theta = \frac{\sin \mathrm{I}}{\sin a} \,.\, \sin \psi. \qquad (c)$$

Lastly, if we subtract each member of the third of the equations (a) from 1, divide both sides by 2, and reduce, we transform it into the following

$$\sin \frac{\mathrm{D}}{2} = \sin 2a \,.\, \cos \frac{\theta}{2}. \qquad (d)$$

These equations afford ready and direct means of computing ψ, θ, and D in succession, from the known values of a and I; the formulæ are adapted to logarithmic evaluation, and are in themselves not inelegant.

§ IV. *Of Reflexion from Curved Surfaces.*

108. The reflexion of a ray from a curved surface is performed as if it took place at a reflecting plane, a tangent to the point of incidence. The reflected ray will therefore lie in the plane which contains the incident ray and the normal or perpendicular at the point of incidence. The general expressions for the course of the ray after reflexion at surfaces of double curvature being considerably complex, and not likely to be of great service to us in the sequel, we shall confine ourselves to the particular case of a surface of revolution (comprehending the cases of a plane, and conoidal surfaces of all kinds) where the plane of incidence is supposed to pass through the axis of revolution.

109. *Proposition.* A ray being incident on any surface of revolution in a plane passing through the axis, to find the direction of the reflected ray.

General investigation of the course of a ray reflected at any curve. Fig. 11.

Q P (fig. 11) being a section of the surface by the plane of incidence, Q N the axis, Q P the incident, and P r the reflected ray, which produced if necessary cuts the axis in q. Draw the tangent P T, the ordinate P M, and the normal P N, which produce to O, and put as follows,

$$x = \mathrm{Q\,M}; \; y = \mathrm{M\,P}; \; p = \frac{dy}{dx}; \; \theta = \text{the angle } \mathrm{M\,Q\,P},$$

or the angle made by the incident ray with the axis; then, since the angle of reflexion is equal to that of incidence, we have $\angle\, r\,\mathrm{P\,O} = \mathrm{O\,P\,Q}$, and therefore $\mathrm{N\,P}\,q = \mathrm{O\,P\,Q}$; consequently $\mathrm{Q\,P\,T} = \mathrm{T\,P}\,q$. Now, $\mathrm{Q}\,q = \mathrm{Q\,M} - \mathrm{M}\,q = \mathrm{Q\,M} - \mathrm{P\,M} \,.\, \tan \mathrm{M\,P}\,q$

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$$= x - y \,.\, \tan \{ \mathrm{TPM} - \mathrm{TP}q \}$$
$$= x - y \,.\, \tan \{ \mathrm{TPM} - \mathrm{TPQ} \}$$
$$= x - y \,.\, \tan \{ \mathrm{TPM} - \mathrm{PTM} + \mathrm{PQM} \}$$
$$= x - y \,.\, \tan \{ 90^\circ - 2\,\mathrm{PTM} + \mathrm{PQM} \}$$

But by the theory of curves we have $\tan \mathrm{PTM} = \frac{dy}{dx} = p$, consequently $\mathrm{PTM} = \text{arc tan } p = \tan^{-1} p$, denoting by $\tan^{-1}$ the inverse function of that expressed by tan; and since $\mathrm{PQM} = \theta$, this expression becomes

$$\mathrm{Q}q = x - y \,.\, \text{cotan} \{ 2 \,.\, \tan^{-1} p - \theta \}$$
$$= x - y \,.\, \text{cotan} \{ 2 \,.\, \tan^{-1} \left(\frac{dy}{dx} \right) - \tan^{-1} \left(\frac{y}{x} \right) \qquad (a)$$

$$\left(\text{Because } \tan \theta = \frac{\mathrm{PM}}{\mathrm{QM}} = \frac{y}{x} \right)$$

This then is the general expression for the distance between the points in which the incident and reflected rays cut the axis.

Now, by Trigonometry, we have (A and B being any two quantities)

$$\text{cotan} \{ 2 \tan^{-1} \mathrm{A} - \tan^{-1} \mathrm{B} \} = \text{cotan} \left\{ \tan^{-1} \frac{2\mathrm{A}}{1 - \mathrm{A}^2} - \tan^{-1} \mathrm{B} \right\}$$
$$= \text{cotan} \,.\, \tan^{-1} \left(\frac{2\mathrm{A} - (1 - \mathrm{A}^2)\,\mathrm{B}}{(1 - \mathrm{A}^2) + 2\mathrm{AB}} \right)$$

that is, since $\text{cotan} \,.\, \tan^{-1} \theta = \frac{1}{\theta}$, the cotangent and tangent being reciprocals of each other, simply

$$\frac{1 - \mathrm{A}^2 + 2\mathrm{AB}}{2\mathrm{A} - (1 - \mathrm{A}^2)\,\mathrm{B}}$$

Applying this to the present case, $\mathrm{A} = \frac{dy}{dx} = p$; $\mathrm{B} = \frac{y}{x}$, and therefore the expression above found for $\mathrm{Q}q$ becomes

$$\left.\begin{aligned} \mathrm{Q}q &= x - y \,.\, \frac{(1 - p^2)\,x + 2py}{2px - (1 - p^2)\,y} \\ &= 2 \,.\, \frac{(x + py)\,(px - y)}{2px - (1 - p^2)\,y} \end{aligned}\right\} \qquad (b)$$

General expressions for the distance of the focus from the radiant point $\mathrm{Q}q$.

These expressions contain the whole theory of the foci and aberrations of reflecting surfaces.

Corol. 1. To find the angle made by the reflected ray with the axis, which we will call θ'.

This is the angle $\mathrm{P}q\mathrm{M}$, which is the complement of $\mathrm{MP}q$. Now we have found above

$$\mathrm{MP}q = 90^\circ - 2 \tan^{-1} p + \theta.$$

Hence

$$\theta' = 2 \,.\, \tan^{-1} p - \theta$$

But $\tan \theta = \frac{y}{x}$, so that substituting we have

$$\tan \theta' = \frac{2px - (1 - p^2)\,y}{(1 - p^2)\,x + 2py}; \qquad (c)$$

110. Angle made by the reflected ray and the axis.

Corol. 2. $$\mathrm{A}q = a' = a + 2\,\frac{(x + py)\,(px - y)}{2px - (1 - p^2)\,y}. \qquad (d)$$

111.

In all the foregoing formulæ we have supposed the origin of the x placed at Q the radiant point. If we would place it elsewhere, as at A, we have only to write $x - a$ for x throughout. The formulæ then become on this hypothesis,

112. Formulæ when the radiant point is not in the origin of the coordinates.

$$\begin{cases} \tan \theta = \dfrac{y}{x - a} & (e) \\ \tan \theta' = \dfrac{2p\,(x - a) - (1 - p^2)\,y}{(1 - p^2)\,(x - a) + 2py} & (f) \end{cases}$$

$$\begin{cases} \mathrm{A}\,\mathrm{Q} = a; \; \mathrm{Q}q = \dfrac{2\,(x - a + py)\,(px - pa - y)}{2p\,(x - a) - (1 - p^2)\,y} & (g) \\ \mathrm{A}q = a' = \dfrac{2\,(x + py)\,(px - y) + \{(1 - p^2)\,y - 2px\}\,a}{2px - (1 - p^2)\,y - 2pa}; & (h) \end{cases}$$

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113. Formulæ when the incident rays are parallel to the axis.

If the incident ray be parallel to the axis, we have only to suppose the point Q infinitely distant; or placing, as in the last article, the origin of the x at a point A at a finite distance, to make a (= A Q) infinite. The above expressions then give $Q\,q = \infty$

$$\left.\begin{aligned} \tan\theta' &= \frac{2p}{1-p^2} \\ A\,q &= x - y\,.\,\frac{1-p^2}{2p} \end{aligned}\right\} \quad (i)$$

114. *Proposition.* To represent the incident and reflected rays by their equations.

The equation of any straight line is necessarily of the form $Y = \alpha X + \beta$. Suppose we take A for the common origin of the coordinates, and, retaining the foregoing notation, representing by x and y the coordinates of the point P in the curve, let X and Y represent those of any point in the incident ray; and, Q being the point in which that ray cuts the axis, and A Q = a, it is evident, first, that when $X = a$, $Y = o$; and secondly, since the ray passes through P, when $X = x$, $Y = y$. Hence we have

$$o = \alpha a + \beta, \qquad \text{and} \qquad y = \alpha x + \beta,$$

whence we get $$\alpha = \frac{y}{x-a} \qquad \beta = -\frac{a\,y}{x-a}; \qquad (1)$$

therefore, the equation of the incident ray is

$$Y = \frac{y}{x-a}\,(X-a); \qquad (2)$$

or, which is the same in a different form,

$$Y - y = \frac{y}{x-a}\,(X-x); \qquad (3)$$

or, since $$\tan\theta = \frac{P\,M}{M\,Q} = \frac{y}{x-a},$$

$$Y = (X-a)\,.\,\tan\theta; \qquad (4)$$

or, again, $$Y - y = (X-x)\,.\,\tan\theta. \qquad (5)$$

Similarly for the reflected ray, it is obvious that if we represent its equation by $Y = \alpha' X + \beta'$, we shall have

$$\alpha' = \frac{y}{x-a'} \qquad \beta' = -\frac{a'\,y}{x-a'}; \qquad (6)$$

and consequently

$$Y = \left(\frac{y}{x-a'}\right).(X-a') = (X-a')\,.\tan\theta'; \qquad (7)$$

$$Y - y = \frac{y}{x-a'}\,.\,(X-x) = (X-x)\,.\tan\theta'; \qquad (8)$$

will be the corresponding forms of the equation of the reflected ray, in which a' and $\tan\theta'$ are given in terms of x, y, a, and $p = \frac{dy}{dx}$ by the equations (g) and (h) or (i).

115. Fig. 11. If the whole figure (fig. 11) be turned about the axis A M, and Q be supposed a radiant point, the rays in the whole conical surface generated by the revolution of Q P will be concentred after reflexion in one and the same point q, which will thus become infinitely more illuminated than by any single ray from an elementary molecule of the surface. The point P will generate an annulus, having M P for its radius; and q is called the *focus* of this annulus, and the distance A q the *focal distance* of the same annulus. This last expression is commonly understood to mean the distance of q from the *vertex*, or point where the curve meets the axis, but we shall use it at present in the more general sense.

Focus. Focal distance. Vertex.

116. Generally speaking, then, the focus varies as the point P in the reflecting annulus varies, unless in that particular case where, by the nature of the curve, the function expressing a' is constant. Let us examine this case.

117. Investigation of the curves which reflect all the incident rays to one point.

Proposition. To find the curve which will have the same focus for every point in its surface of revolution, or on which rays diverging from or converging to any point Q, being incident, shall all after reflexion converge to or diverge from one point q.

The value of Q q assigned in Art. 109, E q, (b) being made constant, affords the equation

$$\frac{(x + p\,y)\,(p\,x - y)}{2\,p\,x - (1-p^2)\,y} = \text{constant} = c.$$

Light. This equation, cleared of fractions, and putting x for $x - c$, (which is merely shifting the origin of the co-ordinates to the distance c from their former origin) becomes Part I.

$$p\{x^2 - y^2 - c^2\} = (1 - p^2)\, x\, y. \qquad (a)$$

To integrate this equation, assume a new variable z, such that $p\,y = x\,z$, and (multiplying the original equation by y) we have

$$p\,y\,(x^2 - y^2 - c^2) = x\,y^2 - x \cdot p^2\,y^2,$$

that is

$$x\,z\,(x^2 - y^2 - c^2) = x\,y^2 - x^3\,z^2,$$

whence we find

$$y^2 = \frac{z\,x^2 - z\,c^2 + z^2\,x^2}{1 + z} = x^2\,z - c^2 \cdot \frac{z}{1 + z}.$$

Differentiating this equation we get

$$2\,y\,d\,y \left(= 2\,p\,y\,d\,x = z\,x\,z\,d\,x \text{ because } p = \frac{d\,y}{d\,x}\right)$$

$$= d\left(x^2\,z - \frac{c^2\,z}{1 + z}\right)$$

$$= 2\,x\,z\,d\,x + x^2\,d\,z - c^2\,d \cdot \left(\frac{z}{1 + z}\right)$$

that is

$$x^2\,d\,z - c^2\,d \cdot \frac{z}{1 + z} = o,$$

or

$$\left\{x^2 - \frac{c^2}{(1 + z)^2}\right\} d\,z = o. \qquad (b)$$

This equation may obviously be satisfied in two ways; the first is, by putting the factor

$$x^2 - \frac{c^2}{(1 + z)^2} = o, \quad \text{or } x = \pm \frac{c}{1 + z}$$

which gives (restoring the value of z, $z = \frac{p\,y}{x}$) merely $x + p\,y = c$; and, eliminating p between this and the original equation (a) we find, on reduction,

$$y^2 + (x - c)^2 = o.$$

This is, however, (as is clear from the way in which it has been obtained,) only a *singular solution* of the differential equation, (see DIFFERENTIAL CALCULUS, *singular solutions*;) and as the value of y which results from it is always imaginary, it affords no curve satisfying the conditions of the problem.

The other way in which the equation (b) can be satisfied, is by putting $d\,z = o$, or $z =$ constant. Let this constant be represented by $-h$; then, since $z = \frac{p\,y}{x}$, we have

The curve is in all cases a conic section.

$$\frac{p\,y}{x} = \frac{y\,d\,y}{x\,d\,x} = -h,$$

which, integrated, gives

$$y^2 = h\,(a^2 - x^2),$$

a being another constant. This is the general equation to the conic sections, and it is obvious, from the properties of these curves, that they satisfy the conditions; because two lines drawn from their foci to any point in their periphery make equal angles with the tangent at that point, and, consequently, a ray proceeding from, or converging to, one focus, and reflected at the curve, must necessarily take a direction to or from the other. But, the foregoing analysis being direct, shows that they possess this property in common with no other curves.

118. Ellipse. Fig. 12. Thus in the case of the ellipse, all rays, (fig. 12,) S P, S P′, &c. diverging from the focus S will after reflexion converge to the other focus H, the interior surface of the ellipse being polished; and all rays Q P, Q P′, &c. converging to S, will after reflexion diverge from H.

Fig. 13. 119. Hyperbola. In the hyperbola, (fig. 13,) rays Q P, Q′ P, &c. converging to one focus S, and incident on the polished convex surface of the curve, will after reflexion converge to the other focus H; and if diverging from S, and reflected on the polished concave surface P P′, will after reflexion diverge from H.

120. Parabola. Fig. 14. In the case of the parabola, rays parallel to the axis, incident on the interior or concave surface, will all be reflected to the focus S, fig. 14; and if reflected at the exterior or convex surfaces, will all after reflexion diverge from S.

121. Circle. Rays converging to, or diverging from, the centre of a sphere will all after reflexion diverge from, or converge to, the same centre.

Let us now apply our general formula (b) (Art. 109) to some particular cases.

Light. Part I.

Proposition. Let the reflecting surface be a plane, or the curve P C a straight line. Required the focus of reflected rays.

122. Focus of a plane surface.

Here we have $x = \text{constant} = a$; $p = \frac{dy}{dx} = \infty$, and the general formula becomes simply

$$Q\,q = a' = \frac{2\,x\,y}{y} = 2\,x = 2\,a.$$

So that the focus of reflected rays is a point on the opposite side of the reflecting plane equally distant from it with the radiant point; and as this is independent of y, or of the situation of the point P, we see that all the rays after reflexion diverge from this point, see fig. 15.

123. Focus of a spherical annulus.

Proposition. To find the focus of any annulus of a spherical reflector.

Let r be the radius of the sphere, and, if we fix the origin of the coordinates at the radiant point, the equation of the generating circle will be

$$r^2 = (x - a)^2 + y^2$$

This, differentiated, gives $(x - a)\,dx + y\,dy = 0$,

consequently $$p = \frac{dy}{dx} = -\frac{x - a}{y};\quad 1 - p^2 = \frac{2\,y^2 - r^2}{y^2}.$$

Hence, substituting in the general expression (b), we find for the focal distance the following value,

$$Q\,q = \frac{2\,a\,\{\,r^2 + a\,(x - a)\,\}}{r^2 + 2\,a\,(x - a)};\qquad (a)$$

which expresses in all cases the distance of the focus of reflected rays from the radiant point.

For optical purposes, however, it is more convenient to know its distance from the centre, or from the surface.

The distance from the centre (E q, fig. 16,) is

$$= Q\,q - Q\,E = \frac{2\,a\,(a\,x - a^2 + r^2)}{2\,a\,x + r^2 - 2\,a^2} - a,$$

or $$E\,q = \frac{a\,r^2}{2\,a\,(x - a) + r^2};\qquad (b)$$

in which positive values of E q lie to the right of E, or the same way with those of x or of Q q.

Focus for central rays in a spherical reflector.

Corol. 1. If we would find the focus of the infinitely small annulus immediately adjoining to the *vertex* C, or C′ of the reflecting spherical surface, or, as it is termed in Optics, the *focus of central rays*, we must put in the case of the vertex C (when the reflexion takes place on a concave surface) $x = a + r$, and in the other case, viz. that where the rays are reflected on the convex surface C′, $x = a - r$. The former gives

$$E\,q = \frac{r\,a}{2\,a + r};\ C\,q = \frac{r\,(a + r)}{2\,a + r};\qquad (c)$$

the latter gives the same results, writing only $-r$ for r.

124. If we bisect the radii C E and C′ E in F and F′, and suppose q and q' to be the foci of central rays reflected at C and at C′, we shall have

$$F\,q = \tfrac{1}{2}\,r - \frac{r\,a}{2a + r} = \frac{\left(\frac{r}{2}\right)^2}{a + \frac{r}{2}},\qquad (d)$$

which gives the following useful analogy,

$$Q\,F : F\,E :: E\,F : F\,q.\qquad (e)$$

Similarly we have Q F′ : F′ E :: E F′ : F′ q; so that the same analogy applies to both cases, and may be regarded as the fundamental proposition in the theory of the foci for central rays. For it is obvious, that if P C were any other curve than a circle, the same must hold good, taking only E the centre of curvature at the vertex.

125. Principal focus.

Corol. 2. If a be infinite, or the incident rays be parallel, we have F $q = 0$, which shows that the *focus of central parallel rays* bisects the radius. This focus, for distinction's sake, is called the *principal focus* of the reflector.

126. Conjugate foci.

Definition. Q and q are termed *conjugate foci.* It is evident that if q be made the radiant point, Q will be its focus; for the rays will pursue the same course backwards.

127.

Corol. 3. Regarding only central rays: the conjugate foci move in opposite directions, and coincide at the centre and surface of the reflector.

For let a vary from ∞ to $-\infty$, then F q will vary as follows: first, while a varies from ∞ to $-\frac{r}{2}$ F, q is

Light. positive, and increases from o to ∞; that is, as Q moves up to F, q moves through C to infinity. As the motion of q continues, Fq then becomes negative; because a is then negative and greater than $\frac{r}{2}$, and a increasing Fq diminishes; therefore q moves from the right towards F, that is in the opposite direction to Q's motion; and when Q is at an infinite distance to the right, q is again at F. Part I. Conjugate foci move in opposite directions.

When Q comes to E, $a = o$. F$q = \frac{r}{2}$, or q is at E also.

When Q comes to C, $a = -r$, F$q = -\frac{r}{2}$, or q is at C also.

It appears by the value of Eq, Equation (b), that a spherical reflector A C B, fig. 17, whose chord (or *aperture*, as it is termed in Optics) is A B, causes the ray reflected at its exterior annulus A to converge to, or diverge from, a point q, different from the focus of central rays. Let f be this latter focus, then we shall have 128. Longitudinal aberration, for any aperture. Fig. 17.

$$\mathrm{E}f = \frac{a\,r}{2\,a + r},\quad \mathrm{C}f = \frac{(a + r)\,r}{2\,a + r};\quad fq = \frac{a\,r^2}{2\,a\,(x - a) + r^2} - \frac{a\,r}{2\,a + r}.$$

This quantity fq is called the *longitudinal aberration* of the spherical reflector. If the rays fall on the convex portion, we need only write $-r$ for r.

Proposition. To express approximately the longitudinal aberration of a spherical reflector whose aperture is inconsiderable with respect to its focal length. 129. Longitudinal aberration for small apertures.

y being the semi-aperture, and $x - a$ being equal to $\sqrt{r^2 - y^2} = r - \frac{y^2}{2\,r}$, (neglecting y^4, and higher powers of y,) we have

$$fq = \text{aberration} = \frac{a\,r^2}{2\,a\,r + r^2 - \frac{a\,y^2}{r}} - \frac{a\,r}{2\,a + r} = \frac{a^2\,y^2}{r\,(2\,a + r)^2};\qquad (f)$$

If we put C$f = f$, we have $f = \frac{r\,(a + r)}{2\,a + r}$, and, consequently, we may eliminate a, the distance of the radiant point, and express the aberration in terms of the aperture, radius of curvature, and distance of the focus of central rays from C, the vertex of the minor; for this gives $a = \frac{r\,(r - f)}{2f - r}$, which, substituted for a in the expression (f) gives 130. Another expression.

$$\text{aberration} = \frac{(r - f)^2 . y^2}{r\,3} = \frac{\mathrm{E}f^2 . (\text{semi-aperture})^2}{(\text{rad.})^3}.\qquad (g)$$

To express the *lateral aberration*, or the quantity by which the reflected ray A qg deviates from the axis, at the focus of central rays, or the value of fg, (fig. 17,) we have 131. Lateral aberration.

$$fg = fq\,.\,\frac{\mathrm{A\,M}}{\mathrm{M}q};\ \text{but A M} = y,\ \text{and M}q = \mathrm{E\,M} - \mathrm{E}q = x - a - \frac{a\,r^2}{2\,a\,(x - a) + r^2}$$

$$= \frac{2\,a\,(x - a)^2 + r^2\,(x - 2\,a)}{2\,a\,(x - a) + r^2};\ \text{so that}$$

$$fg = \frac{2\,a^2\,r}{2\,a + r}\,y \times \frac{a - x + r}{r^2\,(x + 2\,a) + 2\,a\,(x - a)^2}.\qquad (h)$$

When the aperture is very small, this becomes simply 132. Lateral aberration for small apertures.

$$fg = \frac{a^2\,y^3}{r^2\,.\,(r + a)\,(r + 2\,a)}.\qquad (i)$$

When a is infinite, or the incident rays are parallel, we have the following, 133. Aberrations for parallel rays and small apertures.

$$\left.\begin{aligned} fq &= \text{longitudinal aberration} = \frac{y^2}{4\,r} \\ fg &= \text{lateral aberration} = \frac{y^3}{2\,r^2} \end{aligned}\right\}\qquad (j)$$

If the rays fall on the convex side of the sphere we must make r negative, which only changes the signs of the aberrations.

§ V. *Of Caustics by Reflexion, or Catacaustics.*

134. If rays of light be incident on a medium of any other form than that of a conic section, having the radiant point in the focus, they will after reflexion no longer converge to or diverge from any one point, but will be dispersed according to a law depending on the nature of the reflecting curve; the inclination of each reflected ray to the axis varying according to the point of the curve from which it is reflected, and not being the same for any two consecutive rays. Of course each ray will intersect that immediately consecutive to it in some point or other, and the *locus* of these points of continual intersection will trace out a curve to which the reflected rays will all necessarily be tangents, and which is called *a caustic*. If these rays fall on another reflecting curve, they will be again dispersed, and another caustic will originate in the continual intersections of the consecutive rays of the former, and so on to infinity.

Caustics by reflexion defined.

135. Let Q P, Q′ P′, (fig. 18,) be any two contiguous rays incident on consecutive points P, P′ of a reflecting curve P P′, and after reflexion let them pursue the courses P R, P′ R′; and since they are not necessarily parallel, let Y be their point of intersection, then will Y be the point in the caustic Y Y′ Y″ corresponding to the point P in the reflecting curve; and if we determine the points Y′ Y″, &c. from the consecutive points P′ P″, &c. in the same manner, the locus of these, or the curve Y Y′ Y″ will be the whole caustic.

Fig. 18.

136. Since the reflected ray passes through P, whose coordinates are $x\,y$, its equation, as we have already seen (Art. 114), is necessarily of the form

Coordinates of the caustic investigated on any supposition of the law of divergence.

$$\mathrm{Y} - y = \mathrm{P}\,(\mathrm{X} - x)$$

If we regard x, y, P as variable, this will represent any one of the reflected rays P R, and the consecutive ray P′ R′ will be represented by

$$\mathrm{Y} - (y + d\,y) = (\mathrm{P} + d\,\mathrm{P})\ (\mathrm{X} - (x + d\,x)\ )$$

Now since the point Y in which these two rays intersect is common to both, the coordinates X and Y at this point are the same for both; and therefore at this point both these equations coexist, and thereby determine the values of X and Y, or the situation of the point Y. Now the latter of these equations is nothing more than the former *plus* its differential, on the supposition of X and Y remaining constant. Therefore, we have to find X and Y from the two equations,

$$\mathrm{Y} - y = \mathrm{P}\ (\mathrm{X} - x)$$

$$-\,d\,y = (\mathrm{X} - x)\,d\,\mathrm{P} - \mathrm{P}\,d\,x,$$

which gives at once

$$\left.\begin{aligned} \mathrm{X} &= x + \frac{\mathrm{P} - p}{d\,\mathrm{P}}\,d\,x \\ \mathrm{Y} &= y + \mathrm{P}\,.\,\frac{\mathrm{P} - p}{d\,\mathrm{P}}\,d\,x \end{aligned}\right\} \qquad (k)$$

In these equations we have only to substitute for P its value $= \tan\theta'$, or $\dfrac{2\,p\,(x-a) - (1-p^2)\,y}{(1-p^2)\,(x-a) + 2\,p\,y}$; and after executing all the differentiations indicated, or implied, to eliminate x and y by the equations of the curve and the other conditions to which the quantity a may be subjected, an equation between X and Y will result which will be the equation of the caustic.

137. *Proposition.* To determine the caustic when rays diverge from one fixed point in the axis of a given reflecting curve.

Caustic when rays diverge from a fixed point.

In this case a is invariable, and the differentiation of P must be performed on this hypothesis. It will, therefore, simplify the question if we put $a = o$; or suppose the origin of the coordinates in the radiant point, in which case

$$\left.\begin{aligned} \mathrm{P} &= \frac{2\,p\,x - (1-p^2)\,y}{2\,p\,y + (1-p^2)\,x} \\ \frac{d\,\mathrm{P}}{d\,x} &= (1+p^2)\,.\,\frac{(1+p^2)\,(y - p\,x) + 2\,q\,(x^2+y^2)}{\{\,2\,p\,y + (1-p^2)\,x\,\}^2} \\ q &= \frac{d\,p}{d\,x} \\ \mathrm{P} - p &= \frac{(1+p^2)\,(p\,x - y)}{2\,p\,y + (1-p^2)\,x} \end{aligned}\right\} \qquad (l)$$

Where

Light. Part I.

which substituted, we find

$$\left.\begin{aligned} \mathbf{X} &= 2 \cdot \frac{p\,(p\,x - y)^2 - q\,x\,(x^2 + y^2)}{(1 + p^2)\,(p\,x - y) - 2\,q\,(x^2 + y^2)} \\ \mathbf{Y} &= 2 \cdot \frac{(p\,x - y)^2 + q\,y\,(x^2 + y^2)}{-(1 + p^2)\,(p\,x - y) + 2\,q\,(x^2 + y^2)} \end{aligned}\right\} \quad (m)$$

138. Caustic for parallel rays.

Corol. 1. If the incident rays be parallel, or the radiant point at an infinite distance, we may fix the origin of the coordinates where we please; and since in this case the equation of any reflected ray is, by 113 equation (*i*) and 114 equation (8),

$$\mathbf{Y} - y = (\mathbf{X} - x) \cdot \frac{2\,p}{1 - p^2} \qquad (m, 2)$$

we have

$$\mathbf{P} = \frac{2\,p}{1 - p^2};\ \mathbf{P} - p = \frac{p\,(1 + p^2)}{1 - p^2};\ \frac{d\,x}{d\,\mathbf{P}} = \frac{(1 - p^2)^2}{2\,q\,(1 + p^2)}$$

putting q for $\frac{d\,p}{d\,x}$ or $\frac{d^2\,y}{d\,x^2}$.

These substitutions made, we get the following values for the coordinates of the caustic,

$$\mathbf{X} = x + \frac{p}{2\,q}\,(1 - p^2);\ \mathbf{Y} = y + \frac{p^2}{q}. \qquad (n)$$

139. Distance between corresponding points in curve and caustic

Corol. 2. In the general case, if we put f = the line $\mathbf{P}\,y$, or the distance between the point in the curve and the corresponding point in the caustic we have

$$f = \sqrt{(\mathbf{X} - x)^2 + (\mathbf{Y} - y)^2}$$

Which, if we write for $\mathbf{X} - x$ and $\mathbf{Y} - y$, their values above found become

$$f = \sqrt{1 + \mathbf{P}^2} \cdot \frac{\mathbf{P} - p}{d\,\mathbf{P}}\,d\,x \qquad (o)$$

or, writing for $\mathbf{P}$ its value, and executing the operations,

$$f = \frac{-(y - p\,x)\,(1 + p^2)\,\sqrt{x^2 + y^2}}{(y - p\,x)\,(1 + p^2) + 2\,q\,(x^2 + y^2)}. \qquad (p)$$

140.

Corol. 3. In the case of parallel rays, when

$$\mathbf{P} = \frac{2\,p}{1 - p^2};\ \frac{d\,\mathbf{P}}{d\,x} = \frac{2\,q\,(1 + p^2)}{(1 - p^2)^2};\ \mathbf{P} - p = \frac{p\,(1 + p^2)}{1 - p^2};\ \sqrt{1 + \mathbf{P}^2} = \frac{1 + p^2}{1 - p^2}$$

we have

$$f = \frac{p\,(1 + p^2)}{2\,q}. \qquad (q)$$

141.

Corol. 4. Call c the chord of the circle of curvature passing through the origin of the coordinates, or through the radiant point: then, by the theory of curves,

$$c = \frac{2\,(p\,x - y)\,(1 + p^2)}{q\,\sqrt{x^2 + y^2}}$$

so that

$$q\,(x^2 + y^2) = \frac{2\,(p\,x - y)\,(1 + p^2)\,\sqrt{x^2 + y^2}}{c}$$

and substituting this for $q\,(x^2 + y^2)$, in the general expression for f, we eliminate q, and get

$$f = \frac{c\,\sqrt{x^2 + y^2}^2}{4\,\sqrt{x^2 + y^2} - c} = \frac{r\,c}{4\,r - c}$$

putting

$$r = \sqrt{x^2 + y^2}^2.$$

Hence we have

$$f - \tfrac{1}{4}\,c = \frac{(\tfrac{1}{4}\,c)^2}{r - \tfrac{1}{4}\,c}$$

which gives

$$r - \tfrac{1}{4}\,c : \tfrac{1}{4}\,c :: \tfrac{1}{4}\,c : f - \tfrac{1}{4}\,c. \qquad (r)$$

142. Fig. 19.

Hence the following general property. (Smith's *Optics*, ed. 1738, p. 160.)

Q and q being two conjugate foci of an elementary pencil of rays reflected at any curve surface at P, fig. 19. Let V P W be the circle of curvature; (if the curve be a circle, this will be the curve itself.) Let the chords

Light. PV, PW in the direction of the incident and reflected rays be divided in F, f, so that PF and Pf shall each be one quarter of the whole chords, and the relation between Q and q will be expressed by the proportion Part I.

General relation between conjugate points or foci of reflected rays incident on any curve.

$$QF : FP :: Pf : fq. \qquad (s)$$

143. *Corol.* 5. Putting $\frac{P-p}{dP}\,dx = M$, we have $\frac{dX}{dx} = 1 + \frac{dM}{dx}$;

$$\frac{dY}{dx} = p + P\frac{dM}{dx} + M\frac{dP}{dx} = P\left(1 + \frac{dM}{dx}\right).$$

Hence it follows that $$P = \frac{dY}{dX};$$

P therefore is to the caustic, for the coordinates X, Y, what p is to the reflecting curve for the corresponding point whose coordinates are x, y.

144. Length of the caustic investigated. *Corol.* 6. If we put S for the length of the caustic = the arc AHKY, we have $dS = \sqrt{dX^2 + dY^2}$

or $$dS = dX.\sqrt{1+P^2} = (dx + dM)\sqrt{1+P^2}$$

$$dx.\sqrt{1+P^2} + df - M.\frac{P\,dP}{\sqrt{1+P^2}}$$

because $$df = d.M.\sqrt{1+P^2} + M.\frac{P\,dP}{\sqrt{1+P^2}}$$; but $M\,dP = (P-p)\,dx$

so that we have $$dS = df + dx\left\{\sqrt{1+P^2} - \frac{(P-p)P}{\sqrt{1+P^2}}\right\}$$

$$= df + dx.\frac{1+Pp}{\sqrt{1+P^2}};$$

that is, substituting for P its value $$\frac{2px - (1-p^2)y}{2py + (1-p^2)x},$$

$$dS = df + dx.\frac{x+py}{\sqrt{1x^2+y^2}} = df + d.\sqrt{x^2+y^2}$$

and integrating $$S = \text{constant} + f + \sqrt{x^2+y^2}.$$

Caustics always rectifiable. Hence it follows, that the caustic is always a rectifiable curve, and its

$$\left.\begin{aligned}\text{Length AK}y &= QP + Py + \text{constant}\\ \text{But}\quad \text{Arc AKF} &= QC + CF + \text{constant}\end{aligned}\right\}\ \text{consequently, subtracting}$$

$$\text{Arc F}y = (QC + CF) - (QP + PY).$$

Hence it appears, that the caustic is necessarily a rectifiable curve when the reflecting curve is not itself transcendental.

145. Fig. 20. If the rays PR, P′R′, P″R″, &c. after reflexion at the curve PP′P″ fall on another reflector RR′R″ and are reflected in the directions RS, R′S′, R″S″, &c. (fig. 20) their continual intersections will form another caustic ZZ′Z″, and so on *ad infinitum*, which may be determined by a similar analysis. In like manner, whatever be the law according to which the rays QP, Q′P′, &c. are dispersed, we may conceive each to be a tangent to a curve which may be regarded as the caustic of another reflecting curve, and so on. Let VV′V″ be this curve. Since PVQ is a tangent to it, if this curve and the curve PP′P″ be given, the point Q in the axis from which the incident ray QP may be regarded as radiating, is determined in terms of the coordinates of P, and therefore the quantity a may be eliminated altogether. The manner of doing this is shown in the following

146. General relation between two conjugate caustics and their intermediate reflecting curve investigated.

Proposition. To determine the relations between any two consecutive, or, as they may be termed, *conjugate* caustics VV′V″, YY′Y″, and the intermediate reflecting curve PP′P″.

Let V and Y be, as before, any two conjugate points in the caustics, P the reflecting point; then if we put

ξ and η	for the coordinates	of V
x and y	for those	of P
X and Y		of Y

Light. Since the line P V Q is a tangent to the first curve at V, we must evidently have Part I.

$$y - \eta = \frac{d\,\eta}{d\,\xi}\,(x - \xi)$$

and this, combined with the equation between η and ξ, which represents the curve V V′ V″ suffices to determine η and ξ in terms of x, y, or *vice versâ*, x and y in terms of ξ and η.

Again, we have also by Art. 114, equation (2)

$$y - \eta = \frac{y}{x - a}\,(x - \xi)$$

and consequently

$$x - a = y\,.\frac{x - \xi}{y - \eta};\quad a = \frac{\xi\,y - \eta\,x}{y - \eta}.$$

Thus a is given in terms either of x, y, or of η, ξ, whichever we may prefer. It only remains to substitute this in the value of P.

$$\mathrm{P} = \frac{2\,p\,(x - a) - (1 - p^2)\,y}{(1 - p^2)\,(x - a) + 2\,p\,y}$$

which thus becomes

$$\mathrm{P} = \frac{2\,p\,(x - \xi) - (1 - p^2)\,(y - \eta)}{(1 - p^2)\,(x - \xi) + 2\,p\,(y - \eta)};\qquad (t)$$

and this, being free of a, may be substituted in the equations (k) Art. 136, when X and Y will be at once obtained in terms of x, y, ξ, η, the coordinates of the reflecting curve and the preceding caustic.

We shall now proceed to illustrate the theory above delivered by an example or two.

147. Caustic of a cycloid. Required the caustic when the reflecting curve is a cycloid, and the incident rays are parallel to each other and to the axis of the cycloid.

The equation of the cycloid is $$\frac{d\,y}{d\,x} = p = \frac{\sqrt{x}}{\sqrt{2 - x}}$$

taking unity for the radius of the generating circle.

From this we get

$$\frac{1}{q} = (2 - x)\,\sqrt{2\,x - x^2}$$

and therefore $$\frac{p}{q} = 2\,x - x^2;$$

consequently, by the equations (k) of Art. 136, we shall have

$$\mathrm{X} = x + \frac{1 - p^2}{2}\,.\,\frac{p}{q} = 2\,x - x^2$$

$$\mathrm{Y} = y + p\,.\,\frac{p}{q} = y + x\,\sqrt{2\,x - x^2}$$

whence

$$\frac{d\,\mathrm{Y}}{d\,x} = p + \frac{\sqrt{x}}{\sqrt{2 - x}}\,(3 - 2\,x) = 2\,\sqrt{2\,x - x^2} = 2\,\sqrt{\mathrm{X}}$$

Now we have also

$$\frac{d\,\mathrm{X}}{d\,x} = 2\,(1 - x)$$

But since $\mathrm{X} = 2\,x - x^2$, we have $1 - x = \sqrt{1 - \mathrm{X}}$, and therefore

$$\frac{d\,\mathrm{X}}{d\,x} = 2\,\sqrt{1 - \mathrm{X}}$$

So that we have, finally, $$\frac{d\,\mathrm{Y}}{d\,\mathrm{X}} = \sqrt{\frac{\mathrm{X}}{1 - \mathrm{X}}}$$

which shows that the caustic is itself a cycloid of half the linear dimensions of the reflecting curve. Is itself a cycloid.

148. To take one other example, let us suppose the reflecting curve a circle, and the radiant point infinitely distant. Here we have (placing the origin of the coordinates in the centre)

Light. $$x^2 + y^2 = r^2;\ p = -\frac{x}{\sqrt{r^2 - x^2}};\ q = -\frac{r^2}{(r^2 - x^2)^{\frac{1}{2}}},$$ Part I.

Caustic of a circle. consequently, by the equations (k) of Art. 136

$$\left.\begin{aligned} X &= x + \frac{p(1-p^2)}{2q} = \frac{3r^2 - 2x^2}{2r^2}\,x \\ Y &= y + \frac{p^2}{q} = \frac{(r^2 - x^2)^{\frac{3}{2}}}{r^2} = \frac{y^3}{r^2} \end{aligned}\right\};\qquad (u)$$

Then since (supposing, for brevity, $r = 1$, which will not affect the result)

$$4X^2 = 9x^2 - 12x^4 + 4x^6$$

$$4Y^2 = 4 - 12x^2 + 12x^4 - 4x^6$$

Adding, $$4(X^2 + Y^2) = 4 - 3x^2;\ x^2 = \frac{4}{3}(1 - X^2 - Y^2)$$

So that we get, finally, substituting this value of x^2 in that of Y, and reducing,

$$(4X^2 + 4Y^2 - 1)^3 = 27Y^2;\qquad (v)$$

which is the equation of the caustic.

This equation belongs to an epicycloid generated by the revolution of a circle whose radius is $\frac{1}{4}$ that of the reflecting circle on another concentric with the latter, and whose radius is $\frac{1}{2}$ that of the reflecting circle.

Fig. 21. Fig. 21 represents the caustic in this case; Q P being the incident ray, and P Y the reflected. It has a cusp at F, which is the principal focus of rays reflected at the concave surface B C D, and another at F′, which is that of the rays reflected from the convex surface B A D. In the latter case, it is not the rays themselves, but their prolongations backwards which touch the caustic.

149. *Corol.* When y is very small, or immediately adjacent to the cusp F, the form of the caustic approaches indefinitely to that of a semicubical parabola. For, generally,

$$X = \tfrac{1}{2}\sqrt{1 + 3Y^{\frac{2}{3}} - 4Y^2},$$

and when Y is very small, neglecting Y^2 in comparison with $Y^{\frac{2}{3}}$

$$X = \frac{1}{2} + \frac{3}{4}Y^{\frac{2}{3}},\ \text{or}\ Y^2 = \left(\frac{4}{3}\right)^3 \cdot \left(X - \frac{1}{2}\right)^3;\qquad (w)$$

150. It is, as we have seen, only in certain very particular cases, when rays proceeding from one point and reflected at a curve proceed after reflexion all to or from one point. In general they are distributed in the manner described in Art. 145, 146, being all tangents to the caustic. The density of the rays therefore in any point of the caustic is infinitely greater than in the space surrounding it, and in the space between the caustic and the reflecting curve (P C F Y, fig. 18) is greater than in the space without the caustic Q Y F. This is obvious, for in the latter space only the incident rays occur, while in the former are included all the reflected rays as well as the incident ones.

151. This may be easily shown experimentally, in a very satisfactory manner pointed out by Dr. Brewster, by

Fig. 22. bending a narrow strip of polished steel into any concave form, as in fig. 22, and placing it upright on a sheet of white paper. If in this state it be exposed to the rays of the sun, holding the plane of the paper so as to pass nearly but not quite through the sun, the caustic will be seen traced on the paper, and marked by a very bright well-defined line; the part within being brighter than that without, and the light graduating away from the caustic inwards by rapid gradations. If the form of the spring be varied, all the varieties of catacaustics, with their singular points, cusps, contrary flexures, &c. will be seen beautifully developed. The experiment is at once amusing and instructive.

The bright line seen on the surface of a drinking-glass full of milk, or, better still, of ink, standing in sunshine, is a familiar instance of the caustic of a circle just investigated.

152. If the figure 18 be turned round its axis, the reflecting curve will generate a surface of revolution, which, if supposed polished within or without, as the case may be, will become *a mirror*. The caustic will also generate a conoidal surface, to which all the rays reflected by the mirror will be tangents. No mirror, therefore, which is not formed by the revolution of a conic section having the radiant point in its focus, can converge all the reflected rays to one point or *focus*. There will, however, always be one point which receives the reflected rays in a more dense state than any other. This point is the cusp F, as we shall presently see. The deviation of any reflected ray from this point is what is termed its *aberration*.

153. The concentration and dispersion of rays by reflecting and refracting surfaces forming the great business of *practical optics*, it will be necessary to enter at large into this subject; and, first, it will be proper to inquire how far any given reflector will enable us to concentrate the rays which fall on it. To this end let the following problem be proposed.

154. *Proposition.* A reflector of any figure, of a given diameter or *aperture* A B, being proposed, to find the *circle of least aberration*, or the place where a screen must be placed to receive all the rays reflected from the surface, within the least possible circular space (since they cannot be all collected in one point) and the diameter of this circle.

Light. Part I.

A C B (fig. 23) being the mirror, Q the radiant point, G K f k g the caustic, f the cusp or focus for central rays, q the focus of the extreme rays A q, B q, produce these lines till they cut the caustic in Y y. It is clear, then, since all the rays reflected from the portion A C B of the reflector are tangents to points of the caustic between K, f and k, f, that they must all pass through the line Y y. Retaining the notation of the foregoing propositions, (*i. e.* supposing Q x = X; X y = Y.) Let us put Q L = $\mathring{\mathrm{X}}$, L K = $\mathring{\mathrm{Y}}$, Q D = $\mathring{x}$; D A = $\mathring{y}$; and let $\mathring{\mathrm{P}}$, $\mathring{p}$ represent the values of P and p corresponding to the points K and A of the caustic and reflecting curves. The equation of the line A K q y will then be Fig. 23.

$$\mathrm{Y} - \mathring{y} = \mathring{\mathrm{P}}\,(\mathrm{X} - \mathring{x})\,; \qquad (x)$$

Y and X being the coordinates of any point in it. But at the point y, where it cuts the other branch of the caustic, these coordinates are common to the straight line, and to the caustic. At this point, therefore, the above equation, and those expressing the nature of the caustic, must subsist together. Now these are the equations (k) Art. 136, combined with the original equation of the reflecting curve. Eliminating, then, x and y, by the aid of two of them, and determining the values of X, Y from the rest, the problem is resolved.

Now the same equation which gives the value of y, or x y, must also give that of L K, because K is a point in both caustic and the line A K y, as well as y. But, moreover, since A K y is a tangent, the point K is a double point; therefore the final equation in Y must necessarily have two equal roots, besides the value of Y sought; and these being known, the other may be found from a depressed equation. 155.

The method here followed is, apparently, different from that usually employed, which consists in making the value of y as determined by the intersection of the extreme reflected ray A K y, and any other reflected ray (from P) a maximum. But the difference is only apparent, for in the latter method we have to make Y as determined by the two equations (holding good jointly)

$$\mathrm{Y} - \mathring{y} = \mathring{\mathrm{P}}\,(\mathrm{X} - \mathring{x}), \text{ and } \mathrm{Y} - y = \mathrm{P}\,(\mathrm{X} - x)$$

a maximum, or d Y = o. Now in this case the former equation gives d X = o also; and therefore, differentiating the latter, we have

$$-d\,y = (\mathrm{X} - x)\,d\,\mathrm{P} - \mathrm{P}\,d\,x,$$

whence

$$\mathrm{X} - x = \frac{\mathrm{P} - p}{d\,\mathrm{P}}\,d\,x$$

and therefore

$$\mathrm{Y} - y = \mathrm{P}\,.\,\frac{\mathrm{P} - p}{d\,\mathrm{P}}\,d\,x.$$

Now these are nothing more than the equations of Art. 136, expressing the general properties of the caustic; so that this consideration of the maximum only leads by a more circuitous route to the same equations as the method above stated, and is in fact nothing more than a different mode of expressing the caustic.

Let us apply this reasoning to the case when the reflector is spherical. Resuming the equations and notation of Art. 148, and putting a for the extreme value of y, or the *semi-aperture* of the mirror, and b for the corresponding value of x, that of P will be 156. Circle of least aberration in a spherical reflector.

$$\left(\frac{2\,p}{1 - p^2} = \right) \;=\; \frac{2\,a\,b}{b^2 - a^2} \;=\; \frac{2\,a\,b}{1 - 2\,a^2}.$$

Hence the equation (m, 2) Art. 138, of the extreme reflected ray becomes

$$\mathrm{Y} - a = \frac{2\,a\,b}{1 - 2\,a^2}\,(\mathrm{X} - b)$$

whence we get

$$2\,\mathrm{X} = \frac{1}{b}\left(1 + \frac{1 - 2\,a^2}{a}\,.\,\mathrm{Y}\right)$$

Assume z, so that Y = $a^3\,z^3$, z being another unknown quantity, then we have

$$4\,\mathrm{X}^2 = \frac{1}{1 - a^2}\,\{1 + (1 - 2\,a^2)\,a^2\,z^3\}^2.$$

Substituting this for 4 X^2, and for Y^2 its value $a^6\,z^6$ in the equation of the caustic (v) Art. 148, extracting the cube root, and reducing, we get the following equation for finding z,

$$a^2\,z^6 + (2 - 4\,a^2)\,z^3 + (3\,a^2 - 3)\,z^2 + 1 = o.$$

Now this, according to the remark in Art. 155, must have two equal roots, viz. when $x = b$, or Y = a^3, that is, when $z = 1$. Hence this equation must necessarily be divisible by $(z - 1)^2$. Performing the division we find it is so, and the quotient gives

$$a^2\,z^4 + 2\,a^2\,z^3 + 3\,a^2\,z^2 + 2\,z + 1 = o\,; \qquad (y)$$

for determining the remaining values of z.

Light. Part I.

As this investigation is rigorous, nothing having been omitted or neglected as small, we have here the complete solution of the problem, whatever be the aperture of the mirror. If this be supposed small in comparison with the radius, an approximation to the value of z will be had by the series thence derived,

157. Case when the aperture is moderate.

$$z = -\frac{1}{2} - \frac{9}{32}a^2 - \frac{9}{32}a^4 - \frac{1395}{4096}a^6 - \&c.$$

and of course since $Y = a^3 z^3$,

$$Y = -\frac{a^3}{8} - \frac{27}{128}a^5 - \frac{675}{2048}a^7 - \&c. \qquad (z)$$

158. Case where the aperture is small when compared to radius.

The first term of this series is sufficient for most cases which occur in practice, and gives

$$Y = -\frac{a^3}{8} \qquad (a)$$

or, supposing r the radius of curvature of the reflector,

$$Y = -\frac{a^3}{8r^2} \qquad (\beta)$$

The lateral aberration corresponding to the semi-aperture a is, by the equation (j), Art. 133, equal to $\frac{a^3}{2r^2}$; consequently, in the case of small apertures, the radius of the least circle of aberration is equal to $\frac{1}{4}$ of the lateral aberration (at the focus) of the exterior annulus.

159. *Corol.* The least circle of aberration is nearer the mirror than its principal focus, by $\frac{3}{4}fg$ or $\frac{3}{4}$ the longitudinal aberration $= \frac{3}{16} \cdot \frac{a^2}{r}$.

160. Density of reflected rays at any point investigated. Fig. 24.

To complete the theory of caustics, it only remains to examine the degree of concentration of the reflected rays at any assigned point. To this end, let S (fig. 24) be any point, and through it let P S Y q be drawn touching the caustic in Y. Then S may be regarded as lying in a conical surface generated by the revolution of the tangent P Y s q, about the axis; and all the rays in the annulus, generated by the revolution of the element P P′, will be contained in the hollow conoidal solid formed by the revolution of the figure P P′ Y q' q about the same axis. Hence at S the rays will be concentrated: first, in a plane parallel to that of the paper, in the ratio of P P′ to S S′, or P Y to S Y; and, secondly, in a plane perpendicular to that of the paper, or in the ratio of the circumferences of the circles generated by the revolution of P and of S, that is, in the ratio of these radii P M : S T. On both accounts, therefore, the concentration at S will be represented by $\frac{PM}{ST} \times \frac{PY}{SY}$, or $\frac{Pq}{Sq} \times \frac{PY}{SY}$. If, therefore, we represent by 1 the density of the rays immediately on their reflexion at P, their density at S corresponding, will be represented by $\frac{PY \cdot Pq}{SY \cdot Sq}$, and this is true, whatever be the situation of S.

161. 1st case.

But there are now several cases to be distinguished. First, when S is situated in any part of the spaces K H V, N D W, no such tangent can be drawn to cut the reflector within its aperture A B; therefore these spaces receive no rays at all, and the density $= o$ in every point.

162. 2nd case.

Secondly, when S is situated anywhere within the spaces A G B, V H F E, E F D W, only one such tangent can be drawn to cut the reflecting curve between A and B. So that in these spaces the density is simply represented by

$$D = \frac{PY \cdot Pq}{SY \cdot Sq}.$$

163. 3rd case.

Thirdly, within the spaces K G H and M G D two tangents can be drawn from any point S, both touching the branch F k on the same side of the axis as the point S. If we suppose P_1 Y_1 S q_1 and P_2 Y_2 S q_2 to be these tangents, S will receive rays belonging to both these converging conoids, and the density will therefore be the sum of those belonging to either, or

$$D = \frac{PY_1 \cdot Pq_1}{SY_1 \cdot Sq_1} + \frac{PY_2 \cdot Pq_2}{SY_2 \cdot Sq_2}.$$

Fig. 25.

See fig. 25.

164. 4th case.

Fourthly and lastly, within the space F H G D there may be drawn three tangents q_1 S Y_1 P_1, q_2 S Y_2 P_2, and q_3 S Y_3 P_3, all falling within A B, the two first (fig. 26) touching the branch F k on the same side as S, the

Light. Part I.

third on the opposite side. The former belong to cones of rays converging to q_1 q_2, the latter to a cone converging to q_2, but intercepted by S after meeting at q, and again diverging. Hence, in this case, the density will be expressed by $D = \frac{PY_1 \cdot Pq_1}{SY_1 \cdot Sq_1} + \frac{PY_2 \cdot Pq_2}{SY_2 \cdot Sq_2} + \frac{PY_3 \cdot Pq_3}{SY_3 \cdot Sq_3}$. Fig. 26.

Application to particular cases.

It would lead into too great complication to attempt developing the actual value of these fractions in terms of the coordinates of S, and we will therefore merely apply them to some remarkable positions of S.

165. Case 1.

Case 1. S in the axis, beyond the principal focus, or between the mirror and its focus for extreme rays G. Here Y coincides with F, and q also does the same therefore in this case, $D = \left(\frac{PF}{SF}\right)^2$, which shows that the density is inversely as the square of the distance of S from the principal focus.

166. Case 2.

Case 2. S in the axis between the principal focus and the focus for extreme rays G, (*i. e.* in the line GF.) Here $Sq_1 = o$, $Sq_2 = o$, $Sq_3 = o$; so that here all the three several component portions of D are infinite, and of course the density is infinitely greater than on the surface of the reflector.

167 Case 3.

Case 3. S at F. Here not only $Sq = o$, but also SY; therefore at F the density is infinitely greater than in the last case, and is the greatest which exists anywhere.

168. Case 4.

Case 4. S anywhere in the caustic. Here $SY = o$, therefore in this case also D is infinite, or the density infinitely greater than at the surface of the reflector; and as S approaches F, this is still further multiplied by the diminution of all the values of Sq.

169. Case 5.

Case 5. S anywhere in H z D, the circle of least aberration. At the centre z and the circumference H the density is infinite. Between these two positions, finite, diminishing to a minimum, and again increasing according to a law too complicated to be here investigated. It will be observed, that the relations expressed in these articles (160—169) are general, and not restricted to the case where the reflecting surface is spherical.

170. Illumination of a screen exposed to the reflected rays.

In all the foregoing reasoning the point S is supposed to receive the rays perpendicularly. The density of the rays therefore here intended must be understood to mean, The number of rays not incident on a given particular plane surface, but passing through a given infinitely small spherical portion of space, or received upon an infinitely small spherical body at S.

In cases, however, where the aperture is small, a screen perpendicular to the axis will receive the rays from every point very nearly at a perpendicular incidence; and hence the above expressions will in this case represent the intensity of illumination of the several points in such a surface, the screen being, however, supposed to stop none of the incident light.

For further information respecting caustics, the reader is referred to Tschirnäus, Leipsic acts 1682, and *Hist. de l'Acad.*, tom. ii. p. 54, 1688; to De la Hire's *Traité des Epicycloides*, and *Mém. de l'Acad.*, vol. x.; to Smith's *Optics;* Carré, *Mém. de l'Acad.*, 1703; J. Bernouilli, *Opera Omnia*, vol. iii. p. 464; *l'Hôpital Analyse des Infiniment Petits;* Hayes's *Fluxions;* Petit, *Correspondence de l'Ecole Polytechnique*, ii. 553; Malus, *Journal de l'Ecole Polytech.*, vol. vi.; Gergonne, *Annales des Mathématiques*, xi. p. 229; De la Rive, *Dissertation sur les Caustiques, &c.;* Sturm, *Annales des Math.*, xvi.; Gergonne, *ditto.*

OF THE REGULAR REFRACTION OF LIGHT BY UNCRYSTALLIZED MEDIA.

§ VI. *Of the Refraction of Homogeneous Light at Plane Surfaces.*

171. When a ray of light is incident on the surface of any transparent uncrystallized medium, a portion of it is reflected; another portion is *dispersed* in all directions, and serves to render the surface visible; and the remainder enters the medium and pursues its course within it.

172. In the reflexion of light, the law of reflexion, as far as regards the direction of the reflected ray, is the same for all reflecting media; the angle of reflexion being equal to that of incidence for all. In refraction, however, the case is otherwise, and each different medium has its own peculiar law of action on light; some turning a ray incident at a given angle more out of its course than others. Whatever be the nature of the refracting medium, the following general laws are found to hold good, and suffices (when the medium is known) to determine the direction of the refracted ray.

173. 1st. The incident ray, the perpendicular to the surface at the point of incidence, and the refracted ray, all lie in the same plane.

174. 2nd. The incident and refracted rays lie on opposite sides of the perpendicular.

175. 3rd. Whatever be the inclination of the incident ray to the refracting surface, the sine of the angle included between the incident ray and the perpendicular is to the sine of that included between the refracted ray and the perpendicular in a constant ratio.

176. These laws equally hold good for plane and for curved surfaces, and are found to be verified with perfect precision by the most delicate experiments, and all the phenomena of refracted light to take place in exact conformity with the results deduced from them by mathematical reasoning.

Light. Part I.

Let A C B (fig. 23) be the refracting surface, P C p the perpendicular to it at the point of incidence C, S C the incident, and C s the refracted ray. Then we shall have

177.

Fig. 23.

$$\sin PCS : \sin pCs :: \mu : 1,$$

μ being a constant quantity; that is, constant for the same medium A B, though its value is different for different media.

178. It is usual, for brevity, to speak of the sine of incidence, and the sine of refraction, instead of the sines of the angle of incidence, and the angle of refraction.

179. The numerical value of the quantity μ, or of $\frac{\text{sin of incidence}}{\text{sin of refraction}}$ in any medium, must be ascertained before the law of refraction in that medium can be regarded as perfectly known. This may be done experimentally by actually measuring the angle of refraction corresponding to any one given angle of incidence, for the value of the above fraction being thus determined for one incidence holds equally for every other, or by other more easy or more refined modes to be described hereafter. This quantity μ is called the *index* of refraction of the medium A B.

Index of refraction.

180. The medium in which the ray proceeds previous to its incidence on A B is here regarded as a vacuum. If the medium A B be also a vacuum, it is clear that the ray will not change its course; so that the angle of incidence will equal that of refraction, and the value of μ will be equal to 1. This is the lowest value of μ, as no medium has yet been discovered which refracts rays *from* the perpendicular when incident from a vacuum. The greatest value of μ yet known is 3, when the refraction is made into chromate of lead; and between these limits almost every intermediate gradation has been found to belong to some one or other transparent body. Thus for air at its ordinary density $\mu = 1.00028$, while for water it is 1.336, for ordinary crown glass 1.535, for flint glass 1.60, for oil of cassia 1.641, for diamond 2.487, and for the greatest refraction of chromate of lead 3.0.

181. It is a general law in Optics, that the visibility of two points from one another is mutual, whatever be the course pursued by the rays which proceed from one to the other. In other words, that if a ray of light proceeding from A arrives by any course at B, however often reflected, refracted, &c., a ray can also arrive at A from B by retracing precisely the same course in a contrary direction. It follows from this, that if the ray S C incident on the exterior surface of a medium A B, (fig. 23,) pursue after refraction the course C s, then will a ray s C, incident on the exterior surface of the medium, be refracted out of it in the direction C S, being bent *from* the perpendicular. Consequently, since in this case the angle of incidence is the same with the angle of refraction in the former case, and *vice versâ*, we shall have here $\frac{\text{sin incidence}}{\text{sin refraction}} = \frac{1}{\mu}$. Thus we see that the index of refraction *out* of any medium into vacuum is the reciprocal of the index of refraction *into* the medium from the vacuum.

Refraction out of any medium into vacuum.

182. Hence it follows, that a ray can be intromitted into any medium from a vacuum at any angle of incidence; for since sin refr. $= \sin pcs = \frac{1}{\mu}$. sin P C S, the value of μ being greater than 1, the sine of pcs will necessarily be less than that of P C S, and of course less than unity; so that the angle of refraction can never become imaginary. Thus, as the angle of incidence P C S increases from o, or as the ray S C becomes more and more oblique to the surface till it barely grazes it, as at S″ C, the refracted ray becomes also more oblique, but much less rapidly, and never attains a greater obliquity than the situation C s'', in which $\sin pCs'' = \frac{\sin 90^\circ}{\mu} = \frac{1}{\mu}$. This limiting angle, then, is the maximum angle of refraction from vacuum into the medium, and its value in any given medium is found by computing the angle whose sine is the reciprocal of the index of refraction. Thus in water the angle of refraction cannot exceed arc sin $\frac{1}{1.336}$, or 48° 27′ 40″. In crown glass the limit is 40° 39′, in flint 38° 41′, in diamond 23° 42′, while for the greatest refraction of chromate of lead the limit is so low as 19° 28′ 20″.

Limit of the angle of refraction.

183. Conversely, when a ray is incident on the interior surface of the medium, at any angle less than the limiting angle whose sine is $\frac{1}{\mu}$, it will be refracted and emerge according to the law laid down in Art. 181, being bent *from* the perpendicular. But as the angle of incidence p C s increases, the angle of refraction P C S increases more rapidly; and when the former angle has reached the limiting value p C s'', the transmitted ray emerges in the direction C S″, barely grazing the external surface. If the angle of incidence be still further increased, the angle of refraction becomes imaginary; for we have sin P C S $= \mu \times \sin pCs$, and if $\sin pCs > \frac{1}{\mu}$, the sine of P C S must be greater than unity. This shows that the ray cannot emerge; but it does not inform us what becomes of it. To ascertain this, we must have recourse to experiment; from which we learn, that after this limit is passed, the ray, instead of being refracted out of the medium, is turned back and *totally reflected* within it, making the angle of reflexion $pCS''' = pCs'''$.

Limit to the possibility of a ray's egress from any medium.

When the ray cannot emerge it is reflected.

Light. Part I.

When the ray is incident on the exterior surface of the medium, a portion is reflected (R) and the remainder (*r*) refracted. The ratio of R to *r* is smallest at a perpendicular incidence, and increases regularly till the incidence becomes 90°; but even at extreme obliquities, and when the incident ray just grazes the surface, the reflexion is never total, or nearly total, a very considerable portion being always intromitted. On the other hand, when the ray is incident on the interior surface, the reflected portion (R) increases regularly, with a very moderate rate of increase, till the angle of incidence becomes equal to the critical angle, whose sine is $\frac{1}{\mu}$; when it suddenly, and, as it were, *per saltum*, attains the whole amount of the incident light, and the refracted portion (*r*) becomes zero. This sudden change from the law of refraction to that of reflexion—this breach of continuity, as it were, is one of the most curious and interesting phenomena in Optics, and (as we shall see hereafter) is connected with the most important points in the theory of light.

184. This reflexion total.

The reflexion thus obtained, being total, far surpasses in brilliancy what can be obtained by any other means; from quicksilver, for instance, or from the most highly polished metals. It may be familiarly shown by filling a glass (a common drinking-glass) with water, and holding it above the level of the eye, (as in fig. 24, No. 2.) If we then look obliquely upwards in the direction E *a c*, we shall see the whole surface shining like polished silver, with a strong metallic reflexion; and any object, as a spoon, A C B, for instance, immersed in it will have its immersed part C B reflected on the surface as on a mirror, but with a brightness far superior to what any mirror would afford. This property of internal reflexion is employed to great advantage in the camera lucida, and might be turned to important uses in other optical instruments, especially in the Newtonian telescope, to obviate the loss of light in the second reflexion, of which more hereafter.

185. Experiment illustrating this total reflexion. Fig. 24, No. 2.

Some curious consequences follow from this, as to vision under water. An eye placed under perfectly still water (that of a fish, or of a diver) will see external objects only through a circular aperture (as it were) of 96° 55′ 20″ in diameter overhead. But all objects down to the horizon will be visible in this space; and those near the horizon much distorted and contracted in dimensions, especially in height. Beyond the limits of this circle will be seen the bottom of the water, and all subaqueous objects, reflected, and as vividly depicted as by direct vision. In addition to these peculiarities, the circular space above-mentioned will appear surrounded with a perpetual rainbow, of faint but delicate colours, the cause of which we shall take occasion to explain further on. But we need not immerse ourselves in water to see, at least, a part of these phenomena. We actually live under an ocean of air, a feebly refracting medium, it is true, in comparison with water; and our vision of external objects near the horizon is modified accordingly. They are seen distorted from their true form, and contracted in their vertical dimensions; thus the sun at setting, instead of appearing circular, assumes an elliptical, or rather compressed figure, the lower half being more flattened than the upper, and this change of figure is considerable enough to be very evident to even a careless spectator. The spherical form of the atmosphere, and its decrease of density in the higher regions, however, prevent the rest of the appearances above described from being seen in it.

186. Appearances of external objects to a spectator under water.

Elliptical form of setting sun explained.

If a medium be bounded by parallel plane surfaces, a ray refracted through it will have its final direction after both refractions the same as before entering the medium.

187. Refraction through parallel surfaces. Fig. 25, No. 2.

Let A B, D F be the parallel surfaces of the medium, and S C E T a ray refracted through it, P C *p*, Q E *q*, perpendiculars to the surfaces at C and E, then we have

$$\left.\begin{array}{l} \sin \text{S C P} : \sin p \text{ C E } (= \sin \text{C E Q}) :: \mu : 1 \\ \sin \text{C E Q} : \sin q \text{ C T} \quad :: 1 : \mu \end{array}\right\} \text{ and, compounding these proportions,}$$

$$\sin \text{S C P} : \sin q \text{ E T, and therefore S C P} = q \text{ E T, and the ray E T is parallel to S C.}$$

This proposition may be proved experimentally, by placing the plane glass of a sextant (unsilvered) before the object-glass of a telescope directed to a distant object, or before the naked eye, and inclining it at any angle to the visual ray. The apparent place of the object will be unchanged.

Experimental proof.

188. Refraction at the common surface of two media in contact. Fig. 26, No. 2.

Experiment. Let a plate of glass, or any other transparent medium, be placed parallel to the horizon, and on it let any transparent fluid be poured, so as to form a compound medium consisting of two media of different refractive indices, in contact, and bounded by parallel planes; and let an object above this combination, a star, for instance, be viewed by an eye placed below it, or through a telescope. It will be found to appear precisely in the same situation as if the media were removed, whatever be the altitude of the object, or star. It follows from this, that a ray S B (fig. 26, No. 2) incident on such a combination of media, A F and D I, as described, will emerge in a direction H T parallel to the incident ray S B.

189. Law of refraction from one medium int another.

Proposition. Let there be any two media (No. 1 and 2) whose respective indices of refraction from a vacuum into each are μ and μ'. Then if these media are brought into perfect contact, (such as that of a fluid with a solid, or of two fluids with one another,) the refraction from either of them (No. 1) into the other (No. 2) will be the same as that from a vacuum into a medium, whose index of refraction is $\frac{\mu'}{\mu}$, the index of refraction of the second medium divided by that of the first.

Let D E F (fig. 26, No. 2) be the common surface of the two media, and let them be formed into parallel plates A F, D I, as in the experiment last described; then any ray S B incident at any angle on the surface A C will emerge at G I in a direction H T parallel to S B. Let B E H be its path within the media, and draw the perpendiculars P B *p*, Q E *q*, R H *r*, then

Light. Part I.

$$\sin S B P : \sin E B p = \sin B E Q :: \mu : 1$$

$$\sin R H E = \sin q E H : \sin r H T = \sin P B S :: 1 : \mu,$$

and, compounding these proportions,

$$\sin H E q : \sin B E Q :: \mu : \mu'; \qquad \frac{\sin B E Q}{\sin H E q} = \frac{\mu'}{\mu}.$$

Absolute and relative indices of refraction.

But B E Q is the angle of incidence, and H E q that of refraction, at the common surface of the media, consequently the *relative* index, or index of refraction from the first into the second, is equal to the quotient $\frac{\mu'}{\mu}$ of the *absolute* indices μ', μ, of the second and first, or their indices of refraction from vacuum.

190. This demonstration, it is true, holds good only for the case when the angles of incidence and refraction at the common surface are both less than the limits of the angles of refraction from vacuum into each medium. If they exceed these limits, the proposition however still holds good, as may be shown by direct measures of the angles of incidence and refraction in any proposed case. At present, therefore, we must receive it as an experimental truth.

191. *Example.* Required the ratio of the sine of incidence to that of refraction out of water into flint glass. The refractive index of flint glass is 1.60, and that of water 1.336, therefore the refractive ratio required is

$$\frac{1.60}{1.336} = 1.197.$$

192. If the index $\mu = -1$, the general law of refraction coincides with that of reflexion. Thus all the cases of reflexion, as far as the direction of the reflected ray is concerned, are included in those of refraction.

Of the Ordinary Refraction of Light through a System of Plane Surfaces, and of Refraction through Prisms.

193. *Definition.* In Optics, any medium having two plane surfaces, through which light may be transmitted, inclined to each other at any angle, is called a *prism.*

194. *Definition.* The *edge* of the prism is the line, real or imaginary, in which the two plane surfaces meet, or would meet if produced.

195. *Definition.* The *refracting angle* of the prism is the angle on which its two plane surfaces are inclined to each other.

196. *Definition.* The *faces* of a prism are the two plane surfaces.

197. *Definition.* The plane perpendicular to both surfaces, and therefore to the edge of a prism, is called the principal section of the prism, or of the two surfaces. This expression has been used in this general sense already, under the head of reflexion.

To determine the direction of a Ray after Refraction through any System of Plane Surfaces.

198. General problem of the refraction through any system of plane surfaces. Fig. 27.

Construction. Since the direction of the ray is the same whether refracted at the given surfaces, or at others parallel to them, conceive surfaces parallel to the given ones, all passing through one point, and from this point, but wholly exterior to the refracting media, let perpendiculars C P, C P′, C P″, &c. be drawn to each of the surfaces, (fig. 27.) Let S C be the direction of the incident ray. Between C P and C S′ draw C S′ in the plane S C P, so that $\sin P C S' = \frac{1}{\mu} \cdot \sin P C S$, μ being the index of refraction of the first medium from the medium in which the ray originally moved, which we will at present suppose a vacuum, then will S′ C be the direction of the ray after the first refraction. Again, let μ' = the relative refractive index of the second medium *out of the first*, or $\mu\,\mu'$ = its absolute refractive index from a vacuum; draw C S″ in the plane S′ C P′ so as to make $\sin P' C S'' = \frac{1}{\mu'} \cdot \sin P' C S'$, then will S″ C be the direction of the ray after the second refraction, and so on.

199. *General analysis.* Let a = S C P the first angle of incidence, a' = S′ C P′ the angle of incidence on the second surface, I = P C P′ the inclination of the two first surfaces to each other, and putting, moreover,

θ = P S′ P′ = the angle which the planes of the first and second refraction make with each other.

ψ = S P P′ = the angle made by the plane of the first refraction with the principal section of the two first refracting surfaces.

ϕ = S′ P′ P = the angle made by the plane of the second refraction with the same principal section.

ρ = P C S′ the first, and ρ' = P′ C S″ the second angle of refraction.

D = S C S″ the deviation after the second refraction.

Light. We have (conceiving S S′ S″ P P′ to be a portion of a spherical surface having C for its centre) in the spherical triangle S′ P P′ given P S′, P P′, and the included angle, required S′ P′ and the angles P S′ P′, P P′ S′; and, again, in the triangle S S′ S″ given S S′, S S″ and the angle S S′ S″, required S S″ the deviation. Or, in symbols, since ρ and ρ' are the angles of refraction corresponding to the angles of incidence α, α', and the indices of refraction μ, μ', Part I.

$$(B)\quad \left\{\begin{array}{l} \sin\alpha = \mu \,.\, \sin\rho \\ \cos\alpha' = \cos\rho \,.\, \cos \mathrm{I} + \sin\rho \,.\, \sin \mathrm{I} \,.\, \cos\psi \\ \sin\alpha' = \mu' \,.\, \sin\rho' \\ \sin\alpha' \,.\, \sin\theta = \sin \mathrm{I} \,.\, \sin\psi \\ \sin\alpha' \,.\, \sin\phi = \sin\rho \,.\, \sin\psi \\ \cos \mathrm{D} = \cos(\alpha-\rho) \,.\, \cos(\alpha'-\rho') - \sin(\alpha-\rho) \,.\, \sin(\alpha'-\rho') \,.\, \cos\theta. \end{array}\right.$$

From these equations, which, however, are rather more involved than in the case of reflexion, (Art. 99, equation A,) we may determine in all circumstances the course of the ray after the second refraction; and, in like manner, as in the case of reflexion, of any of the eleven quantities α, α', ρ, ρ', μ, μ', I, θ, ϕ, ψ, D, any five being given the remaining six may be found, and we may then go on to the next refraction, and so on as far as we please. It is needless to observe, however, that, except in particular cases, the complication of the formula becomes exceedingly embarrassing when more than two refractions are considered. Such is the general analysis of the problem; but the importance of it in optical researches requires an examination in some detail of a variety of particular cases. 200.

Case 1. When two plane surfaces only are concerned, at both of which the refractions are made in one plane, viz. that of the principal section of the two planes, or of the prism which they include. 201. Case 1. When both refractions are made in one plane.

Let the ray S C (fig. 28) be incident from vacuum on any refracting surface A C of a prism C A D, in the plane of its principal section; draw P C perpendicular to that surface, and draw C S′ so that sin P C S′ : sin P C S : : 1 : μ, then will S′ C be the direction of the refracted ray C D. Again, draw C P′ perpendicular to A D, and take the angle S″ C P′, such that sin P′ C S″ : sin P′ C S′ : : 1 : μ', μ' being the relative index of refraction from the medium A C D into the medium A D E, then will S″ C be parallel to the ray after the second refraction; draw, therefore, D E parallel to S″ C, and D E will be the twice refracted ray. As in the general case, calling S C P, α; S′ C P, ρ; S′ C P′, α'; S″ C P′, ρ'; and P C P′, I, &c. Fig. 28.

we have $\sin\alpha = \mu \,.\, \sin\rho$; $\alpha' = \mathrm{I} + \rho$; $\sin\alpha' = \mu' \,.\, \sin\rho'$

and $\pm \mathrm{D} = \mathrm{S\,\hat{C}\,S''} = \alpha - \rho' + \mathrm{I}$; $\theta = o$; $\phi = o$ $\Big\}$. (a)

The first of these equations gives ρ when μ and α are known; the second gives the value of α' when ρ is found; the third gives ρ' when α' and μ' are known; and the last exhibits the deviation D.

The sign of D is ambiguous. If we regard a deviation from the original direction towards the thicker part of the prism, or from its edge as positive, which for future use will be most convenient, we must use the lower sign 202.

or take $$\mathrm{D} = \rho' - \mathrm{I} - \alpha; \qquad (b)$$

but if *vice versâ*, then the upper sign must be used. We shall adhere to the former notation.

Case 2. If, in case 1, we suppose the medium into which the ray emerges to be the same as that from which it originally entered the prism, (a vacuum, for example,) we have $\mu' = \frac{1}{\mu}$. This is the case of refraction through an ordinary prism of glass, or any transparent substance. In this case, I is the refracting angle of the prism, μ its refractive index, (its absolute refractive index if the prism be placed in vacuo, its relative, if in any other medium,) and the system of equations representing the deviation and direction of the refracted ray becomes 203. Case 2. Both refractions in one plane at the faces of a prism in vacuo.

$$\left\{\begin{array}{l} \sin\alpha = \mu \,.\, \sin\rho \\ \sin\alpha' = \mathrm{I} + \rho \\ \sin\rho' = \mu \,.\, \sin\alpha' \\ \sin \mathrm{D} = \rho' - \alpha - \mathrm{I} \end{array}\right\}. \qquad (c)$$

Corol. 1. The deviation may be expressed in another form, which it will be convenient hereafter to refer to. For we have 204.

$$\begin{aligned} \sin(\mathrm{I} + \mathrm{D} + \alpha) &= \sin\rho' = \mu \,.\, \sin\alpha' = \mu \,.\, \sin(\mathrm{I} + \rho) \\ &= \mu\{\sin\rho \,.\, \cos \mathrm{I} + \cos\rho \,.\, \sin \mathrm{I}\} \\ &= \mu\left\{\sin\rho - 2\sin\rho \,.\, \left(\sin\frac{\mathrm{I}}{2}\right)^2 + 2 \,.\, \cos\rho \,.\, \cos\frac{\mathrm{I}}{2} \,.\, \sin\frac{\mathrm{I}}{2}\right\} \end{aligned}$$

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because $$\cos I = 1 - 2\left(\sin \frac{I}{2}\right)^2 \text{ and } \sin I = 2 . \sin \frac{I}{2} . \cos \frac{I}{2}.$$

Now $\mu . \sin \rho = \sin \alpha$ by the first of the equations (c), hence we get (equation d)

$$\sin (I + D + \alpha) = \sin \alpha + 2\mu . \sin \frac{I}{2} . \cos \left(\frac{I}{2} + \rho\right); \qquad (d)$$

whence, I and α being given, and ρ calculated from the equation $\sin \rho = \frac{1}{\mu} \sin \alpha$, D is easily had.

205. *Corol.* 2. If $\alpha = o$, or if the ray be intromitted perpendicularly into the first surface, we have also $\rho = o$, and the expression (d) becomes simply

$$\sin (I + D) = \mu . \sin I; \qquad (e)$$

whence also $$\mu = \frac{\sin (I + D)}{\sin I}; \qquad (f)$$

Thus we see that if $\mu . \sin I > 1$, or if I, the angle of the prism, be greater than $\sin^{-1} \frac{I}{\mu}$,* the critical angle, or the least angle of total internal reflexion, the deviation becomes imaginary, and the ray cannot be transmitted at such an incidence.

206. 1st mode of determining the index of refraction by experiment. *Corol.* 3. The equation (f) affords a direct method of determining by experiment the refractive index of any medium which can be formed into a prism. We have only to measure the angle of the prism, and the deviation of a ray intromitted perpendicularly to one of its faces. Thus I and D being given by observation, μ is known. This is not, however, the most convenient way; a better will soon appear.

207. *Definitions.* One medium in Optics is said to be *denser* or *rarer* than another, according as a ray in passing from the former into the latter is bent *towards* or *from* the perpendicular. When we speak of the *refractive density* of a medium, we mean that quality by which it turns the ray more or less from its course towards the perpendicular (from a vacuum,) and whose numerical measure is the quantity μ the index of refraction.

208. Limit of the refracting angle of a prism. *Proposition.* Given the index of refraction of a prism, to find the limit of its refracting angle, or that which if exceeded, no ray can be directly transmitted through both its faces.

This limit is evidently that value of I which just renders the angle of refraction ρ' imaginary for all angles of incidence on the first surface, or for all values of α, that is, which renders in all cases

$$\mu . \sin \{ I + \rho \} - 1 \text{ positive,}$$

or, $\sin (1 + \rho) - \frac{1}{\mu}$ positive; that is, (since $I + \rho$ cannot exceed $90°$) which renders in all cases $I + \rho - \sin^{-1}\left(\frac{1}{\mu}\right)$ positive. Now $\rho = \sin^{-1} \frac{\sin \alpha}{\mu}$, and consequently the value of α least favourable to a positive value of the function under consideration is $-90°$, which makes $\rho = -\sin^{-1}\left(\frac{1}{\mu}\right)$, its greatest negative value. Consequently, in order that no second refraction shall take place, I must at least be such that $I - 2\sin^{-1}\left(\frac{1}{\mu}\right)$ shall be positive; that is, I, the angle of inclination of the faces of the prism to each other,

Angle of a prism. or as it is briefly expressed, the *angle of the prism*, must be at least twice the maximum angle of internal incidence.

209. For example, if $\mu = 2$, I must be at least $60°$. In this case no ray can be transmitted *directly* through an equilateral prism of the medium in question.

210. *Corol.* 4. If $\mu > 1$, or if the prism be denser than the surrounding medium, $\mu . \sin I$ is $> \sin I$ and $\sin^{-1} (\mu . \sin I) > I$, so that the value of D (equation (d), Art. 204) is positive, or the ray is bent towards the thicker part of the prism, (see fig. 29.) If $\mu < 1$, or the prism be rarer than the medium, the contrary is the case, (see fig. 30.)

211. Case of minimum deviation. *Problem.* The same case being supposed, (that of a prism in vacuo, or in a medium of equal density on both sides,) required to find in what direction a ray must be incident on its first surface so as to undergo the least possible deviation.

Since $D = \rho' - \alpha - I$; (c) Art. 203, and by the condition of the minimum, $d D = o$, we must have

$$d \rho' = d \alpha.$$

Now the equations (c) give by differentiation

$$d \alpha . \cos \alpha = \mu d \rho . \cos \rho; \quad d \alpha' = d \rho; \quad d \rho' . \cos \rho' = \mu d \alpha' . \cos \alpha',$$

that is $$d \rho' . \cos \rho' = \mu d \rho . \cos \alpha' = d \alpha . \frac{\cos \alpha . \cos \alpha'}{\cos \rho},$$

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* The reader will observe, that by the expression $\sin^{-1} \frac{1}{\mu}$ is meant what in most books would be expressed by arc $\sin = \frac{1}{\mu}$.

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or $$\frac{d\rho'}{da}\ (=1) = \frac{\cos a \,.\, \cos a'}{\cos \rho \,.\, \cos \rho'}\ ;\ \text{or } \cos a \,.\, \cos a' = \cos \rho \,.\, \cos \rho'.$$

That is, squaring,

$$(1 - \sin a^2)\ (1 - \sin a'^2) = (1 - \sin \rho^2)\ (1 - \sin \rho'^2)$$

in which, for sin a and sin ρ' writing their equals, μ . sin ρ and μ . sin a', we get

$$\frac{1 - \mu^2 \,.\, \sin \rho^2}{1 - \sin \rho^2} = \frac{1 - \mu^2 \,.\, \sin a'^2}{1 - \sin a'^2},$$

which gives, on reduction, simply sin ρ^2 = sin a'^2, and therefore $\rho = \pm a'$, that is $I + \rho = I \pm a'$, or $a' = I \pm a'$. The upper sign is unsatisfactory, as it would give $I = 0$. The lower therefore must be taken, which gives $a' = \frac{I}{2}$, which satisfies the conditions of the question. We therefore have

$$a' = \tfrac{1}{2} I\ ;\ \rho = -\tfrac{1}{2} I\ ;\ \sin a = -\mu \,.\, \sin\left(\frac{I}{2}\right);\ \sin \rho' = +\mu \,.\, \sin\left(\frac{I}{2}\right).$$

This state of things is represented in fig. 31, for the case where $\mu > 1$, or where the prism is denser than the surrounding medium, and in fig. 32, for that in which it is rarer, or $\mu < 1$. In both cases, a, being negative, indicates that the incident ray must fall on the side of the perpendicular C P, *from* the edge A of the prism (as S C). In both cases, the equations ρ (= P C S′) = $-\frac{1}{2} I$ (= $-\frac{1}{2}$ P C P′) and a' = P′ C S′ = $+\frac{1}{2}$ P C P′, indicate that the once refracted ray S′ C D bisects the angle P C P′, and therefore that the portion of it C D within the prism makes equal angles with both its faces. In both cases, also, the equality of the angles a and ρ' (without reference to their signs) shows that the incident and emergent rays make equal angles with the faces of the prism, and therefore that it is of no consequence on which face the ray is first received. Fig. 31. Fig. 32.

Corol. 5. In this case, also, we have the actual amount of the deviation 212. Expression for the minimum deviation.

$$D = \rho' - a - I = 2 \sin^{-1}\left(\mu \,.\, \sin \frac{I}{2}\right) - I. \qquad (f)$$

Hence also $$\sin \frac{I + D}{2} = \mu \,.\, \sin \frac{I}{2}.$$

Corol. 6. In the same case, I being given by direct measurement, and D by observation, of the actual minimum deviation of a ray refracted through any prism, the value of μ, its index of refraction, is given at once, for we have 213. Another mode of determining the index of refraction of a prism by experiment.

$$\mu = \frac{\sin\left(\frac{I + D}{2}\right)}{\sin\left(\frac{I}{2}\right)}. \qquad (g)$$

And this affords the easiest and most exact means of ascertaining the refractive index of any substance capable of being formed into a prism.

Example. A prism of silicate of lead, consisting of silica and oxide of lead, atom to atom, had its refracting angle 21° 12′. It produced a deviation of 24° 46′ *at the minimum* in a ray of homogeneous extreme red light: what was the refractive index for that ray? 214. Example.

$$I = 21°\ 12',\ \frac{I}{2} = 10°\ 36',\ D = 24°\ 46',\ \frac{D}{2} = 12°\ 23'$$

$\sin\left(\frac{I}{2} + \frac{D}{2}\right)$	= sin 22° 59′	9.59158
$\sin \frac{I}{2}$	= sin 10° 36′	9.26470
	μ = 2.123	0.32688

Case 3. Let us now take a somewhat more general case, viz. to find the final direction and total deviation of a ray, after any number of refractions at plane surfaces, all the refractions being performed in one plane, and, of course, all the common sections of the surfaces being supposed parallel. 215. Deviation of a ray after several refractions is one plane.

Supposing (as above) I to represent the inclination of the first surface to the second; I′ that of the second to the third, &c.; and I, I′, &c. to be negative when the surfaces incline the contrary way from one certain side assumed as positive, taking also δ, δ', δ'', &c. . . . . $\delta^{(n-1)}$ to represent the several partial bendings of the rays at the first, second, third, nth surface respectively, and the other symbols remaining as before, we have the total deviation, $D = \delta + \delta' + \ldots . \delta^{(n-1)}$. Now we have, since in each case $\theta = 180°$,

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$$\sin a = \mu \,.\, \sin \rho\,; \quad a' = \rho + \mathrm{I}\,; \quad \mu' \,.\, \sin \rho' = \sin a'\,; \quad \delta = a - \rho\,;$$
$$\sin a' = \mu' \,.\, \sin \rho'\,; \quad a'' = \rho' + \mathrm{I}'\,; \quad \mu'' \,.\, \sin \rho'' = \sin a''\,; \quad \delta' = a - \rho'\,; \text{ \&c. \&c.}$$

Hence we get (supposing n to represent the number of surfaces)

$$\sin \rho = \frac{1}{\mu} \,.\, \sin a$$

$$\sin \rho' = \frac{1}{\mu'} \,.\, \sin (\mathrm{I} + \rho)$$

$$\sin \rho'' = \frac{1}{\mu''} \,.\, \sin (\mathrm{I}' + \rho')$$

— — — — — — —

$$\sin \rho^{(n-1)} = \frac{1}{\mu^{(n-1)}} \,.\, \sin (\mathrm{I}^{(n-2)} + \rho^{(n-2)})$$

whence the series of values ρ, ρ', &c. may be continued to the end. These determined, we get a, a', &c. by the equations $a = a\,; \quad a' = \rho + \mathrm{I}\,; \quad a'' = \rho' + \mathrm{I}'\,; \ldots . a^{(n-1)} = \rho^{(n-2)} + \mathrm{I}^{(n-2)}$,

and finally
$$\mathrm{D} = \{ a + a' + \ldots . a^{(n-1)} \} - \{ \rho + \rho' + \ldots . \rho^{(n-1)} \}$$
$$= a + \{ \mathrm{I} + \mathrm{I}' + \ldots . \mathrm{I}^{(n-2)} \} - \rho^{(n-1)} .$$

Now $\mathrm{I} + \mathrm{I}' + \ldots \mathrm{I}^{(n-2)}$ is the inclination of the first to the last surface, or the angle (A) of the compound prism, formed of the assemblage of them all, so that we have in general

$$\mathrm{D} = a + \mathrm{A} - \rho^{(n-1)} \qquad (h)$$

216. Case of minimum deviation after any number of refractions.

Let us now inquire, how a ray must be incident on such a system of surfaces so that its total deviation shall be a minimum.

Since $d\,\mathrm{D} = o$ and I, I', &c. are constant, we must have

$$d\,a = d\,\rho^{(n-1)},$$

but
$$\left.\begin{aligned} \mu \,.\, \sin \rho &= \sin a \\ \mu' \,.\, \sin \rho &= \sin (\rho + \mathrm{I}) \\ &\text{\&c.} \end{aligned}\right\} \ldots \left\{\begin{aligned} &\mu\, d\rho \,.\, \cos \rho = d\, a \,.\, \cos a \\ &\mu' d\rho' \,.\, \cos \rho' = d\,\rho \,.\, \cos (\rho + \mathrm{I}) \\ &\mu^{(n-1)} d\,\rho^{(n-1)} \,.\, \cos \rho^{(n-1)} = d\,\rho^{(n-2)} \,.\, \cos (\rho^{(n-2)} + \mathrm{I}^{(n-2)}) \end{aligned}\right.$$

and multiplying all these equations together

$$\mu\mu \ldots . \mu^{(n-1)} \,.\, \cos \,.\, \rho \cos \rho' \,.\, \ldots \cos \rho^{(n-1)} \,.\, \frac{d\,\rho^{(n-1)}}{d\,a} = \cos a \,.\, \cos (\rho + \mathrm{I}) \ldots . \cos (\rho^{(n-2)} + \mathrm{I}^{(n-2)})$$

or simply

$$\mu\mu' \ldots . \mu^{(n-1)} \,.\, \cos \rho \,.\, \cos \rho' \ldots . \cos \rho^{(n-1)} = \cos a \,.\, \cos a' \,.\, \ldots . \cos a^{(n-1)}; \qquad (i)$$

this equation, combined with the relations already stated, between the successive values of ρ and those of a, afford a solution of the problem; but the final equations to which it leads are of great complexity and high dimensions. Thus, in the case of only three refractions, the final equation in sin ρ or sin ρ', &c. rises to the sixteenth degree; and though its form is only that of an equation of the eighth, yet there appears no obvious substitution by which it can be brought lower. The only case where it assumes a tractable form is that of two surfaces, when the equation (i) which in general may be put under the form

$$\mu^2 \mu'^2 \ldots . \mu^{(n)2} (1 - \sin \varrho^2) (1 - \sin \varrho'^2), \text{ \&c.} = (1 - \mu^2 \,.\, \sin \rho^2) (1 - \mu'^2 \,.\, \sin \rho'^2), \text{ \&c.} \qquad (j)$$

reduces itself by putting $\sin \varrho^2 = x$, and $\sin \rho'^2 = y$,

to
$$(\mu^2 \mu'^2 - 1) - \mu^2 (\mu'^2 - 1)\, x - \mu'^2 (\mu^2 - 1)\, y = 0,$$

which, combined with the equation $\mu' \,.\, \sin \rho' = \sin (\varrho + \mathrm{I})$

or
$$(\mu'^2 y + x - \sin \mathrm{I}^2)^2 = 4\, \mu'^2 \,.\, \cos \mathrm{I}^2 \,.\, x\, y,$$

gives a final equation of a quadratic form for determining x or y, and which in the particular case of $\mu\mu' = 1$, or when the second refraction is made into the same medium in which the ray originally moved before its first incidence, gives the same result we have already found for that case by a similar process. Meanwhile, though we may not be able to resolve the final equations in the general case, the equation (j) affords a criterion of the state of minimum deviation which may prove useful in a variety of cases.

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Case 4. When the planes of the first and second refraction are at right angles to each other, required the relations arising from this condition.

217. Case when the planes of the first and second refraction are at right angles.

In this case we have $\theta = 90^0$, $\cos\theta = 0$, $\sin\theta = 1$, so that the general equation (B, 199) becomes

$$\left.\begin{aligned}\sin a &= \mu \,.\, \sin\rho\\ \sin a' &= \mu' \,.\, \sin\rho'\\ \sin a' &= \sin \mathrm{I} \,.\, \sin\psi\end{aligned}\right\} \text{and } \cos a' = \cos\varrho \,.\, \cos \mathrm{I} + \sin\rho \,.\, \sin \mathrm{I} \,.\, \cos\psi.$$

The last of these equations, by transposition and squaring, becomes

$$\cos a'^2 - 2 \,.\, \cos a' \,.\, \cos\rho \,.\, \cos \mathrm{I} + \cos\varrho^2 \,.\, \cos \mathrm{I}^2 = \sin\rho^2 \,.\, \sin \mathrm{I}^2 \,(1 - \sin\psi^2)$$

in which, substituting for $\sin\psi$ its value $\dfrac{\sin a'}{\sin \mathrm{I}}$ deduced from the third equation, and reducing as much as possible, we obtain

$$\cos a'^2 \,.\, \cos\rho^2 - 2 \,.\, \cos a' \,.\, \cos\rho \,.\, \cos \mathrm{I} + \cos \mathrm{I}^2 = 0,$$

which, being a complete square, gives simply

$$\cos\rho \,.\, \cos a' = \cos \mathrm{I}. \qquad (k)$$

This answers to the equation $\cos a \,.\, \cos a' = \cos \mathrm{I}$, obtained, on the same hypothesis, in the case of reflexion (104); for since the latter case is included in the case of refraction, by putting $\mu = -1$ (Art. 192) we have then $a = -\rho$ and $\cos\rho = \cos a$.

218. *Corol.* 1. If i and i' be the inclinations to the first and second surfaces respectively of that part of the ray which lies between the surfaces, we have

$$i = 90^\circ - \varrho \qquad \text{and } i' = 90^\circ - a',$$

so that the equation above found, gives

$$\sin i \,.\, \sin i' = \cos \mathrm{I},$$

or the product of the sines of the inclination of the ray between the surfaces to either surface is equal to the cosine of the inclination of the two surfaces. The same relation may be expressed otherwise, thus: if we suppose the ray to pass both ways from within, out of the prism, the product of the cosines of its interior incidences on the two surfaces is equal to the cosine of their inclination to each other. In this way of stating it, the case of reflexion is included.

219. *Corol.* 2. We have also in the present case

$$\sin\rho = \frac{1}{\mu} \,.\, \sin a\,; \qquad \sin a' = \sqrt{\frac{\mu^2 \,.\, \sin \mathrm{I}^2 - \sin a^2}{\mu^2 - \sin a^2}}\,; \qquad \sin\rho' = \frac{1}{\mu'}\sqrt{\frac{\mu^2 \,.\, \sin \mathrm{I}^2 - \sin a^2}{\mu^2 - \sin a^2}}$$

and

$$\cos \mathrm{D} = \cos(a - \varrho) \,.\, \cos(a' - \rho')$$

so that a being given, all the rest become known. The last equation corresponds to the equation $\cos \mathrm{D} = \cos 2a \,.\, \cos 2a'$ in the case of reflexion.

§ VII. *Of Ordinary Refraction at Curved Surfaces, and of Diacausties, or Caustics by Refraction.*

220. The refraction at a curved surface being the same as at a plane, a tangent at the point of incidence, if we know the nature of the surface, we may investigate, by the rules of refraction at plane surfaces, combined with the relations expressed by the equation of the surface, in all cases, the course of the refracted ray. We shall confine ourselves to the simple case of a surface of revolution, having the radiant point in the axis.

221. General investigation of the focus of any curve surface of revolution. Fig. 33.

Proposition. Given a radiant point in the axis of any refracting surface of revolution, required the focus of any annulus of the surface.

Let C P be the curve, Q the radiant point, Q q N the axis, P M any ordinate, P N a normal, and P q or q P the direction of the refracted ray, and therefore q the focus of the annulus described by the revolution of P. Then if we put μ for the refractive index, and, assuming Q for the origin of the coordinates, put Q M $= x$, M P $= y$, $r = \sqrt{x^2 + y^2}$, $p = \dfrac{dy}{dx}$, we have

$$\sin \mathrm{QPM} = \frac{x}{r}\,; \qquad \cos \mathrm{QPM} = \frac{y}{r}\,;$$

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$$\sin \mathrm{NPM} = \frac{p}{\sqrt{1+p^2}}; \qquad \cos \mathrm{NPM} = \frac{1}{\sqrt{1+p^2}}$$

consequently $$\sin \mathrm{NPQ} = \sin \mathrm{QPM} \,.\, \cos \mathrm{NPM} + \sin \mathrm{NPM} \,.\, \cos \mathrm{QPM}$$
$$= \frac{x+py}{r \,.\, \sqrt{1+p^2}},$$

and therefore $$\sin \mathrm{NP}q = \frac{1}{\mu} \,.\, \sin \mathrm{NPQ} = \frac{x+py}{\mu r \sqrt{1+p^2}};$$

and $$\cos \mathrm{NP}q = \frac{Z}{\mu r \sqrt{1+p^2}}, \quad \text{if we put } Z = \sqrt{\mu^2 r^2 (1+p^2) - (x+py)^2}; \qquad (a)$$

consequently since $\mathrm{MP}q = \mathrm{NP}q + \mathrm{NPM}$, we get

$$\sin \mathrm{MP}q = \frac{x+py+pZ}{\mu r (1+p^2)}; \quad \text{and } \cos \mathrm{MP}q = \frac{-p(x+py)+Z}{\mu r (1+p^2)};$$

whence $$\tan \mathrm{MP}q = \frac{\sin \mathrm{MP}q}{\cos \mathrm{MP}q} = \frac{x+py+pZ}{-p(x+py)+Z}.$$

Now we have $$\mathrm{M}q = \mathrm{PM} \,.\, \tan \mathrm{MP}q = y \,.\, \tan \mathrm{MP}q = \frac{y(pZ+(x+py))}{Z-p(x+py)}; \qquad (b)$$

consequently $$\mathrm{Q}q = x + y \,.\, \tan \mathrm{MP}q = (x+py) \,.\, \frac{px-y-Z}{p(x+py)-Z}. \qquad (c)$$

222. *Corol.* 1. If we put s for the arc CP of the curve, we have, since $r\,dr = x\,dx + y\,dy = dx\,(x+py)$,

$$Z = \sqrt{\mu^2 r^2 \cdot \left(\frac{ds}{dx}\right)^2 - \left(\frac{r\,dr}{dx}\right)^2} = r \,.\, \sqrt{\mu^2 \left(\frac{ds}{dx}\right)^2 - \left(\frac{dr}{dx}\right)^2}; \qquad (d)$$

223. *Corol.* 2. If $\mu = -1$, in which case the refraction becomes a reflexion, we have

$Z = \sqrt{r^2 (1+p^2) - (x+py)^2} = y - px$, writing for r^2 its value $x^2 + y^2$; so that the general value above found for $\mathrm{Q}q$ reduces itself to

$$\mathrm{Q}q = 2 \,.\, \frac{(x+py)(px-y)}{2px - y(1-p^2)},$$

which is the same as that found in (b) Art. 109, in the case of reflexion.

224. *Corol.* 3. If we put $\mathrm{P} = \tan \mathrm{M}q\mathrm{P} = \operatorname{cotan} \mathrm{MP}q = \dfrac{1}{\tan \mathrm{MP}q}$,

we have $$\mathrm{P} = \frac{-p(x+py)+Z}{x+py+pZ}, \qquad (e)$$

and the equation of the refracted ray, if X and Y be its coordinates, (Q being their origin) will be (since Y lies on the opposite side of the curve from Q)

$$\mathrm{Y} - y = -\mathrm{P} \,.\, (\mathrm{X} - x) \qquad (f)$$

225 In the case of parallel rays these expressions become (by putting first $x + a$ for x, and then making a infinite)

$$\left.\begin{aligned} Z &= a\sqrt{\mu^2 (1+p^2) - 1} \\ \mathrm{P} &= \frac{-p + \sqrt{\mu^2 (1+p^2) - 1}}{1 + p\sqrt{\mu^2 (1+p^2) - 1}} \end{aligned}\right\}; \qquad (g)$$

$$\mathrm{A}q = x + y \,.\, \frac{1 + p\sqrt{\mu^2 (1+p^2) - 1}}{\sqrt{\mu^2 (1+p^2) - 1} - p}. \qquad (h)$$

§ VIII. *Of Caustics by Refraction, or Diacaustics.*

The theory of Diacaustics is in all respects analogous to that of Catacaustics already explained. To find the coordinates X and Y of the point in the diacaustic which corresponds to the point P in the refracting curve, we have only to regard the equation (f) and its differential with respect to x, y, and p alone, as subsisting together, and we get the necessary equations for determining X and Y in terms of x, y, as in the case of reflexion, and 226. Fig. 34.

these are $$X = x + \frac{P + p}{dP} dx; \qquad Y = y - P \cdot \frac{P + p}{dP} dx; \qquad (i)$$

the only difference is in the signs and in the value of P, which, instead of the formula (c, Art. 110,) is here expressed by the more complicated function (e, Art. 223,) and the equation of the diacaustic will be obtained as before by eliminating all but X, Y from these.

It is, evident, moreover, that if we suppose, as in the theory of Catacaustics, $M = \frac{P + p}{dP} dx$; and put 227.

S for the length of the caustic, and f for the line P y, we shall have, exactly as in that theory,

$$f = M \cdot \sqrt{1 + P^2}; \qquad -P = \frac{dY}{dX}; \qquad \text{and } dS = df + dx \cdot \frac{1 - Pp}{\sqrt{1 + P^2}}.$$

See Art. 139, 143, 144.

Now we have, substituting for P its value (e),

$$\sqrt{1 + P^2} = \frac{\mu r (1 + p^2)}{x + py + pZ}; \qquad 1 - Pp = \frac{(x + py)(1 + p^2)}{x + py + pZ}; \qquad (k)$$

and consequently the value of dS becomes

$$dS = df + \frac{(x + py)\, dx}{\mu r} = df + \frac{dr}{\mu}, \text{ because } (x + py)\, dx = r\, dr,$$

and integrating $$S = \text{const} + f + \frac{r}{\mu};$$

so that we have, finally, (fig. 34,) Fig. 34.

$$\text{arc } Fy = (CF - Py) + \frac{1}{\mu}(QC - QP). \qquad (l)$$

In the case of reflexion, $\mu = -1$, but at the same time the sign of f is negative, because in this case the reflected ray lies on the same side of the point of incidence with the incident one; thus both terms of the formula change their sign, and this expression coincides with that found Art. 144. 228

In the case of parallel rays, we must use the value of P found in Art. 225, equation (g). Putting 229. $q = \frac{dp}{dx}$, and executing the operations, we find, then,

$$\left.\begin{aligned} X &= x - \frac{1}{P} \cdot \frac{\mu^2 (1 + p^2) - 1}{\mu^2 q} \\ Y &= y + \frac{\mu^2 (1 + p^2) - 1}{\mu^2 q} \end{aligned}\right\}. \qquad (m)$$

Corol. If we suppose $\mu = \infty$, or the refractive power infinite, the refracted ray will coincide with the normal, and the caustic will be identical with the evolute; and it is evident that the expressions (m), when $\mu = \infty$, resolve themselves into the well-known values of the coordinates of the evolute. 230.

If the rays incident on the refracting curve do not diverge from one point, but be all tangents to a curve V V′ V″, (fig. 35,) we must put $x - a$ for x in the value of P, (eq. (e) Art. 224;) and fix the origin of the coordinates at A, putting A Q $= a$; and if, then, we regard a as variable according to any given law, (or regard $x - a$ at once as a given function of x,) and take the differential of P on this supposition, the equations (i) still hold good, and suffice to define the caustic. 231. Fig. 35.

Problem. The radiant point and refractive index of a medium being given, to determine the nature of the curve surface which shall refract all the rays to one point. 232.

Here we are required to find the relation between x and y, so as to make Q q invariable. Let Q $q = c$, and we have

$$c = (x + py) \cdot \frac{px - y - Z}{p(x + py) - Z}; \qquad \text{where } Z = \sqrt{\mu^2 (x^2 + y^2)(1 + p^2) - (x + py^2)}.$$

This equation gives

$$(x + py)\,(p(x - c) - y) = Z(x - c + py).$$

Light. Squaring both sides, and substituting for Z its value, we get Part I.

$$(x + py)^2 \{ (p(x-c) - y)^2 + (x - c + py)^2 \} = (x - c + py)^2 . \mu^2 (x^2 + y^2)(1 + p^2)$$

which, on executing the operations indicated on the left hand side, becomes totally divisible by $1 + p^2$, and reduces itself to

$$(x + py)^2 (y^2 + (x-c)^2) = \mu^2 (x - c + py)^2 (y^2 + x^2)$$

that is, putting for p its value $\frac{dy}{dx}$, multiplying by dx^2, and extracting the square root,

$$\frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \mu . \frac{(x-c)\,dx + y\,dy}{\sqrt{(x-c)^2 + y^2}},$$

and integrating (each side being a complete differential)

$$\sqrt{x^2 + y^2} = b + \mu . \sqrt{(x-c)^2 + y^2}, \qquad (n)$$

which is the equation of the curve required, and belongs, generally, to a curve of the fourth order.

233. *Corol.* 1. About Q, (fig. 36,) with any radius Q A, arbitrarily assumed, describe a circle A B D E, then if C P be the refracting curve, and we put Q A $= b$, we have Q P $= \sqrt{x^2 + y^2}$, P $q = \sqrt{(x-c)^2 + y^2}$, and the nature of the curve is expressed by the property

Fig. 36.

$$\text{B P} = \mu . \text{P}q, \qquad \text{or, B P : P}q :: \mu : 1.$$

234. *Corol.* 2. If $b = 0$, or the circle A B E be infinitely small, we have Q P : P q : : μ : 1, which is a well known property of the circle. In fact, in this case we have simply

$$x^2 + y^2 = \mu^2 \{ (x-c)^2 + y^2 \}$$

In this, if we change the origin of the coordinates by writing $x + \frac{\mu^2}{\mu^2 - 1} c$ for x, we find

$$y^2 = \left(\frac{\mu}{\mu^2 - 1} c\right)^2 - x^2.$$

The radius of the circle, therefore, is equal to $\frac{\mu}{\mu^2 - 1} \times \text{Q}q$, and the distance of its centre from the radiant point is $\frac{\mu^2}{\mu^2 - 1} \times \text{Q}q$. Take therefore any circle H P C whose centre is E, (fig. 37,) and two points Q, q, such that Q E $= \mu \times$ E C and Q C : C q : : μ : 1. Then if rays diverge from Q, and fall on the surface P H beyond the centre, they will, after refraction into the medium M, all diverge from q.

Fig. 37.

235. *Corol.* 3. If $\mu = -1$, the equation (n), when freed from radicals, is only of the second degree between x and y, and therefore belongs to a conic section. On executing the reduction we get

$$y^2 = \left(1 - \frac{c^2}{b^2}\right) \left(\left(\frac{b}{2}\right)^2 = \left(x - \frac{c}{2}\right)^2\right),$$

which shows that the radiant point Q is in one focus and q in the other, which is the same result as that before found by a different mode of integration.

236. *Corol.* 4. When Q is infinitely distant, and the rays are parallel, we must shift the origin of the coordinates from Q to q, by putting $c - x$ for x, and *afterwards* supposing c infinite. This gives

For parallel rays the curve is a conic section.

$$\sqrt{c^2 - 2cx + x^2 + y^2} = b + \mu \sqrt{x^2 + y^2}.$$

Fig. 38.

Developing the first term in a descending series, we find

$$(c - b) - x + \frac{x^2 + y^2}{2c^2} + \&c. = \mu . \sqrt{x^2 + y^2},$$

Let $c - b = h$, which, since b is arbitrary, is equally general, and may represent any finite quantity, then, as c increases and at length becomes infinite, this equation becomes ultimately

$$h - x = \mu \sqrt{x^2 + y^2}.$$

Let C P be a conic section, q its focus, and A B its directrix, q M $= x$, and P M $= y$, then will Q P $= h - x$ if we take q A $= h$, and the above equation we see expresses that well known property of a conic section, in virtue of which Q P : P q in a constant ratio, (μ : 1.)

237. *Corol.* 5. The curve is an ellipse when Q P $> $ P q, or when the ray is incident from a rarer on a denser medium; and an hyperbola in the contrary case. If Q P $=$ P q, the curve is a parabola; in this case $\mu = 1$, and the rays converge to the focus at an infinite distance, *i. e.* remain parallel.

238 To take a single example of the investigation of the diacaustic curve, from the general expressions above

Light. delivered,—let the refracting surface be a plane, and we shall have, fixing the origin of the coordinates at the radiant point, and supposing the axis of the x perpendicular to the refracting plane A C B, Part I. Caustic of a plane refracting surface.

$$x = \text{constant} = \text{Q C} = a, \quad p = \frac{dy}{dx} = \infty. \qquad \text{Thus we get}$$

$$Z = p\sqrt{(\mu^2 - 1)\,y^2 + \mu^2 a^2}; \qquad \text{P} = -\frac{y}{\sqrt{(\mu^2 - 1)\,y^2 + \mu^2 a^2}};$$

$$\frac{d\,\text{P}}{dx} = -\frac{\mu^2 a^2 p}{\sqrt{(\mu^2 - 1)\,y^2 + \mu^2 a^2}};$$

and therefore by the equations (i) we get, substituting these values, Fig. 39 and 40.

$$\left.\begin{aligned} \mu^2 a^2 (a - \text{X}) &= \{(\mu^2 - 1)\,y^2 + \mu^2 a^2\}^{\frac{3}{2}} \\ \text{Y} &= \frac{1 - \mu^2}{\mu^2} \cdot \frac{y^3}{a^2}. \end{aligned}\right\}$$

Eliminating y from these, we have the equation of the caustic

$$\left(\frac{a - \text{X}}{\mu a}\right)^{\frac{2}{3}} + \left(\frac{\sqrt{1 - \mu^2}}{\mu} \cdot \frac{\text{Y}}{a}\right)^{\frac{2}{3}} = 1$$

This is the equation of the evolute of a conic section whose centre is C, and focus the radiant point Q. If μ be greater than unity, or the refraction be made into a rarer medium from a denser, the conic section is an ellipse, (see fig. 39,) and in the contrary case an hyperbola, (fig. 40.)

§ IX. *Of the Foci of Spherical Surfaces for Central Rays.*

Definitions. The *curvature* of any spherical surface is the reciprocal of its radius, or a fraction whose numerator is unity, and denominator the number of units of any scale of linear measure to which the radius is equal. 239. Curvature defined.

The *proximity* of one point to another is the reciprocal of their mutual distance, or the quotient of unity by the number of units of linear measure in that distance. 240. Proximity.

The *focal distance* of a spherical surface is the distance from the vertex, of the point to which rays converge, or from which rays diverge after refraction or reflexion. 241. Focal distance.

The *principal focal distance*, or *focal length*, is the distance from the vertex of the point to which *parallel and central* rays converge, or from which they diverge after refraction or reflexion. 242. Focal length

The *power* of a surface is the reciprocal of its *principal focal distance*, or *focal length*, estimated as in the definitions of curvature and proximity. 243. Power.

Problem. To find the focus of a spherical refracting surface after one refraction, for central rays. 244. General expressions for the focal distance of any annulus of a spherical refracting surface.

Here, putting a for the distance of the focus of incident rays Q, (fig. 41,) from the centre E, we have

$$(a - x)^2 + y^2 = r^2; \qquad p = \frac{a - x}{y}; \qquad 1 + p^2 = \frac{r^2}{y^2}; \qquad x + py = a;$$

and these substituted in the general expressions Art. 221, give

$$\left.\begin{aligned} y\,Z &= \sqrt{\mu^2 r^2 x^2 + (\mu^2 r^2 - a^2)\,y^2} \\ \text{Q}\,\overline{q} &= a\left\{1 - \frac{r^2}{a(a - x) - y\,Z}\right\} \\ \overline{\text{C}\,q} &= r\left\{1 - \frac{r\,a}{a(a - x) - y\,Z}\right\} \end{aligned}\right\}. \qquad (a)$$

These values of Q q and C q contain the rigorous solution of the problem, whatever be the *amplitude* (y) of the annulus whose focus q is, and we shall accordingly again have recourse to them. At present, however, our concern being only with central rays, we must put $y = 0$, when we find $x = a - r$; $y\,Z = \mu r x = \mu r\,(a - r)$ Focus for central ra

$$\overline{\text{Q}q} = a.\,\frac{(a - r)\,(1 - \mu)}{a - \mu a + \mu r} \qquad \overline{\text{C}\,q} = \frac{\mu r\,(r - a)}{a - \mu a + \mu r}. \qquad (b)$$

Corol. 1. This latter is the focal distance for central rays. Now, since $a - r$ = Q C, this gives the following proportion, 245.

$$\mu.\,\text{Q C} - \text{Q E} : \mu.\,\text{Q C} :: \text{C E} : \text{C}\,q. \qquad (c)$$

Light. Part I.

Corol. 2. If we suppose the focus of incident rays infinitely distant, or $a = \infty$, and take F the place of q for central rays, on that supposition, F will be the principal focus, and we shall have

246. Focus for parallel rays

$$\left.\begin{array}{l} CF = \dfrac{\mu r}{\mu - 1}; \quad \text{that is, } CE : CF :: \mu - 1 : \mu \\ \text{whence we also find} \\ CE : EF :: \mu - 1 : 1, \quad \text{and } CF : FE :: \mu : 1 \end{array}\right\}. \qquad (d)$$

247. These results will be expressed more conveniently for our future reference by adopting a different notation. Let, then,

$R = \dfrac{1}{r}$ = curvature of the surface, and let positive values of r and R correspond to the case where the centre E lies to the right of the vertex C, or in the direction in which the rays proceed.

$D = \dfrac{1}{QC}$ (fig. 42) = proximity of the focus of incident rays to the surface, D being regarded as positive when Q lies to the *right* of C, as in fig. 42, and as negative when to the left, as in fig. 41. Then, since $QE = a$, and since in the foregoing analysis a is regarded as positive when Q is to the *left* of E, we must have (fig. 42) $QE = -a$, and $QC = QE + EC = r - a$, so that

$$D = \frac{1}{r - a}; \qquad a = \frac{1}{R} - \frac{1}{D}. \quad \text{Let also } m = \frac{1}{\mu}:$$

$F = \dfrac{1}{CF}$ = power of the surface:

$f = \dfrac{1}{Cq}$ = proximity of the focus of refracted rays to the surface.

Positive values of F and f, as well as of D and R, being supposed to indicate situations of the points F, f, Q, E, respectively, to the right of C, or in the direction towards which the light travels. This is, in fact, assuming for our positive case that of *converging* rays incident on a *convex* surface of a *denser* medium. We shall have, then,

$$r = \frac{1}{R}; \qquad r' - a = \frac{1}{D}; \qquad a = \frac{1}{R} - \frac{1}{D}; \qquad \mu = \frac{1}{m}.$$

Fundamental equation for the foci of central rays.

But equation (*b*) gives $\dfrac{1}{Cq} = \dfrac{a + \mu(r - a)}{\mu r (r - a)}$, and substituting we shall get

$$f = (1 - m)\, R + m\, D. \qquad (e)$$

This equation comprises the whole doctrine of the foci of spherical surfaces for central rays, and may be regarded as the fundamental equation in their theory.

248. General expression for the power of any surface.

In the case of parallel rays, we have $D = 0$, whether the rays be incident from left to right, or from right to left. In either case, then, f has the same value, viz. $(1 - m) . R$, and the principal focal distance F in either case is the same, being given by the equation

$$F = (1 - m) . R, \qquad (f)$$

which shows, moreover, that the power of any spherical surface is in the direct ratio of its curvature.

249. Hence also we have $f = F + m\, D. \qquad (g)$

250. Fundamental expressions for the central foci in case of reflexion.

In the case of reflexion, where $\mu = -1$, or $m = -1$, these equations become respectively

$$F = 2R; \qquad f = 2R - D; \qquad f = F - D. \qquad (h)$$

Such are the expressions for the central foci in the case of a single surface. Let us now consider that of any system of spherical surfaces.

Problem. To find the central focus of any system of spherical surfaces.

251. Central focus of a system of spherical surfaces investigated. Fig. 43.

Let C′, C″, C‴, &c. be the surfaces. Q′ the focus of rays incident on C′; Q″ that of refracted rays, or the focus of rays incident on C″, and so on. Call also R′, R″, &c. the radii of the first, second, &c. surfaces μ', μ'', &c. their refractive indices, or $\dfrac{\sin \text{inc}}{\sin \text{ref}}$ into each medium from that immediately preceding, $m' = \dfrac{1}{\mu'}$, $m'' = \dfrac{1}{\mu''}$, &c. Also let $D' = \dfrac{1}{C'Q'}$, $D'' = \dfrac{1}{C''Q''}$, &c. and moreover let $C'C'' = t'$, $C''C''' = t''$, &c. t', t'', &c. being regarded as positive when C″, C‴, &c. respectively lie to the right of C′, C″, &c. or in the direction in which the light travels; and if we put $\dfrac{1}{C'Q''} = f'$, $\dfrac{1}{C''Q'''} = f''$, &c. $F' = (1 - m')\, R'$, $F'' = (1 - m'')\, R''$, &c. we shall have by (249)

$$f' = F' + m' D'; \qquad f'' = F'' + m'' D'', \text{ &c.;} \qquad (i)$$

Light. Part I.

but we have also

$$C'Q' = \frac{1}{D'}; \qquad C''Q'' = \frac{1}{D''} = C'Q'' - C'C'', = \frac{1}{f'} - t';$$

and so on; so that we have, besides, the following relations,

$$D' = D'; \qquad D'' = \frac{f'}{1 - f' t'}; \qquad D''' = \frac{f'}{1 - f'' t''}, \text{ \&c.}; \qquad (j)$$

and substituting these values of D'', D''', &c. in the equations (i), and in each subsequent one, introducing the values of f', f'', &c. obtained from those preceding, we shall obtain explicit values of f', f'', &c. to the end.

252. The systems of equations (i) and (j) contain the general solution of the problem, whatever be the intervals between the surfaces. On executing the operations, however, for general values of t', t'', &c. the resulting expressions are found to become exceedingly complex, nor is there any way of simplifying them, the complication being in the subject, not in the method of treating it. For further information on this point, consult Lagrange, (*Sur la Théorie des Lunettes*, Berlin, Acad. 1778.) We shall here only examine the principal cases.

253. Foci of a system of spherical surfaces placed close together.

Problem. To find the focal distance of any system of spherical surfaces placed close together.

Here t', t'', &c. all vanish, and the equations (i) and (j) become simply

$$D' = D'; \qquad D'' = f'; \qquad D''' = f'', \text{ \&c.};$$
$$f' = F' + m' D'; \qquad f'' = F'' + m'' D'', \text{ \&c.};$$

whence by substitution we obtain

$$f'' = F'' + m'' F' + m' m'' D'$$
$$f''' = F''' + m''' F'' + m''' m'' F' + m''' m'' m' D',$$

which it is easy to continue as far as we please.

254. *Corol.* 1. Let the number of surfaces be n, and let M' represent μ', or the *absolute* refractive index out of vacuum into the first medium; $M'' = \mu' \mu''$, or the absolute refractive index from vacuum into the second medium, and so on; μ', μ'', &c. representing only the relative refractive indices from each medium into that succeeding it. Thus we shall have

$$M^{(n)} f^{(n)} = D' + M' F' + M'' F'' + \ldots\ldots M^{(n)} F^{(n)}. \qquad (k)$$

255. *Cor.* 2. For parallel rays, in whichever direction incident, we have $D' = 0$; and the principal focal length of the system, which we will call $\frac{1}{\phi^{(n)}}$, is given by the equation

$$M^{(n)} \phi^{(n)} = M' F' + M'' F'' + \ldots . M^{(n)} F^{(n)}. \qquad (l)$$

256. *Cor.* 3. Hence it appears that $\phi^{(n)}$, the *power of the system*, or its reciprocal focal length for parallel rays, being found by the last equation, the focus for any converging or diverging rays is had at once by the equation

$$M^{(n)} f^{(n)} = M^{(n)} \phi^{(n)} + D'.$$

257. Fundamental expressions for central foci of any system of spherical surfaces.

For brevity and convenience, let us, however, modify our notation as follows: confining the accented letters to the several individual surfaces of which the system consists, let the unaccented ones be conceived to relate to their combined action as a system. Thus, F', F'', ... $F^{(n)}$ representing the individual powers of the respective surfaces; let F, without an accent, denote the resulting power of the system. In this view D' may be used indifferently; accented, as relating to the incidence on the first surface; or unaccented, as expressing the proximity of the focus of incident rays to the vertex of the whole system. Similarly, $M^{(n)}$ may be used without an accent, if we regard the total refractive index of the *system* as that of a ray passing *at one refraction* into the last medium. This supposed, the equations (k) and (l) become

$$MF = M'F' + M''F'' + \ldots\ldots M^{(n)} F^{(n)}; \qquad (m)$$
$$Mf = MF + D; \qquad M(F - f) + D = 0. \qquad (n)$$

258. If the whole system be placed in vacuo, or if the last refraction be made into vacuum, we have $M = 1 = M^{(n)}$, and the equations become

$$\left.\begin{aligned} F &= M'F' + M''F'' + \ldots . M^{(n)} F^{(n)} \\ f &= F + D \end{aligned}\right\}. \qquad (o)$$

259 Lenses defined and distinguished into species.

Definitions. A *lens* in Optics is a portion of a refracting medium included between two surfaces of revolution whose axes coincide. If the surfaces do not meet, and therefore do not include space, an additional boundary is required, and this is a cylindrical surface, having its axis coincident with that of the surfaces.

The *axis* of the lens is the common axis of all the bounding surfaces.

Lenses are distinguished (after the nature of their surfaces) into *double-convex*, with both surfaces convex, (fig. 44;) *plano-convex*, with one surface plane, the other convex, (fig. 45;) *concavo-convex*, (fig. 46;) *double-concave*, (fig. 47;) *plano-concave*, (fig. 48;) and *meniscus*, (fig. 49,) in which the concave surface is less curved than the convex. Also into *spherical*, (when the surfaces are segments of spheres;) *conoidal*, when portions of ellipsoids, hyperboloids, &c

Light. Part I.

These different species are distinguished, algebraically, by the equations of the surfaces, and by the signs of their radii of curvature. In the case of spherical lenses, to which our attention will be chiefly confined, if we

260. Species of lenses how distinguished algebraically. suppose a positive value of the radius of curvature to correspond to a surface whose convexity is turned towards the *left*, or towards the incident rays, and a negative to that whose convexity is turned to the right, or from them, we shall have the following varieties of denomination:

meniscus, concavo-convex } { both radii +, as fig. 46, 49, *a*, or
{ both radii —, as fig. 46, 49, *b*,

plano-convex { radius of first surface +, of second infinite, fig. 45, *b*,
{ radius of first surface infinite, of second —, fig. 45, *a*,

plano-concave { radius of first surface —, of second ∞, fig. 48, *b*,
{ radius of first surface ∞, of second +, fig. 48, *a*,

double-convex: radius of first surface +, of second —, fig. 44,

double-concave: radius of first surface —, of second +, fig. 47,

the rays being supposed in all cases to pass from left to right.

A *compound lens* is a lens consisting of several lenses placed close together.

An *aplanatic* lens is one which refracts all the rays incident on it to one and the same focus.

261. Focus of a single lens. *Problem. To find the power and foci of a single thin lens in vacuo.*

Let R' and R'' be the curvatures of its first and second surfaces respectively, μ the refractive index of the medium of which it consists, $m = \frac{1}{\mu}$; F its power: then we have, since the last refraction is made into vacuum,

$$F = \mu F' + F''; \qquad f = F + D;$$

but, $F' = (1 - m') R'$, and $F'' = (1 - m'') R''$; and as $\mu' = \frac{1}{\mu}$ and $m'' = \mu$, these become respectively

$\frac{1}{\mu}(\mu - 1) R'$ and $-(\mu - 1) R''$, so that the foci of the lens are finally determined by the equations

Fundamental equations.

$$\left.\begin{aligned} F &= (\mu - 1)(R' - R'') \\ f &= F + D \end{aligned}\right\}. \qquad (p)$$

262. Power of a lens. *Corol.* 1. The power of a lens is proportional to the difference of the curvatures of the surfaces in a meniscus or concavo-convex lens; and to their sum, in a double-convex or double-concave.

In plano-convex, or plano-concave lenses, the power is simply as the curvature of the convex or concave surface.

263. *Corol.* 2. In double-convex lenses R' is positive and R'' negative, so that when $\mu > 1$, F is positive, or the rays converge to a focus behind the lens. In plano-convex, $R'' = 0$ and R' is +; or $R' = 0$ and R'' is negative, (260); hence in both cases F is positive and the rays also converge. In meniscus lenses also, R' is +, and R'', though +, is less than R', (fig. 49;) therefore in these, also, the same holds good. In all these cases the *focus is said to be real*, because the rays actually meet there. In double-concave, plano-concave, or concavo-concave lenses, the reverse holds good; the focus lies on the opposite side, or towards the incident rays, and parallel rays, after refraction, diverge from it. In this case, therefore, they never meet, and the focus is called a *virtual focus*.

Fig. 49. Real and virtual foci.

264. *Corol.* 3. If μ be < 1, or the lens be formed of a medium rarer than the ambient medium (which need not be vacuum, provided the whole system be immersed in it,) $\mu - 1$ is negative, and all the above cases are reversed. In this case convex lenses give virtual, and concave, real foci.

265. Positive and negative powers. *Corol.* 4. For lenses of denser media, the powers of double-convex, plano-convex, and menisci are positive; and those of double plano-concave and concavo-convex lenses, negative; *vice versâ* for rarer media.

Corol. 5. The focus of parallel rays is at the same distance, on whichever side of the lens the rays fall. For if the lens be turned above, R' becomes R'', and *vice versâ;* but, since they also change their signs, F remains unaltered.

267. Conjugate foci move in the same direction. *Corol.* 6. The equation $f = F + D$ gives $df = dD$. This shows that the foci of incident and refracted rays move always in the same direction, if the former be supposed to shift its place along the axis; and, moreover, that their proximities to the lens vary by equal increments or decrements for each.

Problem. To determine the central foci of any system of lenses placed close together, the lenses being supposed infinitely thin.

268. Central foci of a system of thin lenses in contact. The general problem of a system of spherical surfaces contains this as a particular case; for we may regard the posterior surface of the first lens, and the anterior of the second, as forming a lens of vacuum interposed between the two lenses, and so for the rest. Thus the system of lenses is resolved into a system of spherical surfaces in contact throughout their whole extent; the alternate media having their refractive indices, or the alternate values of M, unity. If then we call μ', μ'', μ''', &c. the refractive indices of the lenses, we shall have

$$M = 1; \qquad M' = \mu'; \qquad M'' = 1; \qquad M''' = \mu''; \qquad M^{iv} = 1, \text{ \&c.}$$

Light. The compound power F then will (258, o) be represented by

$$F = \mu' F' + F'' + \mu'' F''' + F^{iv} + \mu''' F^{v} + F^{vi} +, \&c.$$

Part I. Superposition of powers. Power of a system of lenses is the sum of the powers of the component individuals.

But
$$F' = (1 - m') R' = \frac{1}{\mu'} (\mu' - 1) R'$$
$$F'' = (1 - m'') R'' = (1 - \mu') R'',$$

because $m' = \frac{1}{\mu'}$ and $m'' = \mu'$. Consequently,

$$\mu' F' + F'' = (\mu' - 1) (R' - R'')$$

and similarly
$$\mu'' F''' + F^{iv} = (\mu'' - 1) (R''' - R^{iv}), \&c.$$

so that we get, finally,

$$F = (\mu' - 1) (R' - R'') + (\mu'' - 1) (R''' - R^{iv}) + \&c.$$

Now, the several terms of which this consists are (by Art. 261) the respective powers of the individual lenses of which the system consists, so that if we put (according to the same principle of notation) L', L'', L''', &c. for the powers of the single lenses, and L for their joint power as a system, we have

$$L = L' + L'' + L''' +, \&c. \qquad (q)$$

an equation which shows that the *power of any system of lenses is the sum of the powers of the individual lenses which compose it;* the word sum being taken in its algebraic sense, when any of the lenses has a negative power. Moreover it is easy to see that we also have $f = L + D$, as in the case of a single lens.

269. Reciprocally, we may regard a system of spherical surfaces forming the boundaries of contiguous media (as in the instance of a hollow lens of glass enclosing water) as consisting of distinct lenses, by imagining the concavity of one medium and the convexity of that in immediate contact with it separated by an infinitely thin film of vacuum, or of any medium having its surfaces equicurve, as in fig. 50; and thus a system of any number (n) of media, whose surfaces are in contact throughout their whole extent, may be conceived replaced by an equivalent system of $2n - 1$ lenses, the alternate ones being vacuum, or void of power. This way of considering the subject has often its use. It, moreover, leads to the result, that *the power of any system of spherical surfaces placed in vacuo is the sum of the powers of the several lenses into which it can be resolved, each placed in vacuo and acting alone.*

Fig. 50.

Power of a system of spherical surfaces expressed.

270. Let us now return to the case of surfaces separated by finite intervals; and, first, let us inquire the foci of a system of surfaces separated by intervals so small that their squares may be neglected. In this case the equations (j), Art. 251, become simply

Foci of a system of surfaces separated by small finite intervals.

$$D' = D; \qquad D'' = f' + f'^2 t'; \qquad D''' = f'' + f''^2 t'', \&c.;$$

and substituting these values in the equations (i), and retaining the notation of Art. 257, we find

$$\left.\begin{aligned} M f = M^{(n)} f^{(n)} = M' F' + M'' F'' + \ldots . M^{(n)} F^{(n)} + D \\ + M' f'^2 t' + M'' f''^2 t'' + \ldots . M^{(n-1)} f^{(n-1)2} t^{(n-1)} \end{aligned}\right\}.$$

Now in this we are to consider that

$$f' = F' + m' D, \; f'' = F'' + m'' F' + m' m'' D', \&c.$$

and the values of f', f'', &c. so expressed, being substituted in the foregoing equation, we find

$$M f = M' F' + M'' F'' + M''' F''' + \&c. \ldots + D \qquad (r)$$
$$+ M' (F' + m' D)^2 t' + M'' (F'' + m'' F' + m'' m' D)^2 t'' +, \&c.$$

271. *Corol.* In the case of two surfaces, supposing $M = 1$, or in the case of a single lens in vacuo, this gives

Case of a single lens of small but finite thickness.

$$f = (\mu - 1) (R' - R'') + D + \frac{1}{\mu} \{ (\mu - 1) R' + D \}^2 t. \qquad (s)$$

For parallel rays, this becomes

$$F = (\mu - 1) (R' - R'') + \frac{(\mu - 1)^2}{\mu} R'^2 . t; \qquad (t)$$

t being here put for t', the interval between the surfaces or *total thickness* of the lens.

272. *Problem. To determine the foci of a lens, whose thickness* t *is too considerable to allow of any of its powers being neglected.*

Focus of a lens of any thickness.

Here we must take the strict formulæ

$$D' = D; \qquad D'' = \frac{f}{1 - f' t}; \qquad f' = (1 - m') R' + m' D; \qquad \text{and } f'' = (1 - m'') R'' + m'' D''.$$

The latter equation gives, on substitution, and recollecting that $m' = \frac{1}{\mu} = m$ and $m'' = \mu$,

Light. Part I.

$$f = f'' = \frac{(\mu - 1)(R' - R'') + D + \frac{\mu - 1}{\mu}\{(\mu - 1)R' + D\}R'' t}{1 - \frac{1}{\mu}\{(\mu - 1)R' + D\}t}; \qquad (u)$$

and for parallel rays

$$F = \frac{\mu(\mu - 1)(R' - R'') + (\mu - 1)^2 . R' R'' t}{\mu - t . (\mu - 1) R'} \qquad (v)$$

273. Foci of a sphere. *Example* 1. *To determine the foci of a sphere.*

Here $R'' = -R' = -R$; $t = \frac{2}{R}$; and the equations (u) and (v) become

$$f = \frac{(2\mu - 2)R + (2 - \mu)D}{(2 - \mu)R - 2D} R; \qquad F = \frac{2\mu - 2}{2 - \mu} . R. \qquad (w)$$

274. *Corol.* 1. If $\mu = 2$, for instance, these values become

$$f = \frac{R^2}{D}; \qquad F = \infty .$$

In this case, then, since f and F express the proximities of the foci to the posterior surface of the sphere, we see that the focus for parallel rays falls on this surface, and that in any other case (as in fig. 51 and 52) q is given by the proportion $QC : CE :: EH : Hq$.

275. *Corol.* 2. Whatever be the value of μ, the focus for parallel rays after the second refraction bisects the distance between the posterior surface of the sphere, and the focus after the first refraction.

276. Foci of a hemisphere. *Example* 2. To determine the foci of a hemisphere, in the two cases; first, when the convex, secondly, when the plane surface receives the incident light.

In the first case, $R' = R$; $R'' = 0$; $t = \frac{1}{R}$: therefore we find

$$f = \frac{(\mu - 1)R + D}{R - D} . R; \qquad F = (\mu - 1)R.$$

277. In the other case, when the rays fall first on the plane side, $R' = 0$, $R'' = -R$, and $t = \frac{1}{R}$, so that

$$f = \frac{\mu(\mu - 1)R + D}{\mu R - D} R; \qquad F = (\mu - 1)R.$$

278. If the thickness of a spherical segment exposed with its convex side to the incident rays be to the radius as μ to $\mu - 1$, or if $t = \frac{\mu}{\mu - 1} \cdot \frac{1}{R} = \frac{1}{(1 - m)R}$, and $R'' = 0$, the expressions (u) and (v) become

$$f = -(\mu - 1) . \frac{R}{D}\{(\mu - 1)R + D\}: \qquad F = \infty$$

In this case the focus for parallel rays falls on the posterior surface of the segment.

279. Focus of any spherical segment, convex side first. Plane side first. In general, for any spherical segment, if exposed with its convex side to the rays, $R'' = 0$, and

$$f = \mu . \frac{(\mu - 1)R + D}{\mu + \{(\mu - 1)R + D\}t}; \qquad F = \frac{\mu(\mu - 1)R}{\mu + (\mu - 1)Rt} .$$

If the plane side be exposed to the rays

$$f = (\mu - 1)R + \frac{\mu D}{\mu - t . D}; \qquad F = (\mu - 1)R.$$

280. Focus of a spherical shell of equal curvatures. If $R' = R''$, or if the lens be a spherical lamina of equal curvatures, the one convex, the other concave,

$$f = \frac{\mu D + (\mu - 1)\{(\mu - 1)R + D\}Rt}{\mu - \{(\mu - 1)R + D\}t}; \qquad F = \frac{(\mu - 1)^2 R^2 t}{\mu - (\mu - 1)Rt} .$$

§ X. *Of the Aberration of a System of Spherical Surfaces.*

281. Focus of a small annulus of a spherical surface investigated.

Problem. To determine the focus of any annulus of a spherical refracting or reflecting surface.

The equations (*a*) of Art. 244, of the last section, in fact, contain a general solution of this problem; but the applications of practical Optics require an approximate solution for annuli of small diameter, or in which y is small compared with r. Conceiving y, then, so small that its fourth and higher powers may be neglected, the expressions in the article cited give

$$x = a - \sqrt{r^2 - y^2} = a - r + \frac{y^2}{2r}; \qquad a - x = r - \frac{y^2}{2r}$$

$$y\,Z = \mu r\,(a - r) + \frac{a\,(\mu^2 r - a)}{2\,\mu r\,(a - r)}\,y^2;$$

and substituting these in the value of $\overline{\mathrm{C}\,q}$, found in the same article, we get for the distance of the focus of refracted rays from the vertex

$$\overline{\mathrm{C}\,q} = \frac{\mu r\,(r - a)}{a - \mu a + \mu r} - \frac{\mu - 1}{2\,\mu} \cdot \frac{a^2\,(a + \mu r)}{(a - r)\,(a - \mu a + \mu r)^2} \cdot \frac{y^2}{r}. \qquad (a)$$

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In conformity, however, with the system of notation adopted in the last section, instead of expressing directly $\overline{\mathrm{C}\,q}$, we will take its reciprocal. As we have hitherto represented the value of this reciprocal for central rays by f, we will continue to do so; and for rays incident at the distance y from the vertex, we will represent the same reciprocal by $f + \Delta f$; Δf then will be that part of f due to the deviation of the point of incidence from the vertex. Now, neglecting y^4, we have

$$\frac{1}{\overline{\mathrm{C}\,q}} = \frac{a - \mu a + \mu r}{\mu r\,(r - a)} + \frac{\mu - 1}{2\,\mu^3} \cdot \frac{a^2\,(a + \mu r)}{r^3\,(a - r)^3}\,y^2 \qquad (b)$$

Now if we put, as we have hitherto done, $\mu = \frac{1}{m}$, $r = \frac{1}{\mathrm{R}}$, $a = \frac{1}{\mathrm{R}} - \frac{1}{\mathrm{D}}$, and substitute these in the above, we shall get the value of $\frac{1}{\mathrm{C}\,q}$, or of $f + \Delta f$, in terms of m, R, and D; and from this, subtracting the term independent of y^2, which is the value of f, we shall get Δf as follows,

$$\Delta f = \frac{m\,(1 - m)}{2}\,(\mathrm{R} - \mathrm{D})^2\,\{\,m\,\mathrm{R} - (1 + m)\,\mathrm{D}\,\}\,y^2. \qquad (c)$$

283. Longitudinal and lateral aberration defined.

Definition. The *longitudinal aberration*, is the distance between the focus for central rays and the focus q of the annulus, whose semidiameter, or *aperture*, is $y = \mathrm{M\,P}$.

The *lateral aberration* at the focus, is the deviation from the axis of the refracted ray, or the portion $f\,k$, intercepted by the extreme ray, of a perpendicular to the axis drawn through the central focus.

284. Relation between them and Δf.

Corol. These aberrations are readily found from the value of Δf above given; for since $\overline{\mathrm{C}\,q} = \frac{1}{f}$, we have $\Delta\,\overline{\mathrm{C}\,q}$ (= longitudinal aberration) $= \Delta\,\frac{1}{f} = -\,\frac{\Delta f}{f^2}$; or, calling ω this aberration,

$$\omega = -\,\frac{\Delta f}{f^2}; \qquad (d)$$

and since $\overline{\mathrm{C}\,q} : \overline{q\,k} :: y : \overline{f\,k}$, or $\frac{1}{f} : \omega :: y : \overline{f\,k}$,

we have $\overline{f\,k}$, or the

$$\text{lateral aberration} = f \,.\, y \,.\, \omega = -\,\frac{\Delta f}{f} \,.\, y; \qquad (e)$$

where $f = (1 - m)\,\mathrm{R} + m\,\mathrm{D}$.

Thus the whole theory of aberration is made to depend on the value of Δf, and we come therefore to consider the various cases of this which present themselves.

285. Case of parallel rays

Case 1. For parallel rays $\mathrm{D} = 0$; and, therefore,

$$\left.\begin{aligned} \Delta f &= \frac{m^2\,(1 - m)}{2}\,\mathrm{R}^3\,y^2; \qquad \omega = -\,\frac{m^2}{2\,(1 - m)}\,\mathrm{R}\,y^2 \\ \text{lateral aberration} &= -\,\frac{m^2}{2}\,\mathrm{R}^2\,y^3 \end{aligned}\right\} \qquad (f)$$

Light. Part I.

286. Case of reflectors.

Case 2. In reflectors, $m = \mu = -1$, and

$$\left.\begin{aligned} \Delta f &= R\,(R-D)^2\,y^2; \qquad \omega = -\frac{R\,(R-D)^2}{(2R-D)^2}\,y^2 \\ &\text{lateral aberration} = -\tfrac{1}{2}\,(R-D)^2\,y^3 \end{aligned}\right\}; \qquad (g)$$

which, for parallel rays, become

$$\Delta f = R^3\,y^2; \qquad \omega = -\tfrac{1}{4}\,R\,y^2; \qquad \text{lateral aberration} = -\tfrac{1}{2}\,R^2\,y^2. \qquad (h)$$

287. Aplanatic foci defined and investigated.

In the general case, if we put either $D = R$, or

$$m\,R - (1+m)\,D = o, \text{ which gives } D = \frac{m}{m+1}\,R; \qquad \frac{1}{D} = (\mu+1)\,.\,\frac{1}{R},$$

the value of Δf, and therefore of the aberration, vanishes. The former case is that of rays converging to the centre of curvature, in which, of course, they undergo no refraction. In the latter, the point is the same with that already determined, Art. 234. It is evident, from what was there demonstrated, that every spherical surface, C P, has two points Q, q in its axis, so related, that all rays converging to or diverging from one of them, shall after refraction *rigorously* converge to or diverge from the other. These points may be called the *aplanatic foci* of the surface; and, to distinguish them, Q may be called the aplanatic focus for incident, and q for refracted rays. To find them in any proposed case, in the axis of any proposed surface C, and *on the concave side* of the surface, take $CQ = (\mu+1) \times$ radius C E of the surface, and $Cq = \left(\frac{1}{\mu}+1\right) \times$ radius. Then will Q and q be the aplanatic foci required. In the case of reflexion, when $\mu = -1$, $CQ = Cq = 0$, and both the aplanatic foci coincide with the vertex of the reflector.

288. Aberration shortens the focus for parallel rays.

Let us next trace the effect of aberration in lengthening or shortening the focus, for all the varieties of position of the focus of incident rays; and, first, when $D = 0$, or for parallel rays, Δf is of the *same*, and therefore ω of the *contrary* sign with R, and therefore with F, which is equal to $(1-m)\,.\,R$. Hence it is evident, that the effect of aberration in this case must be to *shorten the focus of exterior rays*.

289. Effect of aberration in other cases. Fig. 54.

Q in this case is infinitely distant. As it approaches the surface, or as the rays from being parallel become more and more *convergent*, or divergent, the aberration diminishes; but the focus of exterior rays is still always *nearer* the surface than that of central, till Q comes up to the *aplanatic focus* A *for incident rays* on the *concave*, or to the focus F of parallel rays on the *convex side*. When Q is at the former of these points, the aberration is 0; at the latter, infinite.

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When Q is situated anywhere between these points, however, the reverse is the case, and the effect of aberration is to throw the focus for exterior rays farther from the surface than that for central ones. These results are easily deduced from the consideration of all the particular cases, and hold good for all varieties of curvature, and for refracting media of all kinds. In reflectors, the aplanatic foci coincide with the vertex. In these, the focus for exterior rays is shorter than for interior in every case, except when the radiant point is situated between the surface and the principal focus on the *concave* side of the reflecting surface; but between these points, longer.

291. Aberration of any system of spherical surfaces in contact.

Problem. To determine the aberrations of any system of spherical refracting surfaces placed close together.

Retaining the notation of Art. 257, let us suppose the ray, after passing through the first surface, to be incident on the second. Its aberration at this will arise from two distinct causes: first, that after traversing the first surface, instead of converging to or diverging from the focus for central rays, its direction was really to or from a point in the axis distant from that focus by the total aberration of the first surface; and, secondly, that being incident at a distance from the vertex of the second surface, a new aberration will be produced here, which (being, as well as the other, of small amount) the principles of the differential calculus allow us to regard as independent of it, and which being computed separately, and added to it, gives the whole aberration of the two surfaces regarded as a system. The same is true of the small alterations in the values of f', f'', &c. produced by the aberrations. If then we denote by $\delta f''$ the change in the value of f'', produced by the action of the first surface, and by $\delta' f''$, that arising immediately from the action of the second, and by $\Delta f''$, the total alteration produced by both causes, we shall have

$$\Delta f'' = \delta f'' + \delta' f''$$

Now, first, to investigate the partial alteration $\delta f''$ arising from the total alteration $\Delta f'$ in the value of f', or from the aberration of the first surface, we have

$$f'' = (1-m)\,.\,R'' + m''f', \text{ and therefore } \delta f'' = m''\,\Delta f',$$

since, in this case, $\qquad D' = D, \quad D'' = f', \quad D''' = f''$, &c.

Again, to discover the partial variation $\delta' f''$ in f'', arising immediately from the action of the second surface, we have, by the equation (c) at once, putting f' for D'', and neglecting y^4, &c.

$$\delta' f'' = \frac{m''(1-m'')}{2}\,(R''-f')^2\,\{m''R'' - (1+m'')f'\}\,y^2;$$

but we have, by the same equation, also

$$\delta f'' = m''\,\Delta f' = \frac{m''m'(1-m')}{2}\,(R'-D)^2\,\{m'R' - (1+m')D\}\,y^2.$$

Consequently, uniting the two, we have the value of Δf^{II}. Similarly, the value of Δf^{III} may be derived from that of Δf^{II}, by a process exactly the same, and which gives

$$\Delta f^{III} = m^{III} \Delta f^{II} + \frac{m^{III}(1 - m^{III})}{2} (R^{III} - f^{II})^2 \{ m^{III} R^{III} - (1 + m^{III}) f^{II} \} y^2,$$

and so on. Calling, then, as in Art. 257, M', M'', M'''. . . . $M^{(n)}$ the absolute refracting indices of the several media into which the successive refractions are made, and putting $M^{(n)} = M$, we shall have no difficulty in arriving at the following general expression, where Δf denotes the total effect of aberration on the value of f, the reciprocal focal distance of the system,

General expression for Δf.

$$M \cdot \Delta f = \left\{ \begin{array}{l} M' \cdot \frac{m'(1-m')}{2} (R' - D)^2 \{ m' R' - (1 + m') D \} \\ + M'' \cdot \frac{m''(1-m'')}{2} (R'' - f')^2 \{ m'' R'' - (1 + m'') f' \} \\ + M''' \cdot \frac{m'''(1-m''')}{2} (R''' - f'')^2 \{ m''' R''' - (1 + m''') f'' \} \\ + \text{ \&c.} \end{array} \right\} \cdot y^2; \qquad (i)$$

in which it will be recollected that

Successive values of f.

$$\left. \begin{array}{l} f' = (1 - m') R' + m' D \\ f'' = (1 - m'') R'' + m'' (1 - m') R' + m' m'' D \\ f''' = (1 - m''') R''' + m''' (1 - m'') R'' + m''' m'' (1 - m') R' + m''' m'' m' D \\ \text{\&c.} \end{array} \right\}; \qquad (j)$$

and these values being substituted give, if required, an explicit resulting value of Δf in terms of the radii and refractive indices, or their reciprocals, of the surfaces.

292. If the system of surfaces be placed in vacuo, or the last refraction be made into vacuum, $M = 1$, and the second member of the equation (i) exhibits simply the value of Δf. In all cases, the aberration ω is given as before by the equation

$$\omega = - \frac{\Delta f}{f^2}, \text{ and the lateral aberration is } - \frac{\Delta f}{f} y.$$

293. Aberration of a single infinitely thin lens.

To express the aberration of any infinitely thin lens in vacuo, let the terms of the general equation be denoted respectively by Q', Q'', &c., so as to make

$$M \cdot \Delta f = \{ Q' + Q'' + Q''' +, \text{ \&c.} \} y^2. \qquad (k)$$

Then, for the case of a single lens in vacuo, when $m'' = \frac{1}{m'}$, $M' = \frac{1}{m'}$, $M'' = 1$, $M = 1$, we have $\Delta f = Q' + Q''$; and putting, for a moment, $R' - D = B$, $R' - R'' = C$, we find

$$Q' = \frac{1 - m'}{2} y^2 B^2 \{ m' B - D \}$$

$$Q'' = - \frac{1 - m'}{2 m'^3} y^2 (m' B - C)^2 \{ m'^2 B - m' D - C \}$$

whence

$$Q' + Q'' = \frac{1 - m'}{2 m'^3} y^2 C \{ (2 m' B - C) (m'^2 B - m' D) + (C - m' B)^2 \}$$

The expression in brackets, putting for B and C their values, and $\frac{1}{\mu}$ for m', will become

$$\frac{1}{\mu^3} \{ ((2 - \mu) R' + \mu R'' - 2 D) (R' - (1 + \mu) D) + \mu ((\mu - 1) R' - \mu R'' + D)^2 \}.$$

If now we multiply out, arranging according to powers of D, and substitute the result, as also the value of m', $(= \frac{1}{\mu},)$ and of C, $(= R' - R'',)$ in $Q' + Q''$, or Δf; we get

$$\Delta f = (\mu - 1) (R' - R'') \cdot \frac{y^2}{2 \mu} \{ \alpha - \beta D + \gamma D^2 \}$$

General expression for it.

where

$$\left. \begin{array}{l} \alpha = (2 - 2 \mu^2 + \mu^3) R'^2 + (\mu + 2 \mu^2 - 2 \mu^3) R' R'' + \mu^3 R''^2 \\ \beta = (4 + 3 \mu - 3 \mu^2) R' + (\mu + 3 \mu^2) R'' \\ \gamma = 2 + 3 \mu \end{array} \right\}. \qquad (l)$$

Light. Now it has been shown, (Art. 261,) that $(\mu - 1)(R' - R'')$ expresses the power of the lens, so that, putting L for this, we have Part I.

$$\Delta f = \frac{L}{2\mu}(\alpha - \beta D + \gamma D^2)\, y^2. \qquad (m)$$

Such then is the general expression for Δf, the fundamental quantity, from which the aberration ω may be had in any lens by the equation $\omega = -\frac{\Delta f}{f^2}$.

294. Cases in which the aberration of a single lens can be made to vanish.

Corol. 1. The aberration of a lens vanishes when D is so related to R', R'' and μ, as to give

$$\alpha - \beta D + \gamma D^2 = 0; \qquad D = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma}. \qquad (n)$$

Now we find, by substitution and reduction,

$$\beta^2 - 4\alpha\gamma = \mu^2\{(R' + R'')^2 - (2\mu + 3\mu^2)(R' - R'')^2\}$$

and unless this quantity be positive, that is, unless

$$\left(\frac{R' + R''}{R' - R''}\right)^2 > 2\mu + 3\mu^2; \qquad (o)$$

the focus of incident rays cannot be so situated as to render the aberration nothing. But, if the curvatures R' and R'' of the surfaces be such as to satisfy this condition, the value of D may be calculated at once from the equation (*k*.)

295. *Corol.* 2. Whenever, in meniscus or concavo-convex lenses, the difference of the curvatures of the surfaces is small in comparison with their sum, that is, whenever a moderate focal length is produced by great curvatures, the aberration admits of being rendered evanescent by properly placing the focus of incident rays. In a lens of crown glass where $\mu = 1.52$, we have $\sqrt{2\mu + 3\mu^2} = 3.16$; therefore the sum of the curvatures must be at least 3.16 times their difference, to satisfy the condition of possibility. In double-convex or double-concave lenses, R' and R'' having opposite signs, the condition can never be satisfied.

296. No known medium can render the aberration nothing for parallel rays

Corol. 3. If $\alpha = 0$, the aberration vanishes for parallel rays. This condition is, however, only to be satisfied by real values of R' and R'' when μ is equal to or less than $\frac{1}{4}$, and no such media are known to exist.

Corol. 4. The effect of aberration will be to shorten or lengthen the focus for exterior rays, according as the sign of Δf is the same as, or the opposite to, that of f. In particular cases it will, of course, however, depend on the values of μ, R, R', and D which shall take place. The principal case is that of parallel rays, in which $D = 0$, and

297. Cases in which the aberration shortens or lengthens the focus.

$$\Delta f = \frac{y^2}{2\mu} \cdot L\{(2 - 2\mu^2 + \mu^3)R'^2 + (\mu + 2\mu^2 - 2\mu^3)R'R'' + \mu^3 R''^2\}$$

and the focus of external rays will be shorter or longer than that of central ones, according as this quantity has the *same*, or *opposite* sign with L, that is, according as

$$(2 - 2\mu^2 + \mu^3)R'^2 + (\mu + 2\mu^2 - 2\mu^3)R'R'' + \mu^3 R''^2$$

is positive or negative. Now, from what we have already seen in the last corollary, this quantity never can be rendered negative by any real values of R' and R'', unless μ be less than $\frac{1}{4}$. For all other media, therefore, (comprehending all yet known to exist in nature,) every lens, whatever be the curvatures of its surfaces, has the exterior focal length for parallel rays shorter than the central.

298. Case of a glass meniscus.

Corol. 5. In a glass *meniscus*, when the radiant point is on the *convex* side, and the rays *diverge*, we have $4 + 3\mu - 3\mu^2$ a positive quantity; and, R' and R'' being both positive, β is so; hence (D being negative in this case) the term $-\beta D$, and therefore the whole factor $\alpha - \beta D + \gamma D^2$ is positive; and L being also positive, Δf is so; and, therefore, ω, the aberration, negative. Hence, when Q is beyond F, the focus for parallel rays incident the other way, the exterior focus is the shorter; but when between F and C, the longer.

299. Rule, for a large class of lenses, to find the effect of aberration in lengthening or shortening the focus.

Corol. 6. Unless $\left(\frac{R' + R''}{R' - R''}\right)^2 > 2\mu + 3\mu^2$, no real value of D can render $\alpha - \beta D + \gamma D^2$ negative.

It appears, therefore, that in all double-convex or concave lenses, as well as in all meniscus and concavo-convex ones, in which the sum of the curvatures of the surfaces is greater than $\sqrt{2\mu + 3\mu^2}$ times their difference, the factor $\alpha - \beta D + \gamma D^2$ is positive for all values of D, and therefore the aberration ω has in all such lenses the sign *opposite* to that of L. Hence, for all such lenses, we have the following simple and general rule: *the effect of aberration will be to throw the focus of exterior rays more* TOWARDS *the incident light than that of central ones, when the lens is of a positive character, or makes parallel rays* CONVERGE, *but more* FROM *the incident light if of a negative, or if it cause parallel rays to* DIVERGE.

300. *Corol.* 7. All other lenses have, as in the case of single surfaces, aplanatic foci, corresponding to the roots of the equation $\alpha - \beta D + \gamma D^2 = 0$. In general there are two such foci of incident and two of refracted rays: and

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rules might easily be laid down for determining in what positions of the radiant point, with respect to these foci and the lens, the aberration tends to shorten or lengthen the exterior focus; but it is simpler and readier to have recourse at once to the algebraic expressions.

How to proceed in other cases.

Corol. 8. In the case of reflexion, as when rays are reflected between the surfaces of thin lenses of transparent media, we have $m = m'' = \&c. = \mu' = \mu'' = \&c. = -1$; $M' = -1$, $M'' = +1$, &c., and $M = \pm 1$, according as the number of reflexions is even or odd; therefore for n reflexions we have

301. Case of reflexion between any system of transparent surfaces.

$$\left.\begin{aligned} f' &= 2\,R' - D \\ f'' &= 2\,R'' - 2\,R' + D \\ f''' &= 2\,R''' - 2\,R'' + 2\,R' - D \\ &\&c. \end{aligned}\right\}; \qquad (p)$$

and

$$\Delta f = (-1)^{n+1}\left\{\begin{aligned} &R'(R' - D)^2 \\ &- R''(R'' - 2\,R' + D)^2 \\ &+ R'''(R''' - 2\,R'' + 2\,R' - D)^2 \\ &- \&c. \end{aligned}\right\} y^2; \qquad (q)$$

which formulæ serve to determine, in all cases of internal reflexion between spherical surfaces, both the places of the successive foci and the aberrations.

302. *Corol.* 9. If the reflexions take place between equicurve surfaces, having their concavities turned opposite ways, f', f'', &c. are in arithmetical, and therefore their reciprocals, or the focal distances, in harmonic progression.

303. General construction of an aplanatic lens.

Problem. To construct an aplanatic lens, or one which shall refract all rays, for a given refractive index, and converging to or diverging from any one given point, to or from any other.

Let Q and q be the points, the former being the focus of incident, the latter of refracted rays. Let μ = index of refraction; and putting $Q\,q = 2\,c$, and assuming b any arbitrary quantity, construct the curve whose equation is (n), Art. 232. Let H P C, (fig. 36,) be this curve; and with centre q, and any radius $q\,N$ less than $q\,P$, any one of the refracted rays describe the circle H N K. Then since the ray Q P, by the nature of the curve H P C, is after refraction directed to or from q, and, being incident perpendicularly on the second surface, suffers there no flexure, it will, if supposed to emerge from the medium, here continue its course to or from q. If then we suppose the figure C P H N K to revolve round $Q\,q$, it will generate a solid, which, being composed of the proposed medium, is the lens required. If the rays be parallel, as in fig. 38, the curve H P C, as we have seen, is a conic section, which, if the lens be denser than the ambient medium, is an ellipse. Thus, a glass meniscus lens, whose anterior convex surface is elliptic, and posterior spherical, having its centre in the focus of rays refracted by the first surface, is *aplanatic*.

Fig. 36.

Fig. 38.

304. Case when the surfaces of an aplanatic lens are all spherical.

But, without having recourse to the conic sections, the same thing may, in certain cases, be accomplished by spherical surfaces only. For if Q and q (fig. 53) be the *aplanatic foci* of the spherical refracting surface, and if with the centre q and any radius greater than $q\,C$, when the incident rays diverge from Q, as in the lower portion of the figure, but less if they converge to Q as in the upper, we describe a circle K L, or $k\,l$, and turn the whole figure about $Q\,q$ as an axis, the surfaces C P K L, or $c\,p\,k\,l$, will generate the aplanatic lens in question. This also follows evidently from the general formula, (i, Art. 291,) for if $R'' = f'$, the expression of Δf for the lens becomes simply

Fig. 53.

$$\frac{1 - m'}{2}(R' - D)^2\,\{m'\,R' - (1 + m')\,D\}\,y^2,$$

which vanishes when $D = \dfrac{m'}{1 + m'}\,R'$, or when Q is the aplanatic focus of incident rays for the first surface.

More generally, however, the equation $\alpha - \beta\,D + \gamma\,D^2 = 0$, assigns the universal relation between μ, D, R′, R″, which constitutes the lens aplanatic. See *Cor.* 1, Art. 294.

305. Most advantageous form for a single lens for parallel rays determined.

Problem. To assign the most advantageous form for a single lens, or that which, with a given power, has the least possible aberration for parallel rays.

Since the aberration cannot be rigorously made to vanish for parallel rays, when $\mu > \frac{1}{4}$ (Art. 296) we have to make it a minimum. Now $\omega = -\dfrac{\Delta f}{f^2} = -\dfrac{\Delta f}{L^2}$ for parallel rays, or

$$\omega = -\frac{y^2}{2\,\mu}\cdot\frac{a}{L^2}; \text{ and, in general, } d\,\omega = -\frac{y^2}{2\,\mu L^2}\{L\,d\,a - a\,d\,L\}$$

In the present case L is given, therefore we must put $d\,a = 0$, which gives

$$0 = 2\,(2 - 2\,\mu^2 + \mu^3)\,R'\,d\,R' + (\mu + 2\,\mu^2 - 2\,\mu^3)\,(R'\,d\,R'' + R''\,d\,R') + 2\,\mu^3\,R''\,d\,R''.$$

But the condition $d\,L = 0$ gives $d\,R' = d\,R''$; so that our equation becomes, on substitution and reduction,

$$0 = (4 + \mu - 2\,\mu^2)\,R' + (\mu + 2\,\mu^2)\,R'';$$

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that is to say

$$\frac{R''}{R'} = \frac{2\mu^2 - \mu - 4}{2\mu^2 + \mu}. \qquad (r)$$

In the case of a glass lens, taking $\mu = 1.5$, this fraction becomes equal to $-\frac{1}{6}$, which shows that the lens must be double-convex, having the curvature of the posterior surface only $\frac{1}{6}$ that of the anterior, or its radius six times as great. Artists sometimes call such a lens a "*crossed lens.*"

306. Case in which this form is plano-convex.

Corol. 1. If $\mu = 1.6861$, as is nearly the case with several of the precious stones and the more refractive glasses, $R'' = 0$; and the most advantageous figure for collecting all the light in one place is plano-convex, having its convex side turned to the incident rays.

307. Aberrations of various species of lenses determined for parallel rays

Corol. 2. Calling the aberration of a lens of the best figure ω, we shall have $\omega = -\frac{15}{14} y^2 . L$, for glass whose refractive index is 1.5, and the proportional aberrations of other forms will be as follows:

Plano-convex, plane side first (or towards the light)	$4.2 \times \omega$
Plano-convex, curved surface first	$1.081 \times \omega$
Double equi-convex, or concave	$1.567 \times \omega$

308. Aberration of a system of lenses.

Problem, To investigate a general expression for the aberration of any system of infinitely thin lenses placed close together in vacuo.

The general expression for $M \Delta f$, or, since $M = 1$ in the case before us, of Δf, is

$$(Q' + Q'' + Q''' + Q^{iv} +, \&c.)\, y^2,$$

which divides itself into terms originating with the successive lenses in the following manner,

$$\Delta f = (Q' + Q'')\, y^2 + (Q''' + Q^{iv})\, y^2 +, \&c.$$

The first of these quantities we have already considered; let us now, therefore, examine the constitution of the rest. Let then μ' be the refractive index of the first lens, μ'' of the second, μ''' of the third; and let α', β', γ' represent the values of α, β, γ for the first lens, or the expressions in (l, 292,) writing only μ' for μ; also let α'', β'', γ'' represent their values for the second lens, or what the same expressions become when μ'' is put for μ, and R''' and R^{iv} respectively for R' and R'', and so on for the rest of the lenses.

309. Now if we consider the values of Q''' and Q^{iv}, it will be seen that they are composed of the quantities m''', m^{iv}, M''', M^{iv}, R''', R^{iv}, f'' and f''', precisely in the same manner that Q' and Q'' are of m', m'', M', M'', R', R'', D and f'.

Moreover, since by Art. 251 we have

$$\begin{aligned}
f' &= (1 - m')\, R' + m'\, D \\
f'' &= (1 - m')\, R'' + m'\, f' \\
&= (1 - m')\, R'' + m''\,(1 - m')\, R' + m''\, m'\, D \\
&= (\mu - 1)\,(R' - R'') + D, \text{ since } m' = \frac{1}{\mu},\ m'' = \mu. \\
&= L + D; \text{ call this } D''; \text{ (L is the power of the first lens)} \\
f''' &= (1 - m''')\, R''' + m'''\, D'' \\
f^{iv} &= (1 - m^{iv})\, R^{iv} + m^{iv} f''' = L'' + D'' \text{ as before; (L'' is the power of the second lens)} \\
&= L + L' + D; \text{ and so on.}
\end{aligned}$$

And it is clear that $Q''' + Q^{iv}$ will be the same *function* of, *i. e.* similarly composed of, the refractive index and curvatures of the surfaces of the second lens, and of the quantities D'' and f''', that $Q' + Q''$ is of the refractive index and curvatures of the first lens, and of D and f'. It follows, therefore, that the very same system of reductions which led to the equation

$$Q' + Q'' = \frac{L}{2\mu}(\alpha - \beta D + \gamma D^2)$$

being pursued in the case of $Q''' + Q^{iv}$, must lead to the precisely similar equation

$$Q''' + Q^{iv} = \frac{L''}{2\mu''}(\alpha'' - \beta'' D'' + \gamma'' D''^2)$$

General expression for it.

and so on for the remaining lenses; so that we shall have, ultimately, for the whole system (writing L', D', μ' for L, D, μ)

$$\Delta f = \frac{y^2}{2}\left\{\frac{L'}{\mu'}(\alpha' - \beta' D' + \gamma' D'^2) + \frac{L''}{\mu''}(\alpha'' - \beta'' D'' + \gamma'' D''^2) + \&c.\right\}; \qquad (s)$$

in which there are as many terms as lenses.

Light. Part I.

Corol. For parallel rays, $D' = 0$; $D'' = L'$; $D''' = L' + L''$, &c. therefore

310. Case of parallel rays.

$$\Delta f = \frac{y^2}{2}\left\{\begin{array}{l} \dfrac{L'}{\mu'}\,a' + \dfrac{L''}{\mu''}\,(a'' - \beta''\,L' + \gamma''\,L'^2) \\ \qquad + \dfrac{L'''}{\mu'''}\,(a''' - \beta'''\,(L' + L'') + \gamma'''\,(L' + L'')^2) \\ \qquad + \text{ \&c.} \end{array}\right\}. \qquad (t)$$

311. Although the aberration of a single lens for parallel rays admits of being destroyed only on a certain hypothesis of the refractive index, which has no place in nature, yet, by combining two or more lenses, it may be destroyed in a variety of ways. Thus, in the case of two lenses, the expression (t) being put equal to zero, gives an equation involving μ', μ'', L', L'', R', R'', R''', R^{iv}; or (since L' and L'' are given in terms of μ, μ' and R', R'', &c. and since μ', μ'' are given quantities) only the four unknown quantities R', R'', R''', R^{iv}. Now as there are four of these, and only one equation, it may be satisfied in an infinite variety of ways, and the problem of the destruction of the *spherical aberration* (as it is termed) becomes indeterminate.

312. General equation for the destruction of aberration in a double lens for parallel rays.

The equation in the case of two lenses for parallel rays is

$$\begin{aligned} 0 = {} & \frac{L'}{\mu'}\left\{(2 - 2\,\mu'^2 + \mu'^3)\,R'^2 + (\mu' + 2\,\mu'^2 - 2\,\mu'^3)\,R'\,R'' + \mu'^3\,R''^2\right\}; \qquad (u) \\ & + \frac{L''}{\mu''}\left\{(2 - 2\,\mu''^2 + \mu''^3)\,R'''^2 + (\mu'' + 2\,\mu''^2 - 2\,\mu'''^3)\,R'''\,R^{iv} + \mu''^3\,R^{iv2}\right\} \\ & - \frac{L'\,L''}{\mu''}\left\{(4 + 3\,\mu'' - 3\,\mu''^2)\,R''' + (\mu'' + 3\,\mu''^2)\,R^{iv}\right\} + \frac{L'^2\,L''}{\mu''}\left\{2 + 3\,\mu''\right\} \end{aligned}$$

313. Another form of the same equation.

This equation, if L' and L'', the powers of the separate lenses, be assigned, is of a quadratic form in either R', R'', R''', or R^{iv}; it will therefore depend on the supposition adopted to limit the problem, whether these quantities admit real corresponding values. Now the equations $L' = (\mu' - 1)\,(R' - R'')$ and $L'' = (\mu'' - 1)\,(R''' - R^{iv})$ afford the means of eliminating two of them, and the resulting equation (in R' and R''' for instance) is

$$\begin{aligned} 0 = {} & L'\left\{\frac{2 + \mu'}{\mu'}\,R'^2 - \frac{2\,\mu' + 1}{\mu' - 1}\,L'\,R'\right\} \qquad (v) \\ & + L''\left\{\frac{2 + \mu''}{\mu''}\,R'''^2 - \left(\frac{4\,(\mu'' + 1)}{\mu''}\,L' + \frac{2\,\mu'' + 1}{\mu'' - 1}\,L''\right)R'''\right\} \\ & + \frac{\mu'^2\,L'^3}{(\mu' - 1)^2} + \frac{\mu''^2\,L''^3}{(\mu'' - 1)^2} + \frac{3\,\mu'' + 1}{\mu'' - 1}\,L'\,L''^2 + \frac{2 + 3\,\mu'}{\mu''}\,L'^2\,L''; \end{aligned}$$

and, as the unknown quantities R', R''' are not combined by multiplication, the equation when L' and L'' are given is of an ordinary quadratic form with respect to each. This equation will be of use to us hereafter, when we come to treat of the theory of refracting telescopes.

314. If L' and L'' be not given, since either of them is of the first degree in terms of R', R'', &c., the equation (u) is of the third degree in either of the quantities R', R'', &c., or in L', L'', if either R'' or R^{iv} be eliminated. Now as an equation of the third degree must necessarily have at least one real root, we conclude, first, that *in a double lens, if the curvatures of three of the surfaces be given, that of the fourth may be found, so as to destroy the spherical aberration.*

315. Secondly. *That if the curvature of one surface of each lens, and the power of either, or that of the two combined, be given, the power of the other may be found so as to destroy the spherical aberration.* This is evident; for, supposing R' and R''' given, and either L' or L'', or $L' + L''$, also given, the equation (v) becomes an ordinary cubic in which L' or L'', as the case may be, is the only unknown quantity, and therefore necessarily admits a real value.

316 As examples of aplanatic combinations, we may set down the following cases, in which a lens of glass of the refraction 1.50, and of the best form, having the radii of its surfaces respectively + 5.833 and — 35.000 inches, and its focal length 10.000 inches, has its aberration corrected by applying behind it another lens of similar glass, as in fig. 55. This lens is a meniscus. If its curvatures be determined by the condition of giving the maximum of power to the combination, the radii of its surfaces and its focal length will be as follows: radius of first surface, = + 2.054 inches; radius of second surface, = + 8.128; focal length of correcting lens, = + 5.497; focal length of the two combined, = + 3.474. On the other hand, if we determine the second lens by the condition of the resulting combination, having a focal length as nearly 10.000 as is consistent with perfect aplanaticity, we shall find radius of first surface, = + 3.688; radius of second, = + 6.291; focal length of correcting lens, = + 17.829; focal length of the combination, = + 6.407. Fig. 55.

317. The effect of aberration may be very prettily exhibited by covering a large convex lens with a paper

Light. screen full of small round holes, regularly disposed, and, exposing it to the sun, receiving the converged rays on a white paper behind the lens, which should be first placed very near it, and then gradually withdrawn. The pencils which pass through the holes will form spots on the screen, and their disposition will become more and more unequal over the surface, as the screen is further removed; those at the circumference becoming crowded together before the central ones. The manner in which the several spots corresponding to central rays blend together into one image at the focus, and those formed by the exterior ones are scattered round it, gives us a very good idea of the variation of density of the rays in the circle of aberration at or near the principal focus; and if the white screen be waved rapidly to and fro in the cone of rays, so as to pass over the focus at each oscillation, the whole cone will be seen as a solid figure in the air, and the place of the circle of least aberration will become evident to the eye, forming altogether a very pleasing and instructive experiment. Part I.

§ XI. *Of the Foci for Oblique Rays, and of the Formation of Images.*

318. We have hitherto considered rays as converging to, or diverging from, a single point; but as this is not the case with luminous bodies of a sensible diameter, we now proceed to examine the cases of refraction at spherical surfaces, where more than one radiant point is concerned, or where several pencils are incident at once on the surface. We shall take for our positive, or fundamental case, as we have done all along, that of *converging* rays incident on the convex side of a more refractive medium than the ambient one, and derive all others from it by the changes in the sign and relative magnitudes of R, D, &c.

Foci of oblique pencils. In fig. 56, then, let Q and Q′ be the foci of two pencils of convergent rays incident on the spherical surface C C′, whose centre is E. Draw Q E C, Q′ E C′, cutting the surface in C and C′, and, regarding C E Q as the axis of the pencil R Q, S Q, T Q, the focus of refracted rays will be found by taking q, such as that $\frac{1}{C\,q}$, or f, shall be equal to $(1-m)\,R + m\,D$, (247, *e.*) Similarly, regarding C′ E Q′ as the axis of the pencil converging to Q′, the focus q' will be had by the equation

$$\frac{1}{C'\,q'} = f' = (1-m)\,R + m\,D'.$$

Thus when C′ Q′ = C Q, C q' will also equal C q, and, in general, when the locus of the point Q is given, that of q may be found.

319. Images in Optics defined. *Definition.* The image of an object, in Optics, is the locus of the focus of a pencil of rays diverging from, or converging to, every point of it, and received on a refracting surface. Thus, supposing C Q′ to be a line, or surface, every point of which may be regarded as a focus of incident rays, $q\,q'$ is its image.

320. Form of the image of a straight line *Problem. To find the form of the image of a straight line formed by a spherical refracting or reflecting surface.*

Put C E = r; C Q = a; E M = x; M q' = y; E q' = $\sqrt{x^2+y^2}$; C′ Q′ = a'.

Then we have

$$\frac{1}{C'\,q'} = \frac{1-m}{r} + \frac{m}{a'} = \frac{(1-m)\,a' + m\,r}{r\,a'};$$

and therefore

$$C'\,q' = \frac{r\,a'}{(1-m)\,a' + m\,r}; \qquad E\,q' = \frac{m\,r\,(a'-r)}{(1-m)\,a' + m\,r};$$

we have, consequently,

$$x^2 + y^2 = \frac{m^2 r^2 . (a'-r)^2}{\{(1-m)\,a' + m\,r\}^2}.$$

But, by similar triangles, E q' : E M :: E Q′ : E Q, or

$$x^2 + y^2 = \frac{(a'-r)^2 . x^2}{a^2},$$

equating these two values we get

$$\frac{a}{x} = \frac{(1-m)\,a' + m\,r}{m\,r}; \qquad a' = \frac{m}{1-m} \cdot \frac{r\,(a-x)}{x};$$

Is a conic section. so that eliminating a', by substituting this value for it, we get for a final equation between x and y, or for the equation of the image

$$(1-m)^2\,(x^2+y^2) = \left(\frac{r}{a}\right)^2 . (m\,a - x)^2$$

which belongs to a conic section.

321. *Problem. When an oblique pencil is incident on any system of spherical surfaces, to find the focus of refracted rays.*

Light. Part I. Foci of oblique pencils incident on a system of spherical surfaces. Fig. 57.

Take E′, (fig. 57,) the centre of the first surface, and let Q′ be the focus of incident rays. Join Q′ E′ and produce it to C′, then will C′ be the vertex of the surface corresponding to the pencil whose focus is Q′; and taking

$$\frac{1}{C'\,Q''} = \frac{1 - m'}{C'\,E'} + \frac{m'}{C'\,Q'}$$

Q″ will be the focus of refracted rays. Again, join Q″ and E″, the centre of the second surface, produce to C″, and take

$$\frac{1}{C''\,Q'''} = \frac{1 - m''}{C''\,E''} + \frac{m''}{C''\,Q''}$$

and Q‴ will be the focus after refraction at the second surface, and so on.

322. *Corol.* In the case of an infinitely thin lens, when the obliquity is small, it is evident, from this construction, that the focus of oblique rays will lie at the same distance from the lens with that of rays convergent to, or divergent from, a point in the axis at the same distance with the focus of incident rays, but instead of lying in the axis, will deviate from it.

323. *Definition.* The *centre* of a lens is a point in its axis where a line joining the extremities of two parallel radii of its two surfaces cuts the axis. Thus, in the various lenses represented in fig. 58, 59, 60, and 61, E′ A and E″ B being two parallel radii; join B A, and produce, if necessary, till it meets the axis in X, and X is the centre. Centre of a lens.

324. *Corol.* 1. The centre is a fixed point; for, since A E′ and B E″ are parallel, we have E′ X : E′ E″ : : A E′ : B E″ − A E′, in which proportion three terms being invariable, the other is so also.

325. *Corol.* 2. If C′ C″, the interval of the surfaces or thickness of the lens, be put equal to t (i being always positive) and the curvatures be respectively R′ and R″, we have, for the distance of the centre from the first surface, or for C′ X, the following value,

$$C'\,X = \frac{R''}{R' - R''}\,.\,t.$$

326. *Corol.* 3. If a ray be so incident on a lens that its direction after the first refraction shall pass through its centre, it will suffer no deviation. This is evident, because its course within the lens will be A B, and the radii E′ A and E″ B being parallel, the internal angles of incidence on the surfaces are equal, and, therefore, the angles of refraction both ways out of the lens; consequently the two portions of the ray without the lens are parallel Rays through the centre pass undeviated.

327. *Corol* 4 If the thickness of a lens be very small, the ray passing through its centre may be regarded as undergoing no refraction whatever; for the portion A B within the lens being very small, the two portions exterior to the lens (being parallel) may be regarded as one ray. This is, *à fortiori*, still nearer the truth when the obliquity of the ray to the axis is small; because then the portion A B is very nearly coincident in direction with either of the two exterior portions.

328. *Corol.* 5. Hence, to find the focus of refracted rays in the case of a *very thin lens* and for a *pencil of small obliquity*, take X, the centre of the lens, and the focus will lie in the line Q X, at the same distance from the lens as if the axis of the incident pencil were coincident with that of the lens. Focus of a slightly oblique pencil through a thin lens.

Proposition. When a luminary, or illuminated object, is placed before a double or plano-convex, or meniscus lens, at a distance from it greater than its focal length, there will be formed behind the lens an image, similar to the object, but inverted; and the object and image subtend the same angle at the centre of the lens.

329. For the pencil of rays which emanates (either by direct radiation or by reflexion) from any point, as P, of the object, will after refraction be all made to converge to a point p behind the lens, or at least very nearly so. Were the aberration of the lens evanescent, the convergence would be mathematically exact; and since, whenever the aperture of the lens and the obliquity of the pencil are small, the aberration is so very minute, that the space over which the rays are spread may be regarded as a physical point, and every physical point in the object will have a corresponding point in the image. Now, C being the centre of the lens, the line joining P p passes through C; and the same being true of the line joining any other corresponding points of the object and image, it follows, by similar triangles, that the object and image are similar in figure; and as the rays cross at C, the image is inverted, and subtends the same angle p C q at C that the object does on the other side. Fig. 62. An inverted image of an object is formed behind a convex lens.

330. If a screen of white paper be placed at $q\,p$, this image will be rendered visible as a picture of the object. The experiment may be tried with any magnifier or spectacle-glass at a window, when the forms of external objects, the houses, trees, landscape, &c. will be painted on the paper screen with perfect fidelity, forming a miniature of the utmost delicacy and beauty. This is the principle of the common camera obscura, in which the rays from external objects are thrown by an inclined looking-glass downwards, and being received on a convex lens, are brought to their focus on a white horizontal table, in a room where no other light is admitted. On this table a moving picture of all external objects, in their proper forms, colours, and motions, is seen, infinitely more correct and beautiful than the most elaborate painting. See fig. 63, in which P is the object, A B the reflector, B C the lens, and p the image on the table D. Camera obscura explained.

331. If the rays, instead of being received on white paper, be received on a plate of glass emeried on one side, the picture may be seen by an eye placed at the other side of the glass, as well as by one in front of it; for it is a property of such roughened transparent surfaces to scatter the rays which fall on them, not only by reflexion outwards, but by refraction inwards. If the surface be but slightly roughened, however, the picture will appear much less vivid when looked at obliquely than when the eye is placed immediately behind it; and in this

Light. latter situation the emeried glass may even be removed altogether, and the image will still be seen, and even more distinctly, as if a real object stood in the place in all respects similar to the picture. Part I.

332. We may examine the image on the roughened glass with a magnifying glass, or microscope. It will then appear as a delicate painting, accommodating itself to all the inequalities of the surface. But if, in the act of so examining it, the rough glass be removed, the painting remains as if suspended in air, and the objects it represents are seen brought nearer to the eye, and enlarged in their dimensions. In short, we have formed a telescope.

333. If the lens used to form the image be a concave one, or if a convex reflector be used, as in fig. 64 and 65, the rays, after refraction or reflexion, diverge, not from any actual points in which they cross, but from points in which they would cross if produced backwards. There is in this case, then, no real image formed capable of being received on a screen, but what is called *a virtual one*, visible to the eye if properly situated, either unassisted or aided by a magnifier, and situated on the same side of the lens, or on the contrary side of the reflector with the object, and therefore *erect*.

334. The perfection of the image formed by a lens or reflector, its exact resemblance to the object, and the distinctness of its parts, will depend on the exact convergence of all the rays of pencils emanating from every physical point of the object in strict mathematical points, or in as near an approach to such points as may be. If, therefore, a lens of considerable diameter be used, especially if the curvatures of its surfaces be improperly chosen so as to produce much aberration, the image will be confused; for each point of the object will form, not a point, but a small circular spot in the image, over which the rays are diffused; and as these spots overlap and encroach on each other, distinctness is destroyed. For the formation, therefore, of perfect images, the destruction of aberration is the essential condition; and whatever imperfections, either in the figures of the reflecting or refracting surfaces used, or in the materials of which they are composed, tends to throw the rays aside from their strict geometrical direction, must, of course, confound the images. Hence, in the formation of optical images, there are three great points to be attended to: first, perfect polish of the surfaces; secondly, perfect homogeneity in the material employed; thirdly, strict conformity in the figures of the reflecting and refracting surfaces to geometrical rules, and the results of analysis.

335. There is one case where the aberrations of all kinds are rigorously destroyed, and in which the image is perfect. It is when the rays are reflected at a plane surface. For (fig. 66) if P Q be an object placed before a plane reflector A B, and if perpendiculars be let fall from every point of the object to the surface, and on the other side points in these be taken at the same distances respectively behind the surface as p, q, these points will form the image. Now we have seen, that all rays from any point P, reflected at A B, will after reflexion diverge strictly from p its image. Thus, the image is as perfect and free from aberration as the object; and will appear, to an eye placed so as to receive the rays, like a real object placed behind the reflector.

336. *Corol.* The image formed by a plane reflecting surface is similar and equal to the object, and any corresponding lines in both are equally inclined to the reflecting surface. A common looking-glass is the best illustration of this case.

337. *Proposition.* To determine the image of any object formed by a plane refracting surface. Let B C be the surface, P Q the object. From any point Q draw Q C perpendicular to the surface, and, μ being the index of refraction, if we regard the surface as a sphere of infinite radius, we have R its curvature $= 0$, and the equation

$f = (1 - m)\,\mathrm{R} + m\,\mathrm{D}$ becomes simply $f = m\,\mathrm{D}$. Now $f = \frac{1}{\mathrm{C}\,q}$; $\mathrm{D} = \frac{1}{\mathrm{C\,Q}}$; and $m = \frac{1}{\mu}$. Hence this equation, translated into geometrical language, gives $\mathrm{C}\,q = \mu \times \mathrm{C\,Q}$.

338. In the case represented in the figure, the refraction is made out of a denser medium into a rarer, the object being immersed in the denser (as under water), and the eye of a spectator in the rarer (as in air): the image q of the point Q is therefore nearer the surface than Q, (because in this case μ is less than unity.) The same holds good of all other points of the image; so that the whole object will appear raised by refraction, as in the familiar experiment where a shilling is laid in an empty vessel, and the eye withdrawn till the shilling is hidden by the edge, but reappears again, as if raised up, when the vessel is filled with water. On the other hand, to an eye placed under water, external objects would appear farther removed by the effect of refraction.

339. *Corol.* 1. The image of a straight line P Q in the object is a straight line $p\,q$ in the image, less inclined to the surface if the refraction be made from a denser into a rarer medium. Thus, if a stick D A P Q be partly plunged into water, the immersed portion A Q forms the image A q less inclined; so that to a spectator in air, the stick appears broken and bent upwards at A. The appearance is familiar to every one.

340. In refraction at a plane surface, however, the rays do not rigorously diverge from, or converge to, a single point. Therefore the above result is only approximately correct, and supposes the rays to be incident nearly at right angles to the surface. And this leads us to the consideration of oblique vision through refracting surfaces, or in reflectors of any figure.

341. Oblique vision through refracting or reflecting surfaces of any figure. Fig. 68.

The eye sees by the rays which enter it, and judges of the existence of an object, by the fact of rays diverging sensibly from some point in space. If, then, rays diverge rigorously from a point, the eye which receives them is irresistibly led to the belief (unless corrected by experience and judgment) of an object being there; the illusion is complete, and vision perfect. But if such divergence be only approximate, as when the density of the rays which reach the eye in any one direction is very much greater than in directions adjacent on either side, vision is still produced, only less distinct, in proportion to the degree of deviation from strict mathematical divergence of the rays which produce it. Suppose, now, Q to be a radiant point placed anywhere with respect to the refracting or reflecting surface A C B, (fig. 68,) and let A q F B be the caustic formed by the intersection of all the refracted or reflected rays. Let us suppose an eye placed at E, and from thence draw E q a tangent

Light. to the caustic, which continue to the surface C, and join Q C. Then it is obvious, that any small pencil Q C, Q C' diverging from Q, will form a focus at q (Art. 134, &c.) from which it will afterwards diverge, and fall on the eye at E, nearly as if the rays came from a mathematical point; and from what was said in Art. 161 and 162, it appears that the density of rays in the cone q E is infinitely greater than in any adjacent cone having the eye for its base; so that q will appear as an image of Q, more or less confused, in proportion to the degree of curvature of the caustic at q; for it is evident, that if the curvature be great, the assumed concentration of any small finite pencil Q C C' in one mathematical point q, will deviate more from truth than if the caustic approach nearly to a straight line. Part I.

342. *Corol.* As the eye shifts its place, the apparent position of an object seen in a reflecting or refracting surface shifts also, for as E varies, the tangent E q shifts its place on the caustic, and the point of contact q, or the place of the image shifts.

343. Apparent figure of the horizontal bottom of still water. Fig. 39. Fig. 69.

This doctrine may be illustrated by a very familiar instance. If we look through a surface of still water, not very deep, but having a level horizontal bottom, the bottom will not appear a plane, but will seem to rise on all sides, and approach nearer the surface the more obliquely we look. To explain this, let Q be a point in the bottom, and let Q P e be the course of the pencil of rays by which an eye at e sees it (fig. 39) on the visual ray. The point in the caustic to which e P produced is a tangent, is Y; and from the form of the caustic D Y B (see Art. 238) it is obvious, that Y is nearer the surface the more oblique e P is to it. The apparent figure of the bottom will therefore be thus determined. From the eye E (fig. 69) draw any line E g to the point G of the surface; and having drawn P Y parallel to E G, touching the branch D Y B of the caustic having Q, vertically below E for a radiant point in Y, prolong E G to H, making G H = P Y, then will H be the image of the point Q' in the bottom, belonging to the caustic D' H B'; and the locus of H, or the apparent form of the bottom, will be the curve D F H, having a basin-shaped curvature at D, a point of contrary flexure at F, and an asymptote C G K coinciding with the surface.

344. Rules for finding the place, &c. of an image.

But, to return to the case of images formed by rays incident at very small obliquities and nearly central, the following rules for determining their places, magnitudes, and apparent situations in all cases of spherical surfaces, will be convenient to bear in memory, and will need no express demonstration to the reader of the foregoing pages.

345. *Rule* 1. Any image formed, or about to be formed, by converging rays, or from which rays diverge, may be regarded as an object.

346. Rule for reflectors. Fig. 16

Rule 2. In spherical reflectors the object and its image lie on the same side of the principal focus. They move in contrary directions, and meet at the centre and surface of the reflector. The distance of the image from the principal focus and centre is had by the proportion

$$QF : FE :: EF : Fq :: QE : Eq,$$

and the image is erect when the object and surface lie on the same side of the principal focus; but inverted when on contrary sides. The relative magnitudes of the object and image (being as their distances from the centre) are given by the proportion

object : image :: Q F : F E :: distance of the object from the principal focus : focal length of reflector.

347. Rule for lenses.

Rule 3. In thin lenses, of all species, if Q be the place of the object, q of its image, E the centre of the lens, F the principal focus of rays *incident in a contrary direction*, then will the object and image lie on the same, or opposite side of the lens, according as the object and lens lie on the same or opposite sides of the principal focus F. In the former case the image is erect, in the latter inverted, with respect to the object. The distance of the image from the lens, or from the object, is had by the proportions

$$QF : FE :: QE : Eq; \qquad QF : QE :: QE : Qq;$$

and the magnitude of the object is to that of the image as the distance of the object from F is to the focal length, or as

$$QF : FE.$$

34 .

Rule 4. In all combinations of reflectors and lenses, the image formed by one is to be regarded as the object, whose image is to be formed by the next, and so on, till we come to the last.

349. It has been already remarked (Art. 6) that visible objects are distinguished from optical images by this, that from the former light emanates in all directions, whereas in the latter it emanates only in certain directions. This is an important limitation in practical optics. A real object can be seen whenever nothing opaque is interposed between it and the eye. An image can only be seen when the eye is placed in the pencil of rays which goes to form it, or diverges from it. Thus in the case represented in fig. 62, except the eye be placed somewhere in the space D q p H, it will see no part of the image, B q D and A p H being the extreme rays refracted by the lens from the extremities of the object.

Brightness of images.

The brightness of an image is, of course, proportional to the quantity of light which is concentrated in each point of it; and, therefore, supposing no aberration, as the apparent magnitude of the lens or mirror which forms it, seen from the object $\times \frac{\text{area of object}}{\text{area of image}}$. Or, since the area of the object : that of the image :: (distance)^2 of object from lens : (distance)^2 of image; and since the apparent magnitude of the lens seen from the object is as its $\left(\frac{\text{diameter}}{\text{distance from object}}\right)^2$, the brightness or degree of illumination of the image is as the apparent

Light. magnitude of the lens seen from the image, alone, whatever be the distance of the object. Now the apparent magnitude of the lens seen from the image is always much less than a hemisphere. Therefore (even supposing no light lost by reflection or refraction) the illumination of the image is always much less than that of the object. This is the case when the image is received on a screen which reflects all the rays, or when viewed by an eye behind it having a pupil large enough to receive all the rays which have crossed at the image, *à fortiori*, then, when the eye does not receive all the rays, must the apparent intrinsic brightness be less than that of the object. This supposes the object to have a sensible magnitude; but when both the object and its image are physical points, the eye judges only of absolute light; and the light of the image is therefore proportional to the apparent magnitude of the lens, as seen from the object. In the case of a star, for instance, whose distance is constant, the absolute light of the image is simply as the square of the aperture, and this is the reason why stars can be seen in large telescopes which are too faint to be seen in small ones. Part I.

Images are never so bright as their objects

§ XII. *Of the Structure of the Eye, and of Vision.*

350. It is by means of optical images that vision is performed, that we see. The eye is an assemblage of lenses which concentrate the rays emanating from each point of external objects on a delicate tissue of nerves, called the retina, there forming an image, or exact representation of every object, which is the thing immediately perceived or *felt* by the retina.

Description of the eye. Fig. 70. Fig. 70 is a section of the human eye through its axis in a horizontal plane. Its figure is, generally speaking, spherical, but considerably more prominent in front. It consists of three principal chambers, filled with media of perfect transparency and of refractive powers, differing sensibly *inter se*, but none of them greatly different from that of pure water.

Aqueous humour. Its composition. Refractive power. The first of these media, A, occupying the anterior chamber, is called the *aqueous humour*, and consists, in fact, chiefly of pure water, holding a little muriate of soda and gelatine in solution, with a trace of albumen; the whole not exceeding eight per cent.* Its refractive index, according to the experiments of M. Chossat,† and those of Dr. Brewster and Dr. Gordon,‡ is almost precisely that of water, viz. 1.337, that of water being 1.336.

Cornea. Its figure an ellipsoid of revolution. The cell in which it is contained is bounded, on its anterior side, by a strong, horny, and delicately transparent coat *a*, called the *cornea*, the figure of which, according to the delicate experiments and measures of M. Chossat,§ is an ellipsoid of revolution about the major axis; this axis, of course, determines the *axis of the eye;* but it is remarkable, that in the eyes of oxen, measured by M. Chossat, its vertex was never found to be coincident with the central point of the aperture of the cornea, but to lie always about 10° (reckoned on the surface) inwardly, or towards the nose, in a horizontal plane. The ratio of the semi-axis of this ellipse to the excentricity, he determines at 1.3; and this being nearly the same with 1.337, the index of refraction, it is evident, from what was demonstrated in Art. 236, that parallel rays incident on the cornea in the direction of its axis, will be made to converge to a focus situated behind it, almost with mathematical exactness, the aberration which would have subsisted, had the external surface a spherical figure, being almost completely destroyed.

351. Iris. The posterior surface of the chamber A of the aqueous humour is limited by the *iris* $\beta\gamma$, which is a kind of circular opaque screen, or diaphragm, consisting of muscular fibres, by whose contraction or expansion an aperture in its centre, called the *pupil*, is diminished or dilated, according to the intensity of the light. In very strong lights the opening of the pupil is greatly contracted, so as not to exceed twelve hundredths of an inch in the human eye, while in feebler illuminations it dilates to an opening not exceeding twenty-five hundredths,‖ or double its former diameter. The use of this is evidently to moderate and equalize the illumination of the image on the retina, which might otherwise injure its sensibility. In animals (as the cat) which see well in the dark, the pupil is almost totally closed in the daytime, and reduced to a very narrow line; but in the human eye, the form of the aperture is always circular. The contraction of the pupil is involuntary, and takes place by the effect of the stimulus of the light itself; a beautiful piece of self-adjusting mechanism, the play of which may be easily seen by approaching a candle to the eye while directed to its own image in a looking-glass.

352. Crystalline. Its figure. Immediately behind the opening of the *iris* lies the *crystalline lens*, B, enclosed in its *capsule*, which forms the posterior boundary of the chamber A. Its figure is a solid of revolution, having its anterior surface much less curved than the posterior. Both surfaces, according to M. Chossat, are ellipsoids of revolution about their *lesser* axes; but it would seem from his measures, that the axes of the two surfaces are neither exactly coincident in direction with each other, nor with that of the cornea. This deviation would be fatal to distinct vision were the crystalline lens very much denser than the others, or were the whole refraction performed by it. This,

Refraction. Non-coincidence of the axes of its surfaces. however, is not the case; for the mean refractive index of this lens is only 1.384, while that of the aqueous humour, as we have seen, is 1.337; and that of the *vitreous* C, which occupies the third chamber, is 1.339; so that the whole amount of bending which the rays undergo at the surface of the crystalline is small, in comparison with the inclination of the surface at the point where the bending takes place, and, since near the vertex, a

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* Chenevix, *Philosophical Transactions*, vol. xciii. p. 195.
† *Bulletin de la Soc. Philomatique*, 1818, p. 94.
‡ *Edinburgh Philosophical Journal*, vol. i. p. 42.
§ *Sur la Courbure des Milieux Réfringens de l'Œil chez le Bœuf. Annales de Chim.* x. p. 337.
‖ Dr. Young's Lectures on the Mechanism of the Eye, *Philosophical Transactions*, vol. xci.

material deviation in the direction of the axis can produce but a very minute change in the inclination of the ray to the surface, this cause of error is so weakened in its effect, as, probably, to produce no appreciable aberration.

Why not injurious to vision.

The crystalline is composed of a much larger proportion of albumen and gelatine than the other humours of the eye, so much so as to be entirely coagulable by the heat of boiling water. It is somewhat denser towards the centre than at the outside. According to Dr. Brewster and Dr. Gordon, the refractive indices of its centre middle of its thickness, from the centre to the outside, and the outside itself, are respectively 1.3999, 1.3786, and 1.3767, that of pure water being 1.3358. This increase of density is obviously useful in correcting the aberration, by shortening the focus of rays near the centre, according to the rule laid down in Art. 299 for finding the effect of aberration. The effect of the elliptic figure of the surfaces is, however, a matter of pretty complex calculation, and cannot be entered upon in the limits of this essay. Its use is, probably, to correct the aberration of oblique pencils.

353. Composition of crystalline. Denser towards centre.

The posterior chamber C of the eye is filled with the *vitreous humour*, a fluid differing (according to Chenevix) neither in specific gravity nor in chemical composition in any sensible respect from the aqueous; and, as we have already seen, having a refractive index but very little superior to it.

354.

The refractive density of the crystalline being superior to that of either the aqueous or vitreous humour, the rays which are incident on it in a state of convergence from the cornea, are made to converge more, and exactly in their final focus is the posterior surface of the cell of the vitreous humour covered by the retina *d*, a network (as its name imports) of inconceivably delicate nerves, all branching from one great nerve O, called the optic nerve, which enters the eye obliquely at the inner side of the orbit, next the nose. The retina lines the whole of the cavity C up to *i*, where the capsule of the crystalline commences. Its nerves are in contact with, or immersed in, the *pigmentum nigrum*, a very black velvety matter, which covers the *choroid membrane g*, and whose office is to absorb and stifle all the light which enters the eye as soon as it has done its office of exciting the retina; thus preventing internal reflexions, and consequent confusion of vision. The whole of these humours and membranes are contained in a thick tough coat, called the *sclerotica*, which unites with the cornea, and forms what is commonly called the *white of the eye*.

355. Retina.

Sclerotica.

Such is the structure by which parallel rays, or those emanating from very distant objects, are brought to a focus on the retina. But as we require to see objects near, as well as at a distance, and as the focus of a lens or system of lenses for near objects is longer than for distant ones, it is evident that a power of adjustment must reside somewhere in the eye; by which either the retina can be removed farther from the cornea, and the eye lengthened in the direction of its axis, or the curvature of the lenses themselves altered so as to give greater convergency to the rays. We know that such a power exists, and can be called into action by a voluntary effort; and, evidently, by a muscular action, producing fatigue if long continued, and not capable of being strained beyond a certain point. Anatomists, however, as well as theoretical opticians, differ as to the mechanism by which this is effected. Some assert, that the action of the muscles which move the eye in its orbit, called the *recti*, or straight muscles, when all contracted at once, producing a pressure on the fluids within, forces out the cornea, rendering it at once more convex, and more distant from the retina. This opinion, however, which has been advocated by Dr. Olbers, and even attempted to be made a matter of ocular demonstration by Ramsden and Sir E. Home, has been combated by Dr. Young, by experiments which show, at least, very decisively, that the increase of convexity in the cornea has little if any share in producing the effect. An elongation of the whole eye, spherical as it is and full of fluid, to the considerable extent required, is difficult to conceive as the result of any pressure which could be safely applied, as to give distinct vision at the distance of three inches from the eye, (the nearest at which ordinary eyes can see well,) the sphere must be reduced to an ellipsoid, having its axis nearly one-seventh longer than in its natural state; and the extension of the sclerotica thus produced, would hardly seem compatible with its great strength and toughness. Another opinion, which has been defended with considerable success by the excellent philosopher last named, is, that the crystalline itself is susceptible of a change of figure, and becomes more convex when the eye adapts itself to near distances. His experiments, on persons deprived of this lens, go far to prove the total want of a power to change the focus of the eye in such cases, though a certain degree of adaptation is obtained by the contraction of the iris, which, limiting the diameter of the pencil, diminishes the space on the retina over which imperfectly converged rays are diffused, and thus, in some measure, obviates the effect of their insufficient convergence. When we consider that the crystalline lens has actually a regular fibrous structure, (as may be seen familiarly on tearing to pieces the lens of a boiled fish's eye,) being composed of layers laid over each other like the coats of an onion, and each layer consisting of an assemblage of fibres proceeding from two poles, like the meridians of a globe, the axis being that of the eye itself; we have, so far at least, satisfactory evidence of a muscular structure; and were it not so, the analogy of pellucid animals, in which no muscular fibres can be discerned, and which yet possess the power of motion and obedience to the nervous stimulus, though nerves no more than muscles can be seen in them, would render the idea of a muscular power resident in the crystalline easily admissible, though nerves have as yet not been traced into it. On the whole, it must be allowed, that the presumption is strongly in favour of this mechanism; though the other causes already mentioned may, perhaps, conspire to a certain extent in producing the effect, and though the subject must be regarded as still open to fuller demonstration. It is the boast of science to have been able to trace so far the refined contrivances of this most admirable organ; not its shame to find something still concealed from its scrutiny; for, however anatomists may differ on points of structure, or physiologists dispute on modes of action, there is that in what we *do* understand of the formation of the eye so similar, and yet so infinitely superior, to a product of human ingenuity,—such thought, such care, such refinement, such advantage taken of the properties of natural agents used as mere instruments, for accomplishing a given end, as force upon us a conviction of

356. Change of focus of eye for near objects.

Light. deliberate choice and premeditated design, more strongly, perhaps, than any single contrivance to be found, whether in art or nature, and render its study an object of the deepest interest. Part I.

357. Image on the retina the immediate object of vision. The images of external objects are of course formed inverted on the retina, and may be seen there, by dissecting off the posterior coats of the eye of a newly-killed animal, and exposing the retina and choroid membrane from behind, like the image on a screen of rough glass, mentioned in Art. 331. It is this image, and this only, which is *felt* by the nerves of the retina, on which the rays of light act as a stimulus; and the impressions therein produced are thence conveyed along the optic nerves to the sensorium, in a manner which we must rank at present among the profounder mysteries of physiology, but which appears to differ in no respect from that in which the impressions of the other senses are transmitted. Thus, a paralysis of the optic nerve produces, while it lasts, total blindness, though the eye remains open, and the lenses retain their transparency; and some very curious cases of half blindness have been successfully referred to an affection of one of the nerves without the other.* On the other hand, while the nerves retain their sensibility, the degree of perfection of vision is exactly commensurate to that of the image formed on the retina. In cases of cataract, where the crystalline lens loses its transparency, the light is prevented from reaching the retina, or from reaching it in a proper state of regular concentration, being stopped, confused, and scattered by the opaque or semi-opaque portions it encounters in its passage. The image, in consequence, is either altogether obliterated, or rendered dim and indistinct; and the progress of blindness is accordingly. If the opaque lens be extracted, the full perception of light returns; but one principal instrument for producing the convergence of the rays being removed, the image, instead of being formed *on* the retina, is formed considerably behind it, and the rays being received in their unconverged state on it, produce no regular picture, and therefore no distinct vision. But if we give to the rays, before their entry into the eye, a certain proper degree of convergence, by the application of a convex lens, so as to render the remaining lenses capable of finally effecting their exact convergence on the retina, restoration of distinct vision is the immediate result. This is the reason why persons who have undergone the operation for the cataract (which consists either in totally removing, or in putting out of the way an opaque crystalline) wear spectacles of comparatively very short focus. Such glasses perform the office of an artificial crystalline. A similar imperfection of vision to that produced by the removal of the crystalline, is the ordinary effect of old age, and its remedy is the same. In aged persons the exterior transparent surface of the eye, called the *cornea*, loses somewhat of its convexity, and becomes flatter. The *power* of the eye is therefore diminished, (Art. 248 and 255,) and a perfect image can no longer be formed on the retina. The deficient power is however supplied by a convex lens, or *spectacle-glass*, (Art. 268,) and vision rendered perfect or materially improved.

358. Short-sighted persons have their eyes too convex, and this defect is, like the other, remediable by the use of proper lenses of an opposite character. There are cases, however, though rare, in which the cornea becomes so very prominent as to render it impossible to apply conveniently a lens sufficiently concave to counteract its action. Such cases would be accompanied with irremediable blindness, but for that happy boldness, justifiable only by the certainty of our knowledge of the true nature and laws of vision, which in such a case has suggested the opening of the eye and removal of the crystalline lens, though in a perfectly sound state.

359. Malconformations of the cornea. But these are not the only cases of defective vision arising from the structure of the organ, which are susceptible of remedy. Malconformations of the cornea are much more common than is generally supposed, and few eyes are, in fact, free from them. They may be detected by closing one eye, and directing the other to a very narrow, well-defined luminous object, not too bright, (the horns of the moon, when a slender crescent, only two or three days old, are very proper for the purpose,) and turning the head about in various directions. The line will be doubled, tripled, or multiplied, or variously distorted; and careful observation of its appearances, under different circumstances, will lead to a knowledge of the peculiar conformation of the refracting surfaces of the eye which causes them, and may suggest their proper remedy. A remarkable and instructive instance of the kind has recently been adduced by Mr. G. B. Airy, (*Transactions of the Cambridge Philosophical Society*,) in the case of one of his own eyes; which, from a certain defect in the figure of its lenses, he ascertained to refract the rays to a nearer focus in a vertical than in a horizontal plane, so as to render the eye utterly useless. This, it is obvious, would take place if the cornea, instead of being a surface of revolution, (in which the curvature of all its sections through the axis must be equal,) were of some other form, in which the curvature in a vertical plane is greater than in a horizontal. It is obvious, that the correction of such a defect could never be accomplished by the use of spherical lenses. The strict method, applicable in all such cases, would be to adapt a lens to the eye, of nearly the same refractive power, and having its surface next the eye an exact *intaglio* fac-simile of the irregular cornea, while the external should be exactly spherical of the same general convexity as the cornea itself; for it is clear, that all the distortions of the rays at the posterior surface of such a lens would be exactly counteracted by the equal and opposite distortions at the cornea itself.† But the necessity of limiting the correcting lens to such surfaces as can be truly ground in glass, to render it of any real and everyday use, and which surfaces are only spheres, planes, and cylinders, suggested to Mr. Airy the ingenious idea of a double concave lens, in which one surface should be spherical, the other cylindrical. The use of the spherical surface was to correct the general defect of a too convex cornea. That of the cylindrical may be thus explained. Suppose parallel rays incident on a concave cylindrical surface, A B C D, in a direction perpendicular to its axis, as in fig. 71, and let S S′ P P′ Q Q′ T T′, be any laminar pencil of them contained in a parallelepiped infinitely

Remarkable case, successfully remedied by glasses.

Fig. 71.

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* Wollaston, on Semi-decussation of the Optic Nerves, *Philosophical Transactions*, 1824.

† Should any very bad cases of irregular cornea be found, it is worthy of consideration, whether at least a temporary distinct vision could not be procured, by applying in contact with the surface of the eye some transparent animal jelly contained in a spherical capsule of glass; or whether an actual mould of the cornea might not be taken, and impressed on some transparent medium. The operation would, of course, be delicate, but certainly less so than that of cutting open a living eye, and taking out its contents.

thin, and having its sides parallel to the axis. Any of the rays S P, S′ P′, of this pencil lying in a plane A P S perpendicular to the axis, will after refraction converge to, or diverge from, a point X, also in this plane; and, therefore, all the rays incident on P Q, P′ Q′, will after refraction have for their focus the line X Y, in the caustic surface A F G D, and the principal focus of the cylinder will be the line F G, whose distance from the vertex C C′ of the surface, or F C, is the same with the focal length of a spherical surface, formed by the revolution of A B about the axis F C. Thus we see that a cylindrical lens produces no convergency or divergency in parallel rays, incidental in the plane of its axis; while it converges or diverges rays in a plane at right angles to the axis, as a spherical surface of equal curvature would do. If then such a cylindrical surface be conjoined with a spherical one, the focus of the spherical surface will remain unaltered in one plane, but in the other will be changed to that of a lens formed by it, and a spherical surface of equal curvature with the cylinder. Hence by properly placing such a cylindro-spheric lens across the defective eye, its defect will be (approximately, at least) counteracted. It would be wrong to conclude our account of this interesting application of mathematical knowledge to the increase of the comforts and improvement of the faculties of its possessor, in other than his own words. "After some ineffectual applications to different workmen, I at last procured a lens to these dimensions,* from an artist named Fuller, at Ipswich. It satisfies my wishes in every respect. I can now read the smallest print at a considerable distance with the left" (the defective) "eye as well as with the right. I have found that vision is most distinct when the cylindrical surface is turned from the eye: and as, when the lens is distant from the eye, it alters the apparent figure of objects by refracting differently the rays in different planes, I judged it proper to have the frame of my spectacles made so as to bring the glass pretty close to the eye. With these precautions, I find that the eye which I once feared would become quite useless, can be used in almost every respect as well as the other."

360. Blindness, partial or total, may be caused, not only by the opacity of the crystalline lens, but of any other part, or by anything extraneous to the materials of which they consist, interposed between the external transparent surface of the cornea and the retina. In all such cases, if the sensibility of the nerve be uninjured, the restoration of sight is never to be despaired of. In a recent most remarkable case, operated by Mr. Wardrop, and by him recorded in the *Philosophical Transactions for* 1826, blindness from infancy, accompanied with complete obliteration of the pupil, by a contraction of the iris, owing to an unskilful operation, performed at six months of age, was removed, and perfect sight restored after a lapse of forty-six years, by a simple removal of the obstruction, by breaking a hole through the closed membrane. The details of this case are in the highest degree interesting, but we must refer the reader to the volume of the *Philosophical Transactions* cited for the account.

361. Single vision with two eyes. As we have two eyes, and a separate image of every external object is formed in each, it may be asked, *why do we not see double?* and to some, the question has appeared to present much difficulty. To us it appears, that we might with equal reason ask, *why*—having two hands, and five fingers on each, all endowed with equal sensibility of touch and equal aptitude to discern objects by that sense—*we do not feel decuple?* The answer is the same in both cases: it is a matter of habit. Habit alone teaches us that the sensations of sight correspond to any thing external, and to what they correspond. An object (a small globe or wafer suppose) is before us on a table; we direct our eyes to it, *i. e.* we bring its images on both retinæ to those parts which habit has ascertained to be the most sensible and best situated for seeing distinctly; and having always found that in such circumstances the object producing the sensation is one and the same, the idea of unity in the object becomes irresistibly associated with the impression. But while looking at the globe, squeeze the upper part of one eye downwards, by pressing on the eyelid with the finger, and thereby forcibly throw the image on another part of the retina of that eye, and double vision is immediately produced, two globes or two wafers being distinctly seen, which appear to recede from each other as the pressure is stronger, and approach, and finally blend into one as it is relieved. The same effect may be produced without pressure, by directing the eyes to a point nearer to, or farther from them than the wafer; the optic axis in this case being both directed away from the object seen. When the eyes are in a state of perfect rest, their axes are usually parallel, or a little diverging. In this state all near objects are seen double; but the slightest effort of attention causes their images to coalesce immediately. Those who have one eye distorted by a blow, see double, till habit has taught them anew to see single, though the distortion of the optic axis subsists.

Double vision artificially produced.

Another method.

362. Single objects felt double in certain cases. The case is exactly the same with the sense of touch. Lay hands on the globe, and handle it. It is *one*, nothing can be more irresistible than this conviction. Place it between the first and second fingers of the right hand in their natural position. The right side of the first and left of the second finger feel opposite convexities; but as habit has always taught us that two convexities so felt belong to one and the same spherical surface, we never hesitate or question the identity of the globe, or the unity of the sensation. Now cross the two fingers, bringing the second over the first, and place the globe on the table in the fork between them, so as to feel the left side of the globe with the right side of the second finger, and the right with the left of the first. In this state of things the impression is equally irresistible, that we have two globes in contact with the fingers, especially if the the eyes be shut, and the fingers placed on it by another person. A pea is a very proper object for this experiment. The illusion is equally strong when the two fore fingers of both hands are crossed, and the pea placed between them.

363. Force of habit in producing single vision illustrated by experiment. So forcible is the power of habit in producing single vision, that it will bring the two images to apparent coalescence, when the rays which form one of them are really turned far aside from their natural course. To show this, place a candle at a distance, and look at it with one eye (the left suppose) naked, the other having

* Radius of the spherical surface $3\frac{1}{3}$ inches, of the cylindrical $4\frac{1}{2}$.

before it a prism, with a variable refracting angle, (an instrument to be described hereafter, see INDEX,) and, first, let the angle be adjusted to zero, then will the prism produce no deviation, and the object will appear single. Now vary the prism, so as to produce a deviation of 2° or 3° of the rays in a horizontal plane to the right. The candle will immediately be seen double, the image deviated by the prism being seen to the left of the other; but the slightest motion, such as winking with the eyelids, blends them immediately into one. Again, vary the prism a few degrees more in the same direction; the candle will again be doubled, and again rendered single by winking, and directing the attention more strongly to it; and thus may the optic axes be, as it were, inveigled to an inclination of 20° or 30° to each other. In this state of things, if a second candle be placed exactly in the direction of the deviated image of the first, but so screened, that its rays shall not fall on the left eye, and the prism be then suddenly removed in the act of winking, the two candles appear as one. If the deviation of the image seen with the right eye be made to the apparent right, the range within which it is possible to bring them to coalesce is much more limited, as it is much more usual for us to direct by an effort the optic axes towards, than from each other. If the deviation be made but a very little out of the horizontal plane, no effort will enable us to correct it. It is probable that some cases of squinting might be cured by some such exercise in the act of directing the optic axes, if continued perseveringly.

364. A further cause of single vision. Nervous sympathy.

Such is, undoubtedly, a sufficient explanation of single vision with two eyes; yet Dr. Wollaston has rendered it probable that a physiological cause has also some share in producing the effect, and that a semi-decussation of the optic nerves takes place immediately on their quitting the brain, half of each nerve going to each eye, the right half of each retina consisting wholly of fibres of one nerve, and the left wholly of the other, so that all images of objects out of the optic axis are perceived by one and the same nerve in both eyes, and thus a powerful sympathy and perfect unison kept up between them, independent of the mere influence of habit. Immediately in the optic axis, it is probable, that the fibres of both nerves are commingled, and this may account for the greater acuteness and certainty of vision in this part of the eye.

365. Erect vision by an inverted image.

Another point, on which much more discussion has been expended than it deserves, is the fact of our seeing objects erect when their images on the retina are inverted. Erect, means nothing else than having the head farther from the ground, and the feet nearer, than any other part. Now, the earth, and the objects which stand on it, preserve the same relative situation in the picture on the retina that they do in nature. In that picture, men, it is true, stand with their heads downwards; but then, at the same time, heavy bodies fall upwards; and the mind, or its deputy, the nerve, which is present in every part of the picture, judges only of the relations of its parts to one another. How these parts are related to external objects, is known only by experience, and judged of at the instant only by habit.

366. *Punctum cæcum.* Experiment proving its existence.

There is one remarkable fact which ought not to escape mention, even in so brief an abstract of the doctrine of vision as the present, it is, that the spot ○, at which the optic nerve enters the eye, is totally insensible to the stimulus of light, for which reason it is called the *punctum cæcum*. The reason is obvious: at this point the nerve is not yet divided into those almost infinitely minute fibres, which are fine enough to be either thrown into tremors, or otherwise changed in their mechanical, chemical, or other state, by a stimulus so delicate as the rays of light. The effect, however, is curious and striking. On a sheet of black paper, or other dark ground, place two white wafers, having their centres three inches distant. Vertically above that to the *left*, hold the *right* eye, at 12 inches from it, and so that when looking down on it, the line joining the two eyes shall be parallel to that joining the centre of the wafers. In this situation closing the left eye, and looking full with the right at the wafer perpendicularly below it, this only is seen, the other being completely invisible. But if removed ever so little from its place, either to the right or left, above or below, it becomes immediately visible, and starts, as it were, into existence. The distances here set down may perhaps vary slightly in different eyes.

367.

It will cease to be thought singular, that this fact, of the absolute invisibility of objects in a certain point of the field of view of each eye, should be one of which not one person in ten thousand is apprized, when we learn, that it is not extremely uncommon to find persons who have for some time been totally blind with one eye without being aware of the fact. One instance has fallen under the knowledge of the writer of these pages.

368. Eyes of fishes.

In the eyes of fishes, the humours being nearly of the refractive density of the medium in which they live, the refraction at the cornea is small, and the work of bringing the rays to a focus on the retina is almost wholly performed by the crystalline. This lens, therefore, in fishes is almost spherical, and of small radius, in comparison with the whole diameter of the eye. Moreover, the destruction of spherical aberration not being producible in this case by mere refraction at the cornea, the crystalline itself is adapted to execute this necessary part of the process, which it does by a very great increase of density towards the centre. (Brewster, *Treatise on New Philosophical Instruments*, p. 268.) The fibrous and coated structure of the crystalline lens is beautifully shown in the eye of a fish coagulated by boiling.

369.

The same scientific principles which enable us to assist natural imperfections of sight, can be employed in giving additional power to this sense, even in individuals who enjoy it naturally in the greatest perfection. It being once understood, that the image on the retina is that which we really *see*, it follows, that if by any means we can render this image brighter, larger, more distinct than in the natural state of the organ, we shall see objects brighter than in their natural state, enlarged in dimension, and, therefore, capable of being examined more in detail, or more sharply defined and clearly outlined. The means which the principles already detailed put in our power, for the accomplishment of such ends, are the concentration of more rays than enter the natural eye by lenses; the enlargement of the image on the retina, by substituting for the object seen an image of it, either larger than the object itself, or capable of being brought nearer to us; and the destruction of aberration, by properly adapting the figure and materials of our instruments to the end proposed.

370.

Proposition. The apparent magnitude of a rectilinear object is measured by the angle subtended by it at

Light. the centre of the eye, or by the linear magnitude of its image on the retina, and is therefore proportional to $\frac{\text{linear magnitude of object}}{\text{its distance from the eye}}$. Part I.

The *centre* of the eye, in its optical sense, is a point nearly in the centre of the pupil in the plane of the iris, and the image of any external object P Q, being formed at the bottom of the eye at p q, by rays crossing there, must subtend the same angle; so that $p\ q = \text{P Q} \cdot \frac{p\,\text{E}}{\text{P E}}$. Fig. 72.

371. *Corol.* If the object be so distant that the rays from each point of it may be regarded as parallel, the angular diameter of the object is measured by the inclination of rays of its extreme pencils to each other. Whenever, therefore, the eye sees by parallel, or very nearly parallel, rays, the apparent magnitude of the object seen, is measured by the inclination of its extreme pencils, and the object itself is referred to an infinite distance, or to the concave surface of the heavens.

372. *Prop.* When a convex lens is placed between the eye and any object, so as to have the object at a distance from the lens equal to its focal length, it will be distinctly seen by an eye capable of converging parallel rays, and will appear enlarged beyond its natural size.

Let P Q be the object, C the lens, and E the centre of the eye. Since the object is in the focus of the lens, the rays of a pencil diverging from any point P in it, will emerge parallel to P E, and to each other; they will, therefore, after refraction in the eye, be brought to converge on the retina to a point p, such that E p is parallel to P C. Similarly, rays from Q will, after refraction through the lens and eye, converge to q; such that E q is parallel to Q C. Thus, a distinct image will be formed at p q on the retina, and the apparent angular magnitude of the object seen through the lens will be the angle q E p. Now this is equal to P C Q, or the angle subtended by the object at the centre of the lens, and is, therefore, greater than P E Q, or that subtended by it at the centre of the eye, because the lens is between the eye and object. Fig. 73.

373. Hence, the nearer the eye is to the lens, the less will be the difference between the apparent magnitudes of the object, as seen with and without the lens interposed. But if the lens be of shorter focus than the least distance at which the eye can see distinctly, there will be this essential difference between vision with and without the lens, that in the former case the object is seen distinctly, and well-defined; while in the latter, or with the naked eye, it will be indistinct and confused, and the more so the nearer it is brought.

374. Hence, by the use of a convex lens of short focus, objects may be seen distinct, and magnified to any extent we please: for let L be the power, or reciprocal focal length of the lens, and D the greatest proximity of the object to the centre of the eye at which it can be seen distinctly without a lens. Then we shall have L : D : : angle p E q : angle subtended by the object at the proximity D; and, therefore, : : apparent linear magnitude of object seen through the lens : apparent linear magnitude at proximity D, with the naked eye. Therefore $\frac{\text{L}}{\text{D}}$ is the ratio of these magnitudes, or, as it is called, the *magnifying power* of the lens, beyond that of the naked eye, at its greatest proximity. By a convex lens of short focus objects are magnified. Magnifying power.

375. *Corol.* D being given, the magnifying power is as L, or as $(\mu - 1)(\text{R}' - \text{R}'')$. This explains the use of the word power in the foregoing sections. Whatever we have demonstrated of the powers of lenses in the foregoing pages, is true of *magnifying powers*. Thus the sum of the magnifying powers of two convex lenses is the magnifying power of the two combined. If one be concave, its magnifying power is to be regarded as negative, and instead of their sum we must take their difference. Magnifying power of a system of lenses.

376. *Prop.* To express, generally, the *visual angle* under which a small object placed at any distance from a lens, and seen by an eye any how situated, appears, supposing it seen distinctly.

Let P Q, fig. 74, 75, 76, 77, be the object, E the lens, O the eye, and p q the image. Put $\frac{1}{\text{E Q}} = \text{D}$, $\frac{1}{\text{E } q} = f$; $\frac{1}{\text{E O}} = e$; e being reckoned in the same direction from the centre of the lens that D and f are. Then the visual angle under which the image is seen is q O p, and we have, therefore, visual angle $(= \text{A}) = \frac{q\,p}{\text{O}\,q} = \frac{q\,p}{\text{O E} - \text{E}\,q}$. But, $q\,p = \text{Q P} \cdot \frac{\text{E}\,q}{\text{E Q}} = \text{Q P} \cdot \frac{\text{D}}{f} = \text{O} \cdot \frac{\text{D}}{f}$ putting O for Q P the linear magnitude of the object; and, moreover, $\text{O E} - \text{E}\,q = \frac{1}{e} - \frac{1}{f} = \frac{f - e}{f\,e}$, therefore we have, Fig. 74, 75, 76, 77. Visual angle. Vision through convex lenses.

$$\text{A} = \text{O} \cdot \frac{\text{D}}{f} \cdot \frac{e\,f}{f - e} = \text{O} \cdot \frac{e\,\text{D}}{\text{L} + \text{D} - e}$$

when L, as all along, represents the power of the lens. Now O . D is the visual angle of the object, as seen from the centre of the lens; therefore, putting O . D, or $\frac{\text{Q P}}{\text{Q E}} = (\text{A})$ we get

$$\text{A} = (\text{A}) \cdot \frac{e}{\text{L} + \text{D} - e}. \qquad (a)$$

Light. Part I.

In concave lenses, the images of distant objects are formed erect, and on the same side of the lens with the object. If, therefore, such a lens be held between the eye and distant objects at a sufficient distance from the eye for distinct vision, the objects will be seen *erect*, and *diminished* in magnitude. In this case, e is positive, and L and D both negative; therefore $L + D - e$ is a negative quantity, greater (without regard to the sign) than e, and, consequently, A is negative, and less than (A).

377. Through concave.

378. In reflectors.

In reflectors, $f = 2R - D$, and, therefore,

$$A = (A) \cdot \frac{e}{2R - D - e}. \qquad (b)$$

In a convex reflector, e is necessarily negative, at least if the mirror be made of metal, because the eye must be on the side of the surface towards the incident light; and, therefore, $2R - e$ is positive, and $\frac{e}{2R - D - e}$ will be greater or less than unity, according to the value of $2R - D - e$. In concave reflectors, R is negative, and e is also negative for the same reason as in concave; therefore the sign and magnitude of A in this, as well as the former case, may vary indefinitely, according to the place of the eye, the image, and the object. The varieties of these cases are represented in fig. 78 and 79.

379. General principles of telescopes.

If the image, instead of being seen directly by the naked eye, be seen through the medium of another lens or reflector, so placed as to cause the pencils diverging primarily from each point of the object, to emerge finally, either exactly parallel, or within such limits of convergence or divergence as the eye can accommodate itself to, the object will be seen distinctly, and either larger or smaller than it would be seen by the unassisted eye, according to the magnitude of the image, and the power of the lens or reflector used to view it. This is the principle of all telescopes and microscopes. As most eyes can see with parallel rays, they are so constructed as to make parallel pencils emerge parallel; and a mechanical adjustment allows such a quantity of motion of the lenses or reflectors with respect to each other, as to give the rays a sufficient degree of convergence or divergence as may be required.

380. Astronomical telescope.

In the common refracting, or, as it is sometimes called, the astronomical telescope, the image is first formed by a convex lens, and is viewed through a convex lens, placed at a distance from the other nearly equal to the sum of their focal lengths. The lens which forms the image is called the *object-glass*, and that through which it is viewed, the *eye-glass* of the telescope. If the latter be concave, the telescope is said to be of the Galilæan construction, such having been the original arrangement of Galilæo's instruments. The

Fig. 80, 81. situation of the lenses, and the course of the rays in these two constructions, are represented in fig. 80 and 81.

381. In the former construction, let P Q be the object. Draw Q O G through the centres of the object and eye-glass, and this line will be the axis of the telescope. From R any point in the object draw R O p through the centre O of the object-glass, and meeting $p\,q$, a line through q, the focus of the point Q, perpendicular to the axis in p, then will $p\,q$ be the image of P Q. Let P A, P B be the extreme rays of the pencil diverging from P, and incident on the object-glass, and they will be refracted to and cross at p. Hence, unless the diameter of the eye-glass b G a be such, that the ray A p a shall be received on it, the point p will be seen less illuminated than the point p in the centre of the object, and if it be so small that the line B p produced does not meet it, then none of the rays from P can reach the eye at all. Thus, the *field of view*, or angular dimensions of the object seen, is limited by the aperture of the eye-glass. To find its extent, then, join B b, A a, opposite extremities of the object and eye-glass, meeting the image in r and p, and the axis in X, then $r\,p$ is the whole extent of the image which is seen at all, and the angle p O r, which is equal to P O R, is the angular extent of the field of view. Now we have A B : $a\,b$ :: O X : G X, and,

Field of view.

therefore, A B + $a\,b$: A B :: O G : O X, whence we get $OX = \frac{AB}{AB + ab} \cdot OG$; $GX = \frac{a\,b}{AB + ab} \cdot$ O G. But we have, moreover, $X\,q = O\,q - O\,X$; $p\,r = a\,b \cdot \frac{X\,q}{G\,X}$, and angle r O $p = \frac{r\,p}{O\,q}$. To express this algebraically, put

Diameter of object-glass $= \alpha$; Power of object-glass $= L$.
Diameter of eye-glass $= \beta$; Power of eye-glass $= l$.

Then

$$\left.\begin{aligned} OX &= \frac{\alpha}{\alpha + \beta}\left(\frac{1}{L} + \frac{1}{l}\right); \quad GX = \frac{\beta}{\alpha + \beta}\left(\frac{1}{L} + \frac{1}{l}\right) \\ QX &= \frac{1}{\alpha + \beta}\left(\frac{\beta}{L} - \frac{\alpha}{l}\right); \text{ and } p\,r = \frac{\beta\,l - \alpha\,L}{L + l}; \end{aligned}\right\}, \qquad (c)$$

This last is the linear magnitude of the visible portion of the image; and it is, as we see, symmetrical both with respect to the eye-glass and object-glass.

382. Now from this it is easy to deduce both the field of view and magnifying power of the telescope; for the former is equal to the angle subtended by $p\,r$, at the centre of the object-glass, and the latter is obtained from the former, when the angle r G p subtended at the centre of the eye-glass is obtained. But we have

Light. Part I.

$$r\,O\,p = L \cdot \frac{\beta l - \alpha L}{L + l}; \qquad r\,G\,p = l \cdot \frac{\beta l \quad \alpha L}{L + l}$$

therefore $$\text{magnifying power} = \frac{r\,G\,p}{r\,O\,p} = \frac{l}{L}. \qquad (d)$$

Formulæ for field of view and magnifying power.

Hence we see, that the greater the power of the eye-glass is, compared with that of the object-glass, the greater the magnifying power of the telescope; or, in other words, the greater the focal length of the object glass compared with that of the eye-glass.

383. Distance of eye.

The pencils of rays after refraction at the eye-glass will emerge parallel, and therefore proper for distinct vision to an eye properly placed to receive them. Now the eye will receive both the extreme rays $b\,R'$ and $a\,P'$ of the pencils diverging from r and p, if it be placed at their point of concourse E; but since $b\,E$ is parallel to $r\,G$, and $a\,E$ to $p\,G$, we have

$$G\,E = G\,q \times \frac{a\,b}{p\,r}, \qquad \text{or } G\,E = \frac{\beta\,(L + l)}{\beta\,l - \alpha\,L}. \qquad (e)$$

384.

If the eye be placed nearer to, or farther off from, the eye-glass than this distance, it will not receive the extreme rays, and the *field of view*, or visible area of the object, will be lessened. In the construction of convex single eye-pieces, therefore, care must be taken to prolong the tube which carries them, (as in the figure,) so that when the eye is applied close to its end, it shall still be at this precise distance from the glass.

385. Inversion of telescopes.

If the telescope be inverted, and the eye applied behind the object-glass, it is evident that it will remain a telescope, but its magnifying power will be changed to $\frac{L}{l}$; so that, if it magnified before, it will diminish objects now, and the field of view will be proportionally increased. In this way, beautiful miniature pictures of distant objects may be seen.

386. Adjustments.

If the telescope, instead of being turned on objects so distant as that the pencils flowing from them may be regarded as parallel, be directed to near objects, the distance between the object-glass and eye-glass must be lengthened so as to bring the image exactly into the focus of the latter. To accomplish this, the eye-glass is generally set in a sliding tube movable by a rack-work, or by hand. The same mechanism serves also to adjust the telescope for long or short-sighted persons. The former require parallel or slightly divergent rays, the latter very divergent; and to obtain the necessary divergence for the latter, the eye-glass must be brought nearer the object-glass.

387. Galilæan telescope.

The same theory and formulæ apply to the second, or Galilæan, construction, only recollecting that in this case L, the power of the eye-glass, is negative. In this case, therefore, the value of G E is negative, or the eye should be placed between the object-glass and eye-glass; but, as that is incompatible with the other conditions, in order to get as great a field of view as possible, the eye must be brought as near to its proper place as possible, and therefore close to the eye-glass.

388.

In the astronomical telescope objects are seen inverted, in the Galilæan, erect; for, in the former, the rays from the extremities of the object have crossed before entering the eye, in the latter, not.

389. Microscopes.

If the object be brought nearer the object-glass, the magnifying power is increased; because in this case (calling D the proximity of the object) $\frac{l}{L - D}$ expresses the magnifying power, as is easily seen from what has been said Art. 382. Thus a telescope used for viewing very near objects becomes a *microscope*. The ordinary construction of the compound microscope is nothing more than that of the astronomical telescope modified for the use it is intended for. The object-glass has in this instrument a much greater power than the eye-glass, so that, when employed for viewing distant objects, it acts as a telescope inverted, and requires to be greatly shortened. But for near objects, as D increases, $l - D$ diminishes, and the fraction $\frac{l}{l - D}$ may be increased to any amount, by bringing the object nearer to the object-glass, and at the same time lengthening the interval between the lenses, which is expressed by $\frac{1}{L - D} + \frac{1}{l}$. But as this requires two operations, it is usual to leave the latter distance unaltered, and vary, by a screw or rack-work, only the former. Fig. 82 is a section of such an instrument. It is, however, convenient to have the power of lengthening and shortening the distance between the glasses, as by this means any magnifying power between the limits corresponding to the extreme distances may be obtained; and if a series of object-glasses be so selected, that the greatest power attainable by one within the limits of the adjustment in question, shall just surpass the least obtainable by the next, and so on, we may command any power we please. Such a series is usually comprised in a small revolving plate containing cells, each of which can be brought in succession into the axis of the microscope by a simple mechanism.

Fig. 82.

390. Reflecting telescope.

In the reflecting telescope, of the most simple construction, the image is formed by a concave mirror, and viewed by a convex or concave eye-glass, as in refracting telescopes; but since the head of the observer would intercept the whole of the incident light in small telescopes, and a great part of it in large ones, the axis of the reflector itself is turned a little obliquely, so as to throw the image aside, by which it can be viewed with little or no loss of light. The inconvenience of this is a little distortion of the image, caused by the obliquity of the rays;

but as such telescopes are only used of a great size, and for the purpose of viewing very faint celestial objects, in which the light diffused by aberration is insensible, little or no inconvenience is found to arise from this cause. Such is the construction of the telescopes used by Sir William Herschel in his sweeps of the heavens.

Simplest, or Herschelian construction.

To obviate the inconvenience of the stoppage of rays by the head, Newton, the inventor of reflecting telescopes, employed a small mirror, placed obliquely, as in fig. 83, opposite the centre of the large one. Thus parallel rays P A, P B, emanating from a point in the axis of the telescope, are received, before their meeting, on a plane mirror C D inclined at 45° to the axis, and thence reflected through a tube projecting from the side of the telescope to the lens G, and by it refracted to the eye E. It is manifest, that if the image formed by the mirror A B behind C D be regarded as an object, an image equal and similar to it (Art. 335) will be formed at F, at an equal distance from the plane mirror; and this image will be seen through the glass G, just as if it were formed by an object-glass of the same focal length placed in the prolongation of the axis of the eye-tube, beyond the small mirror, (supposed away.) Hence the same propositions and formulæ will hold good in the Newtonian telescope, as in the astronomical and Galilæan, for the magnifying power, field of view, and position of the eye, substituting only 2 R for L, and 2 R − D for L − D, and recollecting that R is negative, as the mirror has its concavity turned towards the light.

391. Newtonian construction.

392. Gregorian telescope. Fig. 84.

The Gregorian telescope, instead of a small plain mirror turned obliquely, has a small convex mirror with its concavity turned towards that of the large one, as in fig. 84; but instead of being placed at a distance from the large one equal to the sum of the focal lengths, the distance is somewhat greater; hence the image $p\,q$, formed in the focus of the great mirror, being at a distance from the vertex of the small one greater than its focal length, another image is formed at a distance, viz. at or near the surface of the great mirror, at $r\,s$. In the centre of the large mirror there is a hole which lets pass the rays to an eye-lens g. The adjustment to parallel or diverging rays, or for imperfect eyes, is performed by an alteration of the distance between the mirrors made by a screw.

393. Cassegrainian.

The Cassegrainian construction differs in no respect from the Gregorian, except that the small mirror is convex and receives the rays before their convergence to form an image. The magnitude of the field, the distance of the eye and of the mirrors from each other, are easily expressed in these constructions; the latter being derived from the former by a mere change of sign in the curvature of the small mirror. Let then R′ and R″ be the curvatures of the two mirrors, then in the Gregorian telescope R′ is negative and R″ positive; and if we put t for the distance between their surfaces, (t being negative, because the second reflecting surface lies towards the incident light) we shall have for an object whose proximity is D

$$D' = D; \qquad f' = 2\,R' - D' = 2\,R' - D; \qquad f'' = 2\,R'' - D''; \qquad D'' - \frac{f'}{1 - f'\,t'},$$

adopting the formulæ and notation of Art. 251. Now these give, by substitution,

$$D'' = \frac{2\,R' - D}{1 - t\,(2\,R' - D)}; \qquad f'' = 2\,R'' - \frac{2\,R' - D}{1 - t\,(2\,R' - D)} =$$

$$= \frac{2\,R'' - 2\,R' + D - 2\,t\,(2\,R' - D)\,.\,R''}{1 - t\,(2\,R' - D)}. \qquad (f)$$

This is the reciprocal distance of the second image from the second reflecting surface. If we wish that the image to be viewed by the eye-lens should fall *just* on the surface of the large mirror, we have only to put $f'' = \frac{1}{-t}$ (because f'' is positive, and t negative.) For parallel rays this gives

$$R'\,R''\,.\,t^2 + (4\,R' - 2\,R'')\,t - 1 = o; \qquad (g)$$

whence t may be found when R′ and R″ are given, or *vice versâ*.

394. The description of other optical instruments, and of the more refined construction of telescopes, &c. must be deferred till we are farther advanced in our account of the physical properties of light, and especially of the different refrangibility of its rays and their colours, which will form the object of the next part.

PART II.

CHROMATICS.

§ I. *Of the Dispersion of Light.*

HITHERTO we have regarded the refractive index of a medium as a quantity absolutely given and the same for all rays refracted by the medium. In nature, however, the case is otherwise. When a ray of light falls obliquely on the surface of a refracting medium, it is not refracted entirely in one direction, but undergoes a separation into several rays, and is *dispersed* over an angle more or less considerable, according to the nature of the medium and the obliquity of incidence. Thus if a sunbeam S C be incident on the refracting surface A B, and be afterwards received on a screen R V, (fig. 85,) it will, instead of a single point on the screen as R, illuminate a space R V *of a greater extent* the *greater* is the angle of incidence. The ray S C, then, which, before refraction was single, is separated into an infinite number of rays C R, C O, C Y, &c. each of which is refracted differently from all the rest. 395 General phenomenon of separation of a ray into colours. Fig. 85.

The several rays of which the dispersed beam consists, are found to differ essentially from each other, and from the incident beam, in a most important physical character. They are of different colours. The light of the sun is white. If a sunbeam be received directly on a piece of paper, it makes on it a white spot; but if a piece of white paper (that is, such as by ordinary daylight appears white) be held in the dispersed beam, as R V, the illuminated portion will be seen to be differently coloured in different parts, according to a regular succession of tints, which is always the same, whatever be the refracting medium employed. 396.

To make the experiment in the most striking and satisfactory manner, procure a triangular prism of good flint-glass, and having darkened a room, admit a sunbeam through a small round hole O P in the window shutter. If this be received on a white screen D at a distance, there will be formed a round white spot, or image of the sun, which will be larger as the paper is farther removed. Now in the beam before the screen place the prism A B C, having one of its angles C downwards and parallel to the horizon, and at right angles to the direction of the sunbeam, and let the beam fall on one of its sides B C obliquely. It will be refracted and turned out of its course, and thrown upwards, pursuing the course F G R, and may be received on a screen E properly placed. But on this screen there will no longer be seen a white round spot, but a long streak, or, as it is called in Optics, *a spectrum* R V of most vivid colours, (provided the admitted sunbeam be not too large, and the distance of the screen from the prism considerable.) The tint of the lower or *least refracted* extremity R is a brilliant red, more full and vivid than can be produced by any other means, or than the colour of any natural substance. This dies away first into an orange, and this passes by imperceptible gradations into a fine pale straw-yellow, which is quickly succeeded by a pure and very intense green, which again passes into a blue, at first of less purity, being mixed with green, but afterwards, as we trace it upwards, deepening to the purest indigo. Meanwhile, the intensity of the illumination is diminishing, and in the upper portion of the indigo tint is very feeble, but it is continued still beyond, and the blue acquires a pallid cast of purplish red, a livid hue more easily seen than described, and which, though not to be exactly matched by any natural colour, approaches most nearly to that of a fading violet: "*tinctus viola pallor.*" 397. Fig. 86.

If the screen on which the spectrum be received have a small hole in it, too small to allow the whole of the spectrum to pass, but only a very narrow portion of it, as X, (fig. 87,) the portion of the beam which goes to form that particular spot X may be received on another screen at any distance behind it, and will there form a spot *d* of the very same colour as the part X of the spectrum. Thus if X be placed in the red part of the spectrum the spot *d* will be red; if in the green, green; and in the blue, blue. If the eye be placed at *d*, it will see through the hole an image of the sun of dazzling brightness; not, as usually, white, but of the colour which goes to form the spot X of the spectrum. Thus we see, that the joint action of all the rays is not essential to the production of the coloured appearance of the spectrum, but that one colour may be insulated from the rest, and examined separately. 398. Insulation of each colour.

If, instead of receiving the ray X *d*, transmitted through the hole X, on a screen immediately behind it, it be intercepted by another prism *a c b*, it will be refracted, and bent from its course, as in X *f g x*; and after this second refraction may be received on a screen *e*. But it is now observed to be no longer separated into a coloured spectrum like the original one R V, of which it formed a part. A single spot *x* only is seen on the screen, the colour of which is uniform, and precisely that which the part X of the spectrum would have had, were it intercepted on the first screen. It appears, then, that the ray which goes to form any single point of the spectrum is not only independent of all the rest, but having been once insulated from them, is no longer capable of further separation into different colours by a second refraction. 399. Second refraction producing no further change of colour.

This simple, but instructive experiment, then, makes us acquainted with the following properties of light: 400.

Light. Part II.

1. A beam of white light consists of a great and almost infinite variety of rays differing from each other in colour and refrangibility.

Rays of light differ in refrangibility.

For the ray S F from any one point of the sun's disc, which if received immediately on the screen would have occupied only a single point on it, or (supposing the hole in the screen to have a sensible diameter) only a space equal to its area, is dilated into a line V R of considerable length, every point of which (speaking loosely) is illuminated. Now the rays which go to V must necessarily have been more refracted than those which go to R, which can only have been in virtue of a peculiar quality in the rays themselves, since the refracting medium is the same for all.

401. 2. White light may be *decomposed*, *analyzed*, or separated into its elementary coloured rays by refraction. The act of such separation is called the *dispersion* of the coloured rays.

402. 3. Each elementary ray once separated and insulated from the rest, is incapable of further decomposition or analysis by the same means. For we may place a third, and a fourth, prism in the way of the twice refracted ray $g\,x$, and refract it in any way, or in any plane; it remains undispersed, and preserves its colour quite unaltered.

403. 4. The dispersion of the coloured rays takes place in the plane of the refraction; for it is found that the spectrum V R is always elongated in this plane. Its breadth is found, on the other hand, by measurement, to be precisely the same as that of the white image D, (fig. 86,) of the sun, received on a screen at a distance O D from the hole, equal to O F + F G + G R, the whole course of the refracted light, which shows that the beam has undergone no contraction or dilation by the effect of refraction in a plane perpendicular to the plane of refraction.

404. Index of refraction regarded as variable.

To explain all the phenomena of the colours produced by prismatic dispersion, or of the *prismatic* colours, as they are called, we need only suppose, with Newton, that each particular ray of light, in undergoing refraction at the surface of a given medium, has the sine of its angle of incidence to that of refraction in a constant ratio, so long as the medium and the ray are the same; but that this ratio varies not only, as we have hitherto all along assumed, with the nature of the medium, but also with that of the ray. In other words, that there are as many distinct species, or at least varieties of light, as there are distinct illuminated points in the spectrum into which a single ray of white light is dispersed. This amounts to regarding the quantity μ, for any medium, not as one and invariable, but as susceptible of all degrees of magnitude between certain limits: one, the least of which, corresponds to the extreme, or least refracted red ray; the other, the greatest value of μ, to the extreme or most refracted violet. Each of these varieties separately conforms to the laws of reflexion and refraction we have already laid down. As in Geometry we may regard a whole family of curves as comprehended under one equation, by the variation of a constant parameter; so in Optics we may include under one analysis all the doctrine of the reflexions, refractions, and other modifications of a ray of white or compound light, by regarding the refractive index μ as a variable parameter.

405. To apply this, for instance, to the experiment of the prism just related: A single ray of white light being supposed incident on the first surface, must be regarded as consisting of an infinite number of coincident rays, of all possible degrees of refrangibility between certain limits, any one of which may be indifferently expressed by the refractive index μ. Supposing the prism placed so as to receive the incident ray perpendicularly on one surface, then the deviation will be given by the equation

$$\mu \,.\, \sin \mathrm{I} = \sin (\mathrm{I} + \mathrm{D})$$

I being the refracting angle of the prism. D therefore is a function of μ, and if μ vary by the infinitely small increment $\delta\mu$, *i. e.* if we pass from any one ray in the spectrum to the consecutive ray, D will vary by δ D, and the relation between these simultaneous changes will be given by the equation resulting from the differentiation of the above with the characteristic δ: thus we get

$$\delta\mu \,.\, \sin \mathrm{I} = \delta \mathrm{D} \,.\, \cos (\mathrm{I} + \mathrm{D}); \qquad \delta \mathrm{D} = \delta\mu \,.\, \frac{\sin \mathrm{I}}{\cos (\mathrm{I} + \mathrm{D})}. \qquad (a)$$

It is evident, then, that as μ varies, D also varies; and, therefore, that no two of the refracted and coloured rays will coincide, but will be spread over an angle, in the plane of refraction, the greater, the greater is the total variation of μ from one extreme to the other.

406. Analysis and synthesis of white light. Fig. 87. Fig. 88.

In order to justify the term *analysis*, or *decomposition*, as applied to the separation of a beam of white light into coloured rays, we must show by experiment that white light may be again produced by the *synthesis* of these elementary rays. The experiment is easy. Take two prisms A B C, $a\,b\,c$ of the same medium, and having equal refracting angles, and lay them very near together, having their edges turned opposite ways, as in fig. 87. With this disposition, a parallel beam of white light intromitted into the face A C of the first prism, will emerge from the face $b\,c$ of the last, undeviated, and colourless, as if no prisms were in the way. Now the dispersion having been fully completed by the prism A B C, the rays in passing through the thin lamina of air B C $a\,c$ must have existed in their coloured and independent state, and been dispersed in their directions; but being refracted by the second prism so as to emerge parallel, the colour is destroyed by the mixture and confusion of the rays. To see more clearly how this takes place in fig. 88, let S R and S V be two parallel white rays, incident on the first prism, and separated by refraction; the former into the coloured pencil R $c\,v$, the latter into a pencil exactly similar to V $r\,c$. Let R c be the *least* refracted ray of the former pencil, and V c the *most* refracted of the latter. These, of course, must meet; let them meet in c, and precisely at c apply the vertex of the second prism, having its side $c\,a$ parallel to C B, but its edge turned in the opposite direction; then will the rays R C and V c, each for itself, and independent of the other, be refracted so as to emerge parallel to its original direction

S R, S V, and the emergent rays will therefore be coincident and superimposed on each other as *c s*. Thus the emergent ray *c s* will contain an extreme red and an extreme violet ray. But it will also contain every intermediate variety; for draw *c f* anywhere between *c* R and *c* V. Then, since the angle which *c f* makes with the surface B C is greater than that made by the extreme violet ray C B, but less than that made by the extreme red, there must exist some value of μ intermediate between its extreme values, which will give a deviation equal to the angle between *c f* and S Y parallel to S R. Consequently, if S Y be a white ray, separated into the pencil Y *v' r'* by refraction, the coloured ray Y *f c* of that particular refrangibility will fall on *c*, and be refracted along *c s*. Every point then of the surface *g f h* will send to *c* a ray of different refrangibility, comprehending all the values of μ from the greatest to the least. Thus all the coloured elements, though all belonging originally to different white rays, will, after the second refraction, coincide in the ray *c s*, and experience proves that so reunited they form white light. White light, then, is re-composed when all the coloured elements, even though originally belonging to separate white rays, are united in place and direction.

407. In the reflexion of light, regarded as a case of refraction, μ has a specific numerical value, and cannot vary without subverting the fundamental law of reflexion. Thus, there is no dispersion into colours produced by reflexion, because all the coloured rays after reflexion pursue one and the same course. There is one exception to this, more apparent than real, when light is reflected from the base of a prism internally, of which more hereafter.

408. Synthesis of white light by a lens.

The recomposition of white from coloured light may be otherwise shown, by passing a small circular beam of solar light through a prism A B C, (fig. 89,) and receiving the dispersed beam on a lens E D at some distance. If a white screen be held behind the lens, and removed to a proper distance, the whole spectrum will be reunited in a spot of white light. The way in which this happens will be evident by considering the figure, in which T E and T D represent the parallel pencils of rays of any two colours (red and violet, for instance) into which the incident white beam S T is dispersed. These will be collected after refraction, each in its own proper focus; the former at F, the latter at G; after which each pencil diverges again, the former in the cone F H, the latter in G H. If the screen then be held at H, each of these pencils will paint on it a circle of its own colour, and so of course will all the intermediate ones; but these circles all coinciding, the circle H will contain all the rays of the spectrum confounded together, and it is found (with the exception of a trifling coloured fringe about the edges, arising from a slight overlapping of the several coloured images) to be of a pure whiteness.

409. All the rays necessary to produce white.

All natural colours imitable by combinations of the prismatic.

That the reunion of *all* the coloured rays is necessary to produce whiteness, may be shown by intercepting a portion of the spectrum before it falls on the lens. Thus, if the violet be intercepted, the white will acquire a tinge of yellow; if the blue and green be successively stopped, this yellow tinge will grow more and more ruddy, and pass through orange to scarlet and blood red. If, on the other hand, the red end of the spectrum be stopped, and more and more of the less refrangible portion thus successively abstracted from the beam, the white will pass first into pale and then to vivid green, blue-green, blue, and finally into violet. If the middle portion of the spectrum be intercepted, the remaining rays, concentrated, produce various shades of purple, crimson, or plum-colour, according to the portion by which it is thus rendered deficient from white light; and by varying the intercepted rays, any variety of colours may be produced; *nor is there any shade of colour in nature which may not thus be exactly imitated, with a brilliancy and richness surpassing that of any artificial colouring*.

Now, if we consider that all these shades are produced on white paper, which receives and reflects to our eyes whatever light happens to fall on it; and that the same paper placed successively in the red, green, and blue portion of the spectrum, will appear indifferently red, or green, or blue, we are naturally enough led to conclude, that

410. Colours not inherent in bodies.

The colours of natural bodies are not qualities inherent in the bodies themselves, by which they immediately affect our sense, but are mere consequences of that peculiar disposition of the particles of each body, by which it is enabled more copiously to reflect the rays of one particular colour, and to transmit, or stifle, or, as it is called in Optics, absorb the others.

411. Proved by experiment.

Such is the Newtonian doctrine of the origin of colours. Every phenomenon of optics conspires to prove its justice. Perhaps the most direct and satisfactory proof of it is to be found in the simple fact, that every body, indifferently, whatever be its colour in white light, when exposed in the prismatic spectrum, appears of the colour appropriate to that part of the spectrum in which it is placed; but that its tint is incomparably more vivid and full when laid in a ray of a tint analogous to its hue in white light, than in any other. For example, vermillion placed in the red rays appears of the most vivid red; in the orange, orange; in the yellow, yellow, but less bright. In the green rays, it is green; but from the great inaptitude of vermillion to reflect green light, it appears dark and dull; still more so in the blue; and in the indigo and violet it is almost completely black. On the other hand, a piece of dark blue paper, or Prussian blue, in the indigo rays has an extraordinary richness and depth of blue colour. In the green its hue is green, but much less intense; while in the red rays it is almost entirely black. Such are the phenomena of pure and intense colours; but bodies of mixed tints, as pink or yellow paper, or any of the lighter shades of blue, green, brown, &c., when placed in any of the prismatic rays, reflect them in abundance, and appear, for the time, of the colour of the ray in which they are placed.

412. Precautions to insure the perfect homogeneity of a ray.

Refraction by a prism affords us the means of separating a ray of white light into the rays of different refrangibility of which it consists, or of analyzing it. But to make the analysis complete, and to insulate a ray of any particular refrangibility in a state of perfect purity, several precautions are required, the chief of which are as follows: 1st. The beam of light to be analyzed must be very small, as nearly as possible approaching to a mathematical ray; for if A B, *a b* be a beam of parallel rays of any sensible breadth (fig. 89) incident on the prism P, the extreme rays A B, *a b* will each be separated by refraction into spectra G B H and *g b h*: B G, *b g* being the violet, and B H, *b h* the red rays of each respectively; and since A B, *a b* are parallel, therefore C G Fig. 89.

Light. 1st. Smallness of the incident pencil.

and cg will be so, and also D H and dh. Hence the red ray D H from B will intersect the violet cg from b, in some point F behind the prism; and a screen E F f placed at F will have the point F illuminated by a red ray from B, and a violet one from b; and therefore (as is easily seen) by all the rays intermediate between the red and violet, from points between B and b. F therefore will be white. If the screen be placed nearer the prism than F, as at K L kl, it is clear that from any point between L and k lines drawn parallel to K C, D L, to any intermediate direction, will fall between C and c, D and d, &c., respectively; and therefore that every point between L and k will receive from some point or other of the surface C d of the prism a ray of each colour, and will therefore be white. Again, any point as x between k and l can receive no violet ray, nor any ray of the spectrum whose angle of deviation is greater than $180° - abx$; for such ray to reach x must come from a part of the prism below b, which is contrary to the supposition of a limited beam A B, ab; but all rays whose angle of deviation is less than $180° - abx$, will reach x from some part or other of the surface B D. Hence the colour of the portion kl of the image on the screen will be white at k, pure red at l, and intermediate between white and red, or a mixture of the *least* refrangible rays of the spectrum at any intermediate point; and, in the same manner, the portion K L will be white at L, violet at K, and at any intermediate point will have a colour formed by a mixture of a greater or less portion of the more refrangible end of the spectrum. If the screen be removed beyond F, as into the situation G g H h, the white portion will disappear, no point between g and H being capable of receiving any ray whose angle of deviation is between $180° - abg$ and $180 - abH$. We may regard the whole image G h as consisting of an infinite number of spectra formed by every elementary ray of which the beam A B ab is composed, overlapping each other, so that the end of each in succession projects beyond that of the foregoing. The fewer, therefore, there are of these overlapping spectra, or the smaller the breadth of the incident beam, the less will be the mixture of rays so arising, and the purer the colours. Removal of the screen to a greater distance from the prism, evidently produces the same effect as diminution of the size of the beam; for while each colour occupies constantly the same space on the screen (for G g = K k) the whole spectrum is diffused over a larger space as the screen is removed, by the divergence of its component rays of different colours, and therefore the individual colours must of necessity be continually more and more separated from each other.

413. 2nd. Small angular divergence of the pencil. Fig. 90.

2ndly. Another source of confusion and want of perfect homogeneity in the colours of the spectrum is the angular diameter of the sun or other luminary, even when the aperture through which the beam is admitted is ever so much diminished. For let S T (fig. 90) be the sun, whose rays are admitted to the prism A B C through a very small hole O in a screen placed close to it. The beam will be dilated by refraction into the spectrum vr. Now, if we consider only the rays of one particular kind, as the red, and regard all the rest as suppressed, it is clear that a red image r of the sun will be formed by them alone on the screen; the rays from every point of the sun's disc crossing at O, and pursuing (after refraction) different courses. If the prism be placed in its situation of minimum deviation, which at present we will suppose, this image will be a circle, and it and the sun will subtend equal angles at O. In like manner, the violet rays (considered apart from the red) will form a circular violet image of the sun, at v, by reason of their greater refrangibility; and every species of ray, of intermediate refrangibility, will form, in like manner, a circular image between r and v. The constitution of the spectrum so arising will therefore be as in fig. 91, a, being an assemblage of images of every possible refrangibility superposed on and overlapping each other. Now, if we diminish the angular diameter of the sun or luminary, each of these images will be proportionally diminished in size; but their number, and the whole extent over which they are spread, will remain the same. They will therefore overlap less and less, (as in fig. 91, b, c;) and if the luminary be conceived reduced to a mere point (as a star) the spectrum will consist of a line d composed of an infinite number of mathematical points, each of a perfectly pure homogeneous light.

Fig. 91.

414. Experimental methods of obtaining homogeneous prismatic rays. Fig. 7.

There are several ways by which the angular diameter, or the degree of divergence of the incident beam may be diminished. Thus, first, we may admit a sunbeam through a small hole, as A, in a screen, and receive the divergent cone of rays behind it on another screen B, (fig. 7,) at a considerable distance, having another small hole B to let pass, not the whole, but only a small portion of the sun's image. The beam B C, so transmitted, will manifestly have a degree of divergence less than that of the beam immediately transmitted from A in the proportion of the diameter of the aperture B to the diameter of the sun's image on the screen B.

415. Fig. 92.

Another and much more commodious method is to substitute for the sun its image formed in the focus of a convex lens of short focus. This image is of very small dimensions, its diameter being equal to focal length of the lens × sine of sun's angular diameter, (or sine of 30′, which is about one 114th part of radius,) so that a lens of an inch focus concentrates all the rays which fall on it within a circle of about the 114th of an inch in diameter, which, for this purpose, may be regarded as a physical point. The disposition of the apparatus is as represented in fig. 92. The rays converged by the lens L to F, afterwards diverge as if they emanated from an intensely bright luminous point placed at F, and a screen with a small aperture Q being placed at a distance from it, and close behind it the prism A B C, the spectrum rv may be received on a screen again placed at a considerable distance behind the prism, each of whose points will be illuminated by rays of a very high degree of purity and homogeneity, and by diminishing the focal length of the lens, and the aperture O, and increasing the distance F O, or O r, this may be carried to any extent we please. It should, however, be remarked, that the intensity of the purified ray, and the quantity of homogeneous light so obtained, are diminished in the same ratio as the purity of the ray is increased.

416. Fig. 93.

A third method of obtaining a homogeneous beam is to repeat the process of analysis on a ray as nearly pure as can be conveniently obtained by refraction through a single prism. Thus, in fig. 93, V R, the spectrum formed by a first refraction at the prism A, is received on a screen which intercepts the whole of it, except that particular colour we wish to insulate and purify, which is allowed to pass through an aperture M N; behind this is placed another prism B, so as to refract this beam a second time. If then the portion

M N were already perfectly pure, it would pass the second prism without undergoing any further separation; but if there be (as there always will) other rays mixed with it, these will be dilated by the subsequent refraction into a second spectrum *v r* of faint light, with a much brighter portion *m n* in the midst; and if the rest of the rays be intercepted, and this portion only allowed to pass through an aperture, the emergent beam *m p* will be much more homogeneous than before its incidence on the second prism,—and in proportion as the distance between the second prism and the screen is increased, the purity of the ray obtained will be greater.

417. Imperfection of prisms, how eluded in practice.

Another source of impurity in the prismatic rays is the imperfection of the materials of our ordinary prisms, which are full of striæ and veins, which disperse the light irregularly, and thus confound together in the spectrum rays which properly belong to different parts of it. Those who are not fortunate enough to possess glass prisms free from this defect (which are very rare, and indeed hardly to be procured for any price) may obviate the inconvenience by employing hollow prisms full of water, or, rather, any of the more dispersive oils. A great part of the inconvenience arising from a bad prism may, however, be avoided by transmitting the rays *as near the edge of it as possible*, so as to diminish the quantity of the material they have to pass through, and therefore their chance of encountering veins and striæ in their passage.

418. Fixed lines in the spectrum.

When every care is taken to obtain a pure spectrum; when the divergence of the incident beam is extremely small, and its dimensions also greatly reduced; when the prism is perfect, and the spectrum sufficiently elongated to allow of a minute examination of its several parts, some very extraordinary facts have been observed respecting its constitution. They were first noticed by Dr. Wollaston, in a Paper published by him in the *Phil. Trans.*, 1802; and have since been examined in full detail, and with every delicacy and refinement which the highest talents and the most unlimited command of instrumental aids could afford, by the admirable and ever-to-be-lamented Fraunhofer. It does not appear that the latter had any knowledge of Dr. Wollaston's previous discovery, so that he has, in this respect, the full merit of an independent inventor. The facts are these: The solar spectrum, in its utmost possible state of purity and tenuity, when received on a white screen, or when viewed by admitting it at once into the eye, is not an uninterrupted line of light, red at one end and violet at the other, and shading away by insensible gradations through every intermediate tint from one to the other, as Newton conceived it to be, and as a cursory view shows it. It is interrupted by intervals absolutely dark; and in those parts where it is luminous, the intensity of the light is extremely irregular and capricious, and apparently subject to no law, or to one of the utmost complexity. In consequence, if we view a spectrum formed by a narrow line of light parallel to the refracting edge of the prism, (which affords a considerable breadth of spectrum without impairing the purity of the colours, being, in fact, an assemblage of infinitely narrow linear spectra arranged side by side,) instead of a luminous fascia of equable light and graduating colours, it presents the appearance of a striped riband, being crossed in the direction of its breadth by an infinite multitude of dark, and by some totally black bands, distributed irregularly throughout its whole extent. This irregularity, however, is not a consequence of any casual circumstances. The bands are constantly in the same parts of the spectrum, and preserve the same order and relations to each other; the same proportional breadth and degree of obscurity, whenever and however they are examined, *provided solar light be used*, and provided the prisms employed be composed of the same material: for a difference in the latter particular, though it causes no change in the number, order, or intensity of the bands, or their places in the spectrum, as referred to the several colours of which it consists, yet causes a variation in their proportional distances *inter se*, of which more hereafter. By solar light must be understood, not merely the direct rays of the sun, but any rays which have the sun for their ultimate origin; the light of the clouds, or sky, for instance; of the rainbow; of the moon, or of the planets. All these lights, when analyzed by the prism, are found deficient in the identical rays which are wanting in the solar spectrum; and the deficiency is marked by the same phenomenon, viz. by the occurrence of the same dark bands in the same situations in spectra formed by these several lights. In the light of the stars, on the other hand, in electric light, and that of flames, though similar bands are observed in their spectra, yet they are differently disposed; and the spectrum of each several star, and each flame, has a system of bands peculiar to itself, and characteristic of its light, which it preserves unalterably at all times, and under all circumstances.

419. Fig. 94.

Fig. 94 is a representation of the solar spectrum as laid down minutely by Fraunhofer, from micrometrical measurement, and as formed by a prism of his own incomparable flint glass. Only the great number of small bands observed by him (upwards of 500 in number) have been omitted, to avoid confusing the figure. Of these bands, or, as he terms them, "fixed lines" in the spectrum, he has selected seven, (those marked B, C, D, E, F, G, H,) as terms of comparison, or as standard points of reference in the spectrum, on account of their distinctness, and the facility with which they may be recognised. Of these, B lies in the red portion of the spectrum, near the end; C is farther advanced in the red; D lies in the orange, and is a strong double line easily recognised; E is in the green; F in the blue; G in the indigo; and H in the violet. Besides these, there are others very remarkable; thus *b* is a triple line in the green, between E and F, consisting of three strong lines, of which two are nearer each other than the third, &c.

420. Utility of the fixed lines in optical determinations.

The definiteness of these lines, and their fixed position, with respect to the colours of the spectrum,—in other words, the precision of the limits of those degrees of refrangibility which belong to the *deficient* rays of solar light,—renders them invaluable in optical inquiries, and enables us to give a precision hitherto unheard of to optical measurements, and to place the determination of the refractive powers of media on the several rays almost on the same footing, with respect to exactness, with astronomical observations. Fraunhofer, in his various essays, has made excellent use of them in this respect, as we shall soon have occasion to see.

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To see these phenomena, we must place the refracting angle of a very perfect prism parallel to a very small linear opening through which a sunbeam is admitted; or, in place of an opening, we may employ a glass cylinder, or semi-cylinder of small radius, to bring the rays to a linear focus behind and parallel to it, from

Light. Part II.

First method of exhibiting the fixed lines.

which the rays diverge, as from a fine luminous line, in the manner described in Art. 415 for a lens. If now the eye be applied close behind the prism, the line will be seen dilated into a broad coloured band, consisting of the prismatic colours in their order; and if the prism be good, and carefully placed in its situation of minimum deviation, and of sufficiently large refracting angle to give a broad spectrum, some of the more remarkable of the fixed lines will be seen arranged parallel to the edges of the spectrum, especially the lines D and F, the former of which appears, in this way of viewing it, to form a separation between the red and the yellow. If the light of the sun be too bright, so as to dazzle the eye, any narrow line of common daylight (as the slit between two nearly closed window-shutters) may be substituted. This was the mode in which the fixed lines were first discovered by Dr. Wollaston.

422. Second method.

But it is difficult and requires acute sight to perceive, in this manner, any but the most conspicuous lines. The reason is, their very small angular breadth; which, in the largest of them, can scarcely, under any circumstances, exceed half a minute, and in the smaller not more than a few seconds. They require, therefore, to be magnified. This may be done by a telescope interposed between the eye and the prism, in the manner represented in fig. 95, in which $L\,l$ is the line of light, from which rays, diverging in all directions, fall on the prism A B C, are refracted by it, and after refraction are received on the object-glass D of the telescope. This object-glass, it should be observed, must be of that kind denominated *achromatic*, to be presently described, (see Index,) and of which it need only be here said, that it is so constructed as to be capable of bringing rays of all colours to foci at one and the same distance from the glass. Now, if we consider only rays of any one degree of refrangibility (the extreme red, for instance) the pencils diverging from every point of $L\,l$ will, after refraction at the two surfaces of the prism, diverge from corresponding points of an image $L'\,l'$ situated in the direction from the base towards the vertex of the prism. Rays of any greater refrangibility will, after refraction at the prism, diverge from a linear image $L''\,l''$ parallel to $L'\,l'$, but farther from the original line $L\,l$. Thus the white line $L\,l$ will, after refraction at the prism, have for its image the coloured rectangle $L'\,L''\,l'\,l''$, which will be viewed through the telescope as if it were a real object. Now every vertical line of this parallelogram will form in the focus of the object-glass a corresponding vertical image of its own colour; and the object-glass being achromatic, all these images are equidistant from it, so that the whole image of the parallelogram $L'\,l''$ will be a similar coloured parallelogram, having its plane perpendicular to the axis of the telescope. This will be viewed as a real object through the eye-glass, and the spectrum will thus be magnified as any other object would be, according to the power of the telescope, (Art. 382.) With this disposition of the apparatus (which is that employed by Fraunhofer) the fixed lines are beautifully exhibited, and (if the prism be perfect) may be magnified to any extent. The slightest defect of homogeneity in the prism, however, as may be readily imagined, is fatal. With glass prisms of our manufacture it would be quite useless to attempt the experiment; and those who would repeat it in this country should employ prisms of highly refractive liquids, enclosed in hollow prisms of good plate glass. The eye-pieces of telescopes, not being usually achromatic, a slight change of focus is still required, when the lines in the red and violet portions of the spectrum are to be viewed. This (if an inconvenience) might be obviated by the use of an achromatic eye-piece.

Fig. 95.

423. Third method.

That an actual image of the spectrum, with its fixed lines, is really formed in the focus of the object-glass, as described, may be easily shown, by dismounting the telescope, and receiving the rays refracted by the object-glass on a screen in its focus. This, indeed, affords a peculiarly elegant and satisfactory mode of exhibiting the phenomena to several persons at once. An achromatic object-glass of considerable focal length (6 feet, for instance) should be placed at about twice its focal length from the line of light, and (the prism being placed immediately before the glass) the image will be formed at about the same distance, 12 feet behind it, ($f = L + D$; $L = \frac{1}{6}$; $D = -\frac{1}{12}$; $f = \frac{1}{6} - \frac{1}{12} = +\frac{1}{12}$) and being received on a screen of white paper or emeried glass may be examined at leisure, and the distances of the lines from each other, &c. measured on a scale. But by far the best methods of performing these measurements are those practised by Fraunhofer, viz. the adaptation of a micrometer to the eye-end of the telescope, (~~see *Micrometer*, in a subsequent part of this Article,~~) for ascertaining the distances of the closer lines; and the giving the axis of the telescope, together with the prism which is connected with it, a motion of rotation in a horizontal plane, the extent of which is read off by verniers and microscopes on an accurately graduated circle, in the same way as in astronomical observations. The apparatus employed by him for this purpose, and which is applicable to a variety of useful purposes in optical researches, is represented in fig. 96.

Fig. 96.

424. Colours of the spectrum.

The fixed lines in the spectrum do not mark any precise limits between the different colours of which it consists. According to Dr. Wollaston, (*Phil. Trans.*, 1802,) the spectrum consists of only four colours, red, green, blue, and violet; and he considers the narrow line of yellow visible in it in his mode of examination already described (looking through a prism at a narrow line of light with the naked eye) as arising from a mixture of red and green. These colours, too, he conceives to be well defined in the spaces they occupy, not graduating insensibly into each other, and of, sensibly, the same tint throughout their whole extent. We confess we have never been able quite satisfactorily to verify this last observation, and in the experiments of Fraunhofer, (which we had the good fortune to witness, as exhibited by himself at Munich,) where, from the perfect distinctness of the finest lines in the spectrum, all idea of confusion of vision, or intermixture of rays is precluded, the tints are seen to pass into each other by a perfectly insensible gradation; and the same thing may be noticed in the coloured representations of the spectrum published in the first essay of that eminent artist, and executed by himself with extraordinary pains and fidelity. The existence of a pale straw yellow, not of mere linear breadth, but occupying a very sensible space in the spectrum, is there very conspicuous, and may also be satisfactorily shown by other experiments to be hereafter described, when we come to speak of the absorption of light. In short, (with the exception of the fixed lines, which Newton's instrumental means did not enable him to see,) the spectrum is, what that illustrious philosopher originally described it, a graduated succession of tints, in which all

the seven colours he enumerates can be distinctly recognised, but shading so far insensibly into each other that a positive limit between them can be nowhere fixed upon. Whether these colours be really compound or not, whether some other mode of analysis may not effect a separation depending on some other fundamental difference between the rays than that of the degree of their refrangibility, is quite another question, and will be considered more at large hereafter. At present it may be enough to remark, that all probability, drawn from everyday experience, is in favour of this idea, and leads us to believe that orange, green, and violet are mixed colours; and red, yellow, and blue, original ones; the former we everyday see imitated by mixtures of the latter, but never *vice versâ*. This doctrine has been accordingly maintained by Mayer, in a curious Tract published among his works. (See the Catalogue of Optical Writers at the end of this Article.) A very different doctrine has, however, been advanced by Dr. Young, (*Lectures on Natural Philosophy*, i. 441,) in which he assumes red, green, and violet, as the fundamental colours. The respective merits of these systems will be considered more at large hereafter. (See Index, Composition of Colours.)

425. Media differ in dispersive power.

Media, as we have seen, differ very greatly in their refractive power, or in the degree in which prisms of one and the same refracting angle composed of different substances, deflect the rays of light. This was known to the optical philosophers who preceded Newton. This great man, on establishing the general fact, that one and the same medium refracts differently the differently coloured rays, might naturally have been led to inquire experimentally whether the amount of this difference of action were the same for all media. He appears to have been misled by an accidental circumstance in the conduct of an experiment, in which the varieties of media in this respect ought to have struck him,* and in consequence adopted the mistaken idea of a *proportional action* of all media on the several homogeneous rays. Mr. Hall, a gentleman of Worcestershire, was the first to discover Newton's mistake; and having ascertained the fact, of the different *dispersive powers* of different kinds of glass, applied his discovery successfully to the construction of an achromatic telescope. His invention, however, was unaccountably suffered to fall into oblivion, (though it is said that he made several such telescopes, some of which still exist,) and the fact was re-discovered and re-applied to the same great purpose by Mr. Dollond, a celebrated optician in London, on the occasion of a discussion raised on the subject by some *à priori* and paradoxical opinions broached by Euler.

426. Differences of dispersion explained. Fig. 97. Fig. 98.

If a prism of flint glass and one of crown, of equal refracting angles, be presented to two rays of white light, as A B C, *a b c*, (fig. 97;) S C and *s c* being the incident rays, C R, C V the red and violet rays refracted by the flint, and *c r*, *c v* those refracted by the crown; it is observed, *first*, that the deviation produced in either the red or violet ray by the flint glass, is much greater than that produced by the crown; *secondly*, that the angle R C V, over which the coloured rays are dispersed by the flint prism, is also much greater than the angle *r c v*, over which they are dispersed by the crown; and, *thirdly*, that the angles R C V, *r c v*, or the *angles of dispersion*, are *not* to each other as Newton supposed them to be, in the same ratio with the *angles of deviation* T C R, *t c r*, but in a much higher ratio; the dispersion of the flint prism being much more than in proportion to the deviation produced by it. And if, instead of taking the angles of the prism equal, the refracting angle of the crown prism be so increased as to make the deviation of the red ray equal to that produced by the flint prism, the deviation of the violet will fall considerably short of such equality. In consequence of this, if the two prisms be placed close together, with their edges turned opposite ways, as in fig. 98, so as to oppose each other's action, the red ray, being equally refracted in opposite directions, will suffer no deviation; but the violet ray, being more refracted by the flint than by the crown prism, will, on the whole, be bent towards the thicker part of the flint prism, and thus an uncorrected colour will subsist, though the refraction (for one ray, at least) is corrected. *Vice versâ*, if the dispersion be corrected, that is, if the refracting angle of the crown prism, acting in opposition to the flint, be so further increased as to make the difference of the deviations of the red and violet rays produced by it equal to the difference of their deviations produced by the flint, the deviation produced by it will now be greater than that produced by the flint; and the total deviation, produced by both prisms acting together, will now be in favour of the crown.

427. Refraction without separation into colours.

By such a combination of two prisms of different media a ray of white light may therefore be turned aside considerably from its course, without being separated into its elementary coloured rays. It is manifest, that (supposing the angles of the prisms small, and that both are placed in their positions of minimum deviation) the deviations to produce this effect must be in the inverse ratio of the dispersive powers of the two media; for supposing μ, μ' to be the refractive indices of the prisms for extreme red rays, and $\mu + \delta\mu$, $\mu' + \delta\mu'$ for extreme violet, A and A′ their refracting angles, and D and D′ their deviations, we have, generally, in the position of minimum deviation

$$\left.\begin{aligned} \mu \,.\, \sin\frac{A}{2} &= \sin\frac{A+D}{2}, \text{ whence } \delta\mu \,.\, \sin\frac{A}{2} = \tfrac{1}{2}\delta D \,.\, \cos\frac{A+D}{2} \\ \mu' \,.\, \sin\frac{A'}{2} &= \sin\frac{A'+D'}{2} \qquad \delta\mu' \,.\, \sin\frac{A'}{2} = \tfrac{1}{2}\delta D' \,.\, \cos\frac{A'+D'}{2} \end{aligned}\right\};$$

whence, since the prisms oppose each other,

* He counteracted the refraction of a glass, by a water prism. There ought to have been a residuum of uncorrected colour; but, unluckily, he had mixed sugar of lead with the water to increase its refraction, and the high dispersive power of the salts of lead (of which, of course, he could not have the least suspicion) thus robbed him of one of the greatest discoveries in physical optics.

Light. Part II.

$$\tfrac{1}{2}\,\delta\,(D - D') = \frac{\delta\mu \,.\, \sin\frac{A}{2}}{\cos\left(\frac{A+D}{2}\right)} - \frac{\delta\mu' \,.\, \sin\frac{A'}{2}}{\cos\left(\frac{A'+D'}{2}\right)}.$$

Putting this equal to zero, we have

$$\frac{\delta\mu}{\delta\mu'} \cdot \frac{\sin\frac{1}{2}A}{\sin\frac{1}{2}A'} = \frac{\cos\frac{1}{2}(A+D)}{\cos\frac{1}{2}(A'+D')};$$

and, eliminating $\sin\frac{1}{2}A$ and $\sin\frac{1}{2}A'$ from this, by means of the two original equations from which we set out, we get

$$\frac{\delta\mu}{\delta\mu'} \times \frac{\mu'}{\mu} = \frac{\cos\frac{1}{2}(A+D)}{\cos\frac{1}{2}(A'+D')} \times \frac{\sin\frac{1}{2}(A'+D')}{\sin\frac{1}{2}(A+D)} = \frac{\tan\frac{1}{2}(A'+D')}{\tan\frac{1}{2}(A+D)}.$$

Now if we call p, p' the dispersive powers of the media, or the proportional parts of the whole refractions of the extreme red ray, to which the dispersion is equal, we shall have

$$p = \frac{\delta\mu}{\mu - 1} \quad p' = \frac{\delta\mu'}{\mu' - 1} \text{ and } \frac{p}{p'} = \frac{\delta\mu}{\delta\mu'} \times \frac{\mu' - 1}{\mu - 1},$$

so that

$$\frac{p}{p'} = \frac{\mu}{\mu'} \cdot \frac{\mu' - 1}{\mu - 1} \cdot \frac{\tan\frac{1}{2}(A'+D')}{\tan\frac{1}{2}(A+D)} = \frac{\mu' - 1}{\mu - 1} \cdot \frac{\sin\frac{1}{2}A'}{\sin\frac{1}{2}A} \cdot \sqrt{\frac{1 - \mu^2 \,.\, (\sin\frac{1}{2}A)^2}{1 - \mu'^2 \,.\, (\sin\frac{1}{2}A')^2}}. \qquad (a)$$

Such is the strict formula, which, when A and A′ are verv small, becomes

$$\frac{p}{p'} = \frac{(\mu' - 1)\,A'}{(\mu - 1)\,A}; \quad \text{or, since } (\mu - 1)\,A = D, \text{ and } (\mu' - 1)\,A' = D'; \qquad \frac{p}{p'} = \frac{D'}{D}.$$

Dispersive powers compared by experiment.

428. The formula just obtained, furnishes us with an experimental method of determining the ratio of the dispersive powers of two media. For if we can by any means succeed in forming them into two prisms of such refracting angles, that, when placed in their respective positions of minimum deviation, a well defined bright object, viewed through both, shall appear well defined and free from colour at its edges; then, by measuring their angles, and knowing also from other experiments their refractive indices, the equation (a) gives us immediately the ratio in question.

Coloured fringes bordering objects seen through prisms explained. Fig. 99.

429. When we view through a prism any well defined object, either much darker or much lighter than the ground against which it is seen projected, as, for instance, a window bar seen against the sky, its edges appear fringed with colours and ill defined. The reason of this may be explained as follows:

Let A B, fig. 99, be the section of a horizontal bar seen through the prism P held with its refracting edge downwards, and first let us consider what will be the appearance of the upper edge B of the object. Since we see by light, and not by darkness, the thing really seen is not the dark object, but the bright ground on which it stands, or the bright spaces B C, A D above and below. Now the bright space B C above the object being illuminated with white light, will, after refraction at the prism, form a succession of coloured images $b\,c$, $b'\,c'$, $b''\,c''$, &c., superposed on and overlapping each other. They are represented in the figure as at different distances from P, but this is only to keep them distinct. In reality, they must be supposed to lie upon and interfere with each other. The least refracted $b\,c$ of these is red, and the most refracted $b''\,c'''$ violet, and any intermediate one (as $b'\,c'$) of some intermediate colour, as yellow for instance. Beyond b'' no image exists, so that the whole space below b'' will appear dark to an eye situated behind the prism. On the other hand, above b the images of every colour in the spectrum coexist, the bright space $b\,c$ being supposed to extend indefinitely above B. Therefore the space above b in the refracted image will appear perfectly white. Between b and b'' there will be seen, first, a general diminution of light, as we proceed from b towards b'', because the number of superposed luminous images continually decreases; secondly, an excess in all this part, of the more refrangible rays in the spectrum above what is necessary to form white light, for beyond b no red image exists, beyond b' no yellow, and so on; the last which projects beyond all, at b'', being a pure unmixed violet. Thus the light will not only decrease in intensity, but by the successive subtraction of more and more of the less refrangible end of the spectrum will acquire a bluer and bluer tint, deepening to a pure violet, so that the upper edge of the dark object will appear fringed with a blue border, becoming paler and paler till it dies away into whiteness. The reverse will happen at the lower edge A. The bright space A D forms, in like manner, a succession of coloured images, $a\,d$, $a'\,d'$, $a''\,d''$, of which the least deviated $a\,d$ is red, the most $a''\,d''$ violet, and the intermediate ones of the intermediate colours. Therefore the point a, which contains only the extreme red, will appear of a sombre red; a', which contains all the rays from red to yellow (suppose), of a lively orange red; and in proportion as the other images belonging to the more refrangible end of the spectrum come in, this tendency to an excess of red will be neutralized, and the portion beyond a'', containing all the colours in their natural proportions, will be purely white. Hence, the lower edge of the dark object will appear bordered with a red fringe, whose tint fades away into whiteness, in the same way as the blue fringe which borders the upper edge. These fringes, of course, destroy the distinctness of the outlines of objects, and render vision through a prism confused. The confusion ceases, and objects resume their natural well defined outlines, if illuminated with homogeneous light, or if viewed through coloured glasses which transmit only homogeneous rays.

Light. The eye can judge pretty well, by practice, of the destruction of colour, and indistinctness in the edges of objects, when prisms are made to act in opposition to one another, as above described; but (owing to causes presently to be considered) the compensation is never perfect, and there always remains a small fringe of uncorrected purple on one side, and green on the other, when the eye is best satisfied; so that observations of dispersive powers by this method are liable to a certain extent of error, and, indeed, precision in this department of optical science is very difficult to obtain. Part II. 430.

To determine the dispersive power of a medium, having formed it into a prism, and measured by the goniometer, or otherwise, its refracting angle, and ascertained its refractive index, the next step is to find the refracting angle of a prism of some standard medium, which shall exactly compensate its dispersion, so as to produce a refraction as nearly as possible free from colour. But as it is impossible to have a series of standard prisms with every refracting angle which may be requisite, it becomes necessary to devise some means of varying the refracting angle of one and the same prism by insensible gradations. Many contrivances may be had recourse to for this. Thus, first, we may use a prism composed of two plates of parallel glass, united by a hinge, or otherwise, and enclosing between them a fluid, which may be prevented from escaping either by capillary attraction, if in very small quantity, or by close-fitting metallic cheeks, forming a wedge-shaped vessel, if in larger. This contrivance, however, is liable to a thousand inconveniencies in practice. Secondly, we may use two prisms of the same kind of glass, one of which has one of its faces ground into a convex, and the other into a concave cylinder, of equal curvatures, having their axes parallel to the refracting edges. These being applied to each other, and one of them being made to revolve round the common axes of the two cylindric surfaces upon the other, the plane faces will evidently be inclined to each other in every possible angle within the limits of the motion, (see fig. 100, *a*, *b*, exhibiting two varieties of this construction.) The idea, due, we believe, to Boscovich, is ingenious, but the execution difficult, and liable to great inaccuracies. 431. To determine the dispersion of a medium. Prisms with variable refracting angles described. Another construction Fig. 100.

The following method succeeds perfectly well, and we have found it very convenient in practice. Take a prism of good flint glass, whose section is a right angled triangle, A B C, having the angle A about 30° or 35°, C being the right angle, and whose length is twice the breadth of the side A C; and, having ground and polished the side A C, and the hypothenuse of the prism to true planes, cut it in half, so as to form two equal prisms with one face in each a square, and whose refracting angles (A, A′) cannot, of course, be otherwise than exactly equal. Cement the square faces together very carefully with mastic, so that the edges A, A′, shall be on opposite sides of the square surface, which is common to both; and then, making the whole solid to revolve round an axis perpendicular to the common surface, and passing through its centre, grind off all the angles of the squares in the lathe, and the whole will be formed into a cylindrical solid, with oblique, parallel, elliptical, plane ends, as in fig. 101. Then separate the prisms, (by warming the cement,) and set each of them in a separate brass mounting, as in fig. 102, so as to have their circular faces in contact, and capable of revolving freely upon each other about their common centre. The lower one is fixed in the centre of the divided circle D E, while the mounting of the upper or moveable one carries an arm with an adjustable vernier reading off to tenths of degrees, or, if necessary, to minutes. The whole apparatus is set in a swing frame between plates, which grasp the divided plate by a groove in its edge, allowing a motion in its own plane, and a capability of adjusting it to any required position, so as to admit of the compound prism deviating an incident ray in every possible plane, and under every possible situation, with respect to the faces of the prisms. It is evident, that in the position here represented, where the prisms oppose each other, (and at which the vernier must be set to read off zero,) the refracting angle is rigorously nothing; and when turned round 180°, since the prisms then conspire, their combined angle must be double that of each. In intermediate situations, the angle between the planes of their exterior faces must, of course, pass through every intermediate state, and (by spherical trigonometry) it is readily shown, that if θ be the reading off of the vernier, or the angle of rotation of the prisms on each other from the true zero, the angle of the compound prism will be had by the equation 432. Third construction. Fig. 101, 102.

$$\sin \frac{A}{2} = \sin \frac{\theta}{2} \cdot \sin (A) \qquad (b)$$

where (A) is the refracting angle of each of the simple prisms, and A the angle of the compound one.

To use this instrument, place the prism A′, whose dispersive power is to be compared with the medium of which the standard prism (A) is formed, with its edge downwards and horizontal, before a window, and, selecting one of the horizontal bars properly situated, fix it so that the refraction of this bar shall be a minimum, or till, on slightly inclining the prism backwards and forwards, the image of the bar appears stationary. Then take the standard compound prism, adjust it to zero, and set it vertically on its frame behind the first prism. Move its index a few degrees from zero, and turn the divided circle in its own plane, till the refraction so produced by the second prism is contrary to that produced by the first. The colour will be found less than before; continue this till the colour is nearly compensated, then, by means of the swing motion, and of the motion round the vertical axis, adjust the apparatus so that two of the window bars, a horizontal and a vertical one, seen through both prisms, shall appear to make a right angle with each other, (an adjustment, at first, rather puzzling, but which a little practice renders very easy.) Then complete the compensation of the colour; verify the position of the standard prism, (by the same test,) and finally read off the vernier, and the required angle A of the compensating prism is easily calculated by the equation (*b*). This calculation may be saved by tabulating the values of A corresponding to those of θ, (the value of (A) being supposed known by previous exact measures,) or, by graduating the divided circle at once, not into equal parts of θ, but according to such computed values of A, so as to read off at once the value of the angle required. 433. How used.

Light. Part II.

434. Another method proposed by Dr. Brewster.

A simpler, perhaps, on the whole, a better, method of comparing the dispersions of two prisms, is one proposed and applied extensively by Dr. Brewster, in his ingenious Treatise *On New Philosophical Instruments*, a work abounding with curious contrivances and happy adaptations. It consists in varying, not the refracting angle of the standard prism, but the direction in which its dispersion is performed. It is manifest, that if we can produce from a line of white light, by means of a standard prism any how disposed, a coloured fringe, in which the colours occupy the same angular breadth as in that produced by a prism of unknown dispersion; then, the latter, being made to refract this fringe *in a direction perpendicular to its breadth*, and *opposite to the order of its colours*, must destroy all colour and produce a compensated refraction; and therefore if the position of the standard prism which produces such a fringe be known, the dispersion of the other may be calculated. To accomplish this, let A B be a horizontal luminous line of considerable length, and let it be refracted *downwards*, but *obliquely* in the direction A a, B b, by a standard prism whose dispersion is greater than that of the prism to be measured. Then it will form an oblique spectrum $a\,b\,b'\,a'$, $a\,b$ being the red, and $a'\,b'$ the violet; and the angular breadth of this coloured fringe will be $a\,m = a\,a' \times$ sin inclination of the plane of refraction to the horizon. Now, let the prism whose dispersion is to be measured be made to refract this coloured band *vertically upwards*; then, if the plane of the first refraction be so inclined to the horizon that the angle subtended by $a\,m$ at the eye shall be just equal to the angle of dispersion of the other prism, all the colours of the rectangular portion $b\,c\,a'\,d$ will be made to coalesce in the horizontal line A′ B′, which will appear therefore free from colour, except at its extremities A′ B′, where the coloured triangles $a\,c\,a'$, $b\,d\,b'$ will produce a red termination A′ A″ and a blue one B′ B″ at the respective ends of the line to which they correspond. Hence, if, the second prism remaining fixed, with its edge downwards and parallel to the horizon, the other or standard prism be turned gradually round in the plane perpendicular to its principal section, a position must necessarily be found where the twice refracted line A′ B′ will appear free from colour both above and below. In this position let it be arrested, and the angle of inclination of its edge to the horizon read off, its complement is the angle $a\,a'\,m$, which we will call θ. Let us now suppose each prism adjusted to its position of minimum deviation, and (as it is a matter of indifference which is placed first) let the prism to be examined or the fixed prism be placed next the object.* Then, D′ and D being the total deviations produced by the fixed and revolving prisms on the extreme red ray, we must have

Fig. 103.

$$\delta D' - \delta D \,.\, \sin\theta = o; \qquad \text{or } \delta\mu' \,.\, \sin\frac{A'}{2} \,.\, \sec\frac{A'+D'}{2} = \delta\mu \,.\, \sin\frac{A}{2} \,.\, \sec\frac{A+D}{2} \,.\, \sin\theta,$$

whence we obtain

$$\frac{p'}{p} = \frac{\delta\mu'}{\delta\mu} \cdot \frac{\mu-1}{\mu'-1} = \frac{\mu'}{\mu} \cdot \frac{\mu-1}{\mu'-1} \cdot \frac{\tan\frac{1}{2}(A+D)}{\tan\frac{1}{2}(A'+D')} \,.\, \sin\theta; \qquad (c)$$

where the angles $\frac{1}{2}(A+D)$ and $\frac{1}{2}(A'+D')$ are given by the equations

$$\sin\tfrac{1}{2}(A+D) = \mu \,.\, \sin \times \tfrac{1}{2}A; \qquad \sin\tfrac{1}{2}(A'+D') = \mu' \,.\, \sin\tfrac{1}{2}A';$$

from which formula, θ being known, and also the angles and refractive indices of the two prisms, the ratio of their dispersions is found.

435. Absolute dispersive powers, how obtained. 1st. By measuring the spectrum on a screen.

By these, or other similar methods, may the dispersions of any media be compared with those of any other taken as a standard. If the media be solid, they must be formed into prisms; if fluid, they must be enclosed in hollow prisms of truly parallel plates of glass, whose angles must be accurately determined, (and one of which will serve for any number of fluids.) But to ascertain directly the dispersion of that standard prism, we must pursue a different course. The first method which obviously presents itself, is to measure the actual length of the solar spectrum cast by a prism of given refracting angle; but the light of the spectrum dies away so indefinitely at both ends, and its visible extent varies so enormously with the brightness of the sun, and the more or less perfect exclusion of extraneous light, that nothing certain can be concluded from such measures. Yet, if the brighter rays of the spectrum be destroyed, and the eye defended from all offensive light by a glass which permits only the extreme red and violet rays to pass, (see Index, *Absorption*,) some degree of accuracy may be obtained by this means. A method founded on this principle has been described by the writer of these pages in the *Transactions of the Royal Society of Edinburgh*, vol. ix. as follows: Let A and B be two vertical rectangular slits in a screen placed before an open window, the one being half the length of the other, and at a known distance from each other. The eye being guarded as above described, let the slits be refracted by the prism (in its minimum position) from the longer towards the shorter. Then will a red and violet image of each a, b, and a', b' be seen. Now let the prism be removed from the slits, (or *vice versâ*,) still preserving its position of minimum deviation, till the violet image of the longer slit exactly falls upon and covers the red image of the shorter, as in the position $a'\,b$ of the figure. Then it is obvious, that the distance between the slits, divided by their distance from the prism, is the sine of the total angle of dispersion, or is equal to δD,

Fig. 104. 2nd. Another method.

and this being known

$$\delta\mu = \frac{\delta D}{2} \quad \frac{\cos\frac{1}{2}(A+D)}{\sin\frac{1}{2}A},$$

and therefore $\frac{\delta\mu}{\mu-1}$, or p, the dispersive power, is obtained.

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* Dr. Brewster has chosen a somewhat different position, (*Treatise*, &c. p. 296,) with a view to simplify the formulæ; but it does not appear to us that any advantage is gained in that respect by his arrangement.

Light. Part II.

But all these methods are only rude approximations, as the great discrepancies of the results hitherto obtained by them abundantly prove; thus, the dispersions of various specimens of flint glass, obtained by the method last described, come out no less than one-sixth larger than those previously given by Dr. Brewster. The only method which can really be relied on is that practised by Fraunhofer, (where the media can be procured in a state of sufficient purity and quantity for its application;) and consists in determining, with astronomical precision, by direct measures, the values of μ for the several points of definite refrangibility in the spectrum, marked, either by the fixed lines, or by the phenomena of coloured flames or absorbent media. (See Index, *Flames—Absorption.*) By taking advantage of the properties of the latter, a red ray, of a refrangibility strictly definite, may be insulated with great facility; and as it lies so near the extremity of the spectrum as not to be perceptible till all the brighter rays are extinguished, it is invaluable as a fixed term in optical researches, and will always be understood by us in future, when speaking of the *commencement of the spectrum*, or the *extreme red*, even though a red ray still less refrangible should be capable of being discerned by careful management, and in favourable circumstances. In like manner, by the simple artifice of putting a little salt into a flame, a yellow ray of a character perfectly definite is obtained, which, it is very remarkable, occupies precisely the place in the scale of refrangibility where in the solar spectrum the dark line D occurs, (Art. 418, 419.) These, and the fixed lines there mentioned, leave us at no loss for rays identifiable at all times and in all circumstances, (with a good apparatus,) and enable us to place the doctrine of refractive and dispersive powers on the footing of the most accurate branches of science.

436. Method employed by Fraunhofer.

Use of the fixed lines.

The following table, extracted from Fraunhofer's *Essay on the Determination of Refractive and Dispersive Powers, &c.* contains the absolute values of the index of refraction μ for the several rays whose places in the spectrum correspond to the seven lines B, C, D, E, F, G, H, assumed by him as standards (see Art. 419, &c.) for several different specimens of glass of his own manufacture, and for certain liquids. These values, for distinction's sake, we may designate by the signs μ (B), μ (C), μ (D), &c. 437.

Table of the refractive indices of various glasses and liquids for seven standard rays.

Refracting medium.	Specific gravity.	Values of μ (B)	μ (C)	μ (D)	μ (E)	μ (F)	μ (G)	μ (H)
Flint glass, No. 13.....	3.723	1.627749	1.629681	1.635036	1.642024	1.648260	1.660285	1.671062
Crown glass, No. 9.....	2.535	1.525832	1.526849	1.529587	1.533005	1.536052	1.541657	1.546566
Water	1.000	1.330935	1.331712	1.333577	1.335851	1.337818	1.341293	1.344177
Water, another experiment	1.000	1.330977	1.331709	1.333577	1.335849	1.337788	1.341261	1.344162
Solution of potash	1.416	1.399629	1.400515	1.402805	1.405632	1.408082	1.412579	1.416368
Oil of turpentine.....	0.885	1.470496	1.471530	1.474434	1.478353	1.481736	1.488198	1.493874
Flint glass, No. 3	3.512	1.602042	1.603800	1.608494	1.614532	1.620042	1.630772	1.640373
Flint glass, No. 30.....	3.695	1.623570	1.625477	1.630585	1.637356	1.643466	1.655406	1.666072
Crown glass, No. 13	2.535	1.524312	1.525299	1.527982	1.531372	1.534337	1.539908	1.544684
Crown glass, letter M....	2.756	1.554774	1.555933	1.559075	1.563150	1.566741	1.573535	1.579470
Flint glass, No. 23 .... Prism of 60° 15′ 42″	3.724	1.626596	1.628469	1.633667	1.640495	1.646756	1.658848	1.669686
Flint glass, No. 23 .... Prism of 45° 23′ 14″	3.724	1.626564	1.628451	1.633666	1.640544	1.646780	1.658849	1.669680

The above table renders very evident a circumstance which has long been recognised by experimental opticians, and which is of great importance in the construction of telescopes, viz. the *irrationality*, (as it has been termed,) or want of proportionality of the spaces occupied in spectra formed by different media by the several coloured rays, or by those whose refrangibilities, by any one standard medium, lie between given limits. If we fix upon water, for example, as a standard medium, (and we see no reason why it should not be generally adopted as a term of reference in this, as in other physical inquiries—of course at a given temperature—that of its maximum density, for instance,) it is obvious, that any ray may be identified by stating its index of refrangibility by water; thus, a scale of refrangibilities, which, for brevity, we shall term *the water scale*, is established; and so soon as we know the refractive index of a ray from vacuum into water, we have its place in the water spectrum, its colour, 438.

Identification of a ray by its place in a water spectrum.

Light. and its other physical properties (so far as they depend on the refrangibility of the ray) determined. Thus 1.333577 being known to be the refractive index for a ray in water, that ray can be no other than the particular ray D, whose colour is pale orange-yellow, and which is totally deficient in solar light, and peculiarly abundant in the light of certain flames. Now let x be the refractive index of any ray whatever for water, or its place in the water scale. Then it is evident, that its refractive index for any other medium must of necessity be a function of x, because the value of x determines this and all the other properties of the ray. Hence we must have between μ and x some equation which may be generally represented by $\mu = \mathrm{F}(x)$; $\mathrm{F}(x)$ denoting a function of x. Part II.

439. Function of refrangibility. To determine the form of this function, we must consider, that if A be the very small angle of a prism, and D the deviation produced by it at the minimum, we have $\mu \cdot \frac{\mathrm{A}}{2} = \frac{\mathrm{A}+\mathrm{D}}{2}$, or $\mathrm{D} = (\mu - 1)\,\mathrm{A}$. Hence, supposing A the refracting angle constant, the deviation is proportional to $\mu - 1$. Now, since in all media, as well as in water, the deviations observe, at least, *the same order*, being always least for the red and greatest for the violet, it follows, that in all media $\mu - 1$ increases as x increases; so that, supposing x_0 to be the index of refraction in the water scale for the first visible red ray, or the commencing value of x, and μ_0 the index for the same ray in the other medium, $(\mu - 1) - (\mu_0 - 1)$, or $\mu - \mu_0$ must increase with $x - x_0$; and since they vanish together, we may represent the one in a series with indeterminate coefficients, and powers of the other, thus

$$\mu - \mu_0 = \mathrm{A}(x - x_0) + \mathrm{B}(x - x_0)^2 + \mathrm{C}(x - x_0)^3 + \&c.;$$

or, which comes to the same thing, a, β, γ, &c., representing other indeterminate coefficients, ($x_0 - 1$ being constant,)

$$\frac{\mu - \mu_0}{\mu_0 - 1} = a \cdot \left(\frac{x - x_0}{x_0 - 1}\right) + b \cdot \left(\frac{x - x_0}{x_0 - 1}\right)^2 + \&c. \qquad (d)$$

440. Hypothesis of constant dispersion in all media. The simplest hypothesis we can form respecting the values of a, b, &c. is that which makes $a = 1$, and b, and all the other coefficients vanish. This gives $\frac{\mu - \mu_0}{\mu_0 - 1} = \frac{x - x_0}{x_0 - 1}$.

We have before used $\delta\mu$ to denote what is here signified by $\mu - \mu_0$, viz. the difference between the refractive indices of any ray in the spectrum, and that at its commencement; and we have denoted by $\frac{\delta\mu}{\mu - 1}$ the same quantity which is here expressed by $\frac{\mu - \mu_0}{\mu_0 - 1}$. This then is the expression, in our present notation, of the dispersive power of the medium; and the equation now under consideration therefore indicates, that, on the hypothesis made, the dispersive power of the medium must necessarily be the same with that of water; and of course (supposing this hypothesis to be founded in the nature of light) all media must have the same dispersive power. This, as we have already seen, is not the case.

Not the law of nature.

Nor that of proportional dispersions. The next simplest hypothesis is that which admits a as an arbitrary constant determined by the nature of the medium, but still makes b, c, &c. $= o$. This reduces the equation to

$$\frac{\mu - \mu_0}{\mu_0 - 1} = a \cdot \frac{x - x_0}{x_0 - 1};$$

consequently (if μ' and x' be any other corresponding values of μ and x) we must have also

$$\frac{\mu' - \mu_0}{\mu_0 - 1} = a \cdot \frac{x' - x_0}{x_0 - 1}, \text{ and therefore } \frac{\mu' - \mu}{\mu_0 - 1} = a \cdot \frac{x' - x}{x_0 - 1}; \text{ whence we have } \frac{\mu' - \mu}{x' - x} = a \cdot \frac{\mu_0 - 1}{x_0 - 1}.$$

Hence, if this hypothesis be correct, and μ, x and μ', x' be any two pairs of corresponding refractive indices for rays however situated, the fraction $\frac{\mu' - \mu}{x' - x}$ must be invariable. The foregoing table, however, shows very distinctly that this is far from being the case. Thus, if we take the flint glass, No. 13, the comparison of the two rays B and C gives for the value of the fraction in question 2.562; and if we compare in like manner the rays C and D, D and E, E and F, F and G, G and H respectively, we obtain the values 2.871, 3.073, 3.193, 3.460, 3.726; the great deviation of which from equality, and their regular progression, leaves no doubt of the incompatibility of the hypothesis in question, as a general law, with nature. If we institute the same comparison for the other media in the table, we shall find the greatest diversity prevail; and if, instead of water, we assume any other as a standard, the same incompatibility will be found. Thus if the flint glass, No. 13, be compared with oil of turpentine, we find for the values of the series of fractions in question, 1.868, 1.844, 1.783, 1.843, 1.861, 1.899, which first diminish to a minimum and then increase again, &c.

441. It follows from this, that the proportion which the several coloured spaces (or the intervals B C, C D, D E, &c.) bear to each other in spectra formed by different media, is not the same in all. Thus taking the green ray E for the middle colour, and calling all that part of the spectrum which lies on the red side of E the *red*,

Light. and all on the other side the *blue* portions, the ratio of the spaces occupied by the red and blue in any spectrum will be represented by the fraction $\frac{\mu\,(H) - \mu\,(E)}{\mu\,(E) - \mu\,(B)}$. Now the values of this in the several media of the foregoing table are set down in the following list:

Part II. Incommensurability of the coloured spaces in spectra of different media.

Flint, No. 23.....	2.0922	Crown, M.....	1.9484
Flint, No. 30.....	2.0830	Crown, No. 9	1.8905
Flint, No. 3	2.0689	Crown, No. 13....	1.8855
Flint, No. 13.....	2.0342	Solution of potash.	1.7884
Oil of turpentine ..	1.9754	Water.....	1.6936

Here we see that the same coloured spaces which in the flint No. 23 are in the ratio of 21 : 10, in the water spectrum are only in the ratio of 17 : 10 (nearly,) so that the blue portion of the spectrum is considerably more extended in proportion to the red in the flint glass than in the water spectrum.

442. Secondary spectra. From this it follows, that if two prisms be formed of different media (such as flint glass and water) of such refracting angles as to give spectra of equal total lengths, and these be made to refract in opposition to each other, although the red and violet rays will, of course, be united in the emergent beam, yet the intermediate rays will still be somewhat dispersed, the water prism refracting the green, or middle rays more than in proportion to the extremes; consequently, if a white *luminous* line be the object examined through such a combination, instead of being seen after refraction colourless, it will form a coloured spectrum of small breadth compared with what either prism separately would form, and having one side of a purple and the other of a green tint. Any *dark* object viewed against the sky (as a window bar) will be seen fringed with purple and green borders, the green lying on the same side of the bar with the vertex of the flint prism; because in such a combination, green must be considered as the most, and purple as the least, refrangible tint; and the flint prism, of necessity, having the least *refraction* in this case, the *most refrangible* fringe will lie towards its vertex, that being the *least refracted* side of the bar; for the same reason that, when seen through a single prism, a dark object on a white ground appears fringed with blue on its least refracted edge. (Art. 429.)

443. This result accords perfectly with observation. Clairaut, and, after him, Boscovich, Dr. Blair, and Dr. Brewster, have severally drawn the attention of opticians to these coloured fringes, or, as they may be termed, *secondary spectra*, and demonstrated their existence in the most satisfactory manner. Dr. Brewster, in particular, has entered into a very extensive and highly valuable series of experiments, described in his Treatise on new philosophical instruments, and in his paper on the subject in the *Edinburgh Transactions*; from which it follows, that when a compound prism, consisting of any of the media in the following list refracting in opposition to each other, unites the red and violet rays, the green will be deviated from their united course by the combination, in the direction of the refraction of that medium which stands before the other in order:

Dr. Brewster's table of media according to action on green light.

1. Sulphuric acid.	31. Gum juniper.	61. Oil of nutmegs.
2. Phosphoric acid.	32. Rock salt.	62. ——— carraway.
3. Sulphurous acid.	33. Calcareous spar.	63. ——— lemon.
4. Phosphorous acid.	34. Oil of ambergris.	64. Amber.
5. Super-sulphuretted hydrogen.	35. ——— juniper.	65. Oil of spearmint.
6. Water.	36. ——— spermaceti.	66. ——— hyssop.
7. Ice.	37. ——— rape seed.	67. ——— poppy.
8. White of egg.	38. ——— olives.	68. ——— penny-royal.
9. Rock crystal.	39. Zircon.	69. ——— sage.
10. Nitric acid.	40. Flint glass.	70. ——— turpentine.
11. Prussic acid.	41. Oil of rhodium.	71. Canada balsam.
12. Muriatic acid.	42. ——— rosemary.	72. Oil of lavender.
13. Nitrous acid.	43. ——— fenugreek.	73. Muriate of antimony.
14. Acetic acid.	44. Balsam of capivi.	74. Oil of cloves.
15. Malic acid.	45. Nut oil.	75. ——— sweet fennel seeds.
16. Citric acid.	46. Oil of savine.	76. Red-coloured glass.
17. Fluor spar.	47. ——— rue.	77. Orange-coloured glass.
18. Topaz, (blue.)	48. ——— beech-nut.	78. Opal-coloured glass.
19. Beryl.	49. Nitrate of potash.	79. Acetate of lead, (melted.)
20. Selenite.	50. Diamond.	80. Oil of amber.
21. Leucite.	51. Resin.	81. ——— sassafras.
22. Tourmaline.	52. Gum copal.	82. ——— cummin.
23. Borax.	53. Castor oil.	83. ——— anise seeds.
24. Borax, (glass of.)	54. Oil of chamomyle.	84. Essential oil of bitter almonds.
25. Ether.	55. ——— dill-seeds.	85. Carbonate of lead.
26. Alcohol.	56. ——— wormwood.	86. Balsam of Tolu.
27. Gum Arabic.	57. ——— marjoram.	87. Sulphuret of carbon.
28. Crown glass.	58. ——— bergamot.	88. Sulphur.
29. Oil of almonds.	59. ——— peppermint.	89 Oil of cassia.
30 Tartrate of potash and soda.	60 ——— thyme.	

Light. Part II.

It is evident from this table, that (generally speaking) the more refractive a medium is, the greater is the extent of the blue portion of its spectrum compared with the red.

444. 445. Achromatic refraction.

If two prisms of the proper refracting angles, composed of media not very remote from each other in this list, be made to oppose each other, the secondary spectrum will be small, and the refraction almost perfectly colourless. Such a combination is said to be *achromatic*, (a-$\chi\rho\omega\mu a$.)

446. Dispersive powers of higher orders. Tertiary spectra.

The existence of the secondary spectrum, while it renders the attainment of perfect achromaticity impossible, by the use of two media only, shows, also, that in a theoretical point of view we are not entitled to neglect the coefficients b, c, &c. of the equation (d), Art. 439. The law of nature probably requires the series to be continued to infinity; and if, by way of uniting three rays, we employ prisms of three media, *tertiary spectra*, and after them still others in succession, would doubtless be found to arise. These, however, will be small in comparison of each other.

447. Computation of their coefficients.

The table (Art. 437) gives us the means of computing the coefficients on which they depend for the particular media there stated. If we put $\frac{\mu - \mu_o}{\mu_o - 1} = \mathrm{P}$, and $\frac{x - x_o}{x_o - 1} = p$, and suppose P, P', P'', p, p', p'', &c. to be the values of P and p corresponding to the several values of μ and x set down in the table, we shall have, for determining a, b, c, &c. in any one of those media, the equations

$$\mathrm{P} = a\,p + b\,p^2 + c\,p^3 + \&c. \quad \mathrm{P}' = a\,p' + b\,p'^2 + c\,p'^3 + \&c. \quad \mathrm{P}'' = a\,p'' + b\,p''^2 + c\,p''^3 + \&c.$$

and as many such equations must be used as there are coefficients to determine. Confining ourselves at present to two, we find $\mathrm{P} = a\,p + b\,p^2$; $\mathrm{P}' = a\,p' + b\,p'^2$, whence

$$a = \frac{\mathrm{P}\,p'^2 - \mathrm{P}'\,p^2}{p\,p'\,(p' - p)}; \qquad b = -\frac{\mathrm{P}\,p' - \mathrm{P}'\,p}{p\,p'\,(p' - p)};$$

and, since it is desirable to select rays as far removed from each other in the spectrum as possible, we shall take μ_o and x_o from the column μ (B); and determine P and p by the values in the column μ (E), and P', p' by those under μ (H). The results will be as follows:

Refracting media.	Dispersive powers of the first order, that of water being 1.000.	Dispersive powers of the second order, that of water being 0.000.
Flint glass, No. 13 ..	$a = +$ 1.42580	$b = +$ 7.57705
Crown glass, No. 9 ..	0.88419	2.34915
Water.....	1.00000	0.00000
Solution of potash ..	0.99626	1.13262
Oil of turpentine	1.06149	4.58639
Flint glass, No. 3....	1.29013	7.63048
Flint glass, No. 30 ..	1.37026	8.44095
Crown glass, No. 13..	0.87374	2.49199
Crown glass, letter M.	0.90131	3.49000
Flint glass, No. 23 ..	1.37578	8.66904

448. General conditions of achromaticity.

Problem. To determine the analytical relation which must hold good in order that two prisms may form an *achromatic* combination; that is, may refract a white ray without separating the extreme colours.

Resuming the equations and notation of Art. 215, since the prisms are placed in vacuo, we have to substitute μ, $\frac{1}{\mu'}$, μ' and $\frac{1}{\mu'}$ for μ, μ', μ'', μ''', in those equations respectively, and we shall have

$$\left.\begin{aligned} \mu\,.\,\sin\rho &= \sin\alpha \\ \alpha' &= \mathrm{I} + \rho \\ \sin\rho' &= \mu\,.\,\sin\alpha' \end{aligned}\right\}(1); \qquad \left.\begin{aligned} \mu'\,.\,\sin\alpha''' &= \sin\rho''' \\ \rho'' &= -\mathrm{I}'' + \alpha''' \\ \sin\alpha'' &= \mu'\,.\,\sin\rho'' \end{aligned}\right\}(2);$$

and $$\alpha'' = \mathrm{I}' + \rho'; \qquad \mathrm{D} = \alpha + \mathrm{I} + \mathrm{I}' + \mathrm{I}'' - \rho'''.$$

Now, since by hypothesis the incident and emergent rays are both colourless, we must have $\delta\alpha = 0$, and $\delta\,\mathrm{D} = 0$, that is $\delta\rho''' = 0$, the sign δ being supposed to refer to the variation of the place of the ray in the spectrum. Hence the two systems of equations (1) and (2) are exactly similar, in their form; the former as relates to ρ, α, α', ρ', and the latter as to α''', ρ''', ρ'', α''. Now, the first system gives

$$\delta\mu\,.\,\sin\rho + \mu\,\delta\rho\,.\,\cos\rho = 0; \qquad \delta\alpha' = \delta\rho; \qquad \delta\rho'\cos\rho' = \delta\mu\,.\,\sin\alpha' + \mu\,\delta\alpha'.\cos\alpha';$$

whence, by elimination and reduction, we find

$$\delta\rho' = \frac{\sin\mathrm{I}}{\cos\rho\,.\,\cos\rho'}\,\delta\mu; \qquad (e)$$

Light. and, consequently, by reason of the analogy of the two systems of equations pointed out above, Part II.

$$\delta \alpha'' = - \frac{\sin I''}{\cos \alpha''' . \cos \alpha''} \delta \mu' \qquad (f)$$

But, since $\alpha'' = I' + \rho'$, we have $\delta \rho' = \delta \alpha''$, so that we finally get

$$\frac{\cos \rho . \cos \rho'}{\cos \alpha''' . \cos \alpha''} = - \frac{\sin I}{\sin I''} . \frac{\delta \mu}{\delta \mu'}. \qquad (g)$$

The property expressed by this equation may be thus stated. Conceive the ray to pass both ways outwards from a point in its course between the two prisms; then, in order that the combination may be achromatic, *the products of the cosines of its incidences on the surfaces of each prism must be to each other in the ratio compounded of that of the sines of their respective refracting angles, and the differences of their refractive indices for red and violet rays;* besides which, they must refract in opposition to each other, or I and I'' their refracting angles must have opposite signs.

449. Progress of dispersion by a single prism traced

The combination of this equation with the system of equations above stated, expressing the conditions of refraction by the prism, and their relative position with regard to each other (which is included in the equation $\alpha'' = I' + \rho'$) suffice, algebraically speaking, to resolve every problem which can occur, of this kind; but the final equations are for the most part too involved to allow of direct solution. Nevertheless, the results we have arrived at will furnish occasion for remarks of moment; and, first, since ρ' is the angle of refraction from the second surface of the first prism, $\delta \rho'$ is the angular breadth of the spectrum produced by it; this is, therefore, proportional, *cæteris paribus*, to the product of the secants of the angles of refraction at its two surfaces. Let us trace the progress of the variation of this, as the incident ray changes its inclination to the first surface, beginning with the case when it just grazes the surface *from* the back towards the edge. In this case $\alpha = 90°$, $\sin \rho = \frac{1}{\mu}$, consequently ρ, and therefore $I + \rho$ or α', and therefore ρ' are all finite, and at their maximum. Hence $\cos \rho . \cos \rho'$ is finite, and at its minimum; and therefore $\delta \varrho'$, or the breadth of the spectrum, is also finite, but a maximum. As the incident ray becomes more inclined to the surface ρ, and therefore α' and ϱ' diminish, and the denominator of $\delta \rho'$ increases, so that the breadth of the spectrum diminishes, and reaches a minimum when $\cos \rho . \cos \rho'$ attains its maximum; that is, when $d\rho . \tan \rho + d\rho' . \tan \rho' = 0$. Now this equation, substituting and reducing gives, for determining the value of ρ, and therefore of α, or the incidence when the spectrum is a minimum,

Position of least dispersion determined.

$$\mu^2 . \sin (I + \rho) . \cos (I + 2\rho) + \sin \rho = 0. \qquad (h)$$

Hence we see that the position which gives a minimum of breadth to the spectrum is very different from that which gives a minimum of deviation, being given by the above equation, which is easily resolved by a table of logarithms, and which shows at once that ρ must be greater than $45° - \frac{I}{2}$.

After attaining the position so determined, the breadth of the spectrum again increases, and continues to do so till the rays can be no longer transmitted through the prism. At this limit the emergent ray just grazes the posterior face of the prism from its thinner towards its thicker part $\varrho' = 90°$, $\cos \rho' = 0$. At this limit, therefore, the dispersion becomes infinite. All these stages are easily traced by turning a prism round its edge between the eye and a candle; or, better, between the eye and the narrow slit between two nearly closed window-shutters.

450. Distortion of spectrum at extreme incidences. Fig. 105.

Hence, as the incident ray varies from the position S E (fig. 105) to S' E, and therefore the refracted from F G to F' G', the breadth of the spectrum commences at a maximum, but finite value, diminishes to a minimum and then increases to infinity. The distribution of the colours in the spectrum, or the breadths of the several coloured spaces in any state of the data, will moreover differ according to the values of ρ, ρ' and $\sin I$; for the equation (*e*), by assigning to $\delta \mu$ the values which correspond in succession to the intervals between red and orange, orange and yellow, yellow and green, &c. will give the corresponding values of $\delta \rho'$, or the apparent breadths of these spaces. Now the denominator $\cos \rho . \cos \rho'$ is an implicit function of μ, and therefore varies when the initial ray is taken in different parts of the spectrum. The variation is trifling when the angles ρ, ρ' are considerable; but near the limit, when the ray can barely be transmitted, it becomes very great, the spectrum is violently distorted, and the violet extremity greatly lengthened in proportion to the red. The effect is the same as if the nature of the medium changed and descended lower in the order of substances in the table Art. 443.

451. Achromatic combinations of one medium. Fig. 106.

From what has just been said, we see the possibility of achromatising any prism, however large its refracting angle, by any other of the same medium, however small may be its angle; for since, by properly presenting a prism to the incident ray, its dispersion may be increased to infinity; if made to refract in opposition to another whose dispersion has any magnitude, however great, it may be made to counteract, or even overcome it. Thus in fig. 106 the dispersion of the second prism α, of small refracting angle, being increased by the effect of its inclined position, is rendered equal and opposite to that of the prism A, whose refracting angle is large.

452. Subordinate spectra.

When the prisms differ greatly in their angles, however, the second must be very much inclined, so as to bring it near to the limit of transmission. In this case, its law of dispersion, as just shown, will be greatly disturbed, and rendered totally different from what obtains in the other prisms; so that perfect achromaticity

Light. Part II.

cannot be produced; but when the extreme red and violet rays are united, the green will be too little refracted by the second prism, and a purple and green spectrum will arise, as in the case of prisms of different media. To this spectrum Dr. Brewster (who was the first to place it in evidence) has given the name of a *tertiary spectrum*; but it appears to us, that this term had better be reserved for the spectra mentioned in Art. 446, and those now in question may be called *subordinate spectra.*

If a small rectangular object be viewed through such a combination as above described, in which the prism A is placed in its position of minimum deviation, and achromatised by a second a, whose angle is less than that of A, but not so small as to introduce this cause of colour, it will appear distorted in figure; for the sides parallel to the edges of the prisms will undergo no change in their apparent length, while the breadth of the rectangle will appear magnified. For the first prism, by reason of its position, does not alter the angular dimensions of objects seen through it; but the second changes their angular breadth in the ratio of $d\rho'''$ to $d\alpha''$, that is (by differentiation) in the ratio of $\frac{\cos\alpha \,.\, \cos\alpha'}{\cos\rho \,.\, \cos\rho'}$ to unity, a ratio which increases rapidly as the inclination of the prism increases, and ρ' approaches a right angle.

453. Amici's prismatic telescope.

M. Amici has taken advantage of these properties to construct a species of achromatic telescope, which, at first sight, appears very paradoxical, being composed merely of four prisms of the *same* kind of glass, with plane surfaces. To understand its construction, conceive a small square object op placed with the side o parallel to the refracting edges of a pair of prisms so adjusted, and perpendicular to their principal sections, *i. e.* to the plane of the paper. Then, after refraction through both, it will be seen by an eye at E, as a real object $o'p'$, having its length o unaltered, but magnified in breadth. Now, if we add a second pair of prisms, similar to the first, and similarly disposed with respect to each other, so as to form a second achromatic combination, but having the plane of their principal sections at right angles to the former, producing a refraction perpendicular to the plane of the paper, or parallel to the *length* of the distorted square, this will be in like manner seen as a real and colourless object, but again distorted, its side $o'p'$ remaining unaltered, but o' being magnified. Thus, by the effect of the first distortion, the breadth of the square is magnified, and, by that of the second, its length, and in the same ratio; and therefore the final result will be an image undistorted, achromatic, and magnified. The writer of these pages had the pleasure of witnessing the very good performance of one of these singular telescopes, magnifying about four times in the hands of its inventor, at Modena, in 1826. It is evident, that, by superposing several such telescopes on each other, the magnifying power may be increased in geometrical progression. It is equally clear, that, by using prisms of two different media to form the several binary combinations, the *spectra of distortion* may be made to counteract the *secondary spectra*, arising from the difference in the scales of dispersion in the two media; and thus an achromaticity, almost mathematically perfect, might be obtained. It is worthy of consideration, whether, for the purpose of viewing very bright objects, as the sun, for instance, this species of telescope might not prove of considerable service. It would have the advantage of being its own darkening glass, of not bringing the rays to a focus, and therefore of requiring no extraordinary care in the figuring of the surfaces; and, in short, of being exempt from all those inconveniencies which oppose the perfection of telescopes of the usual constructions, as applied to this particular object.

subordinate

454. Conditions of achromaticity for several prisms of small angles

Proposition. To find the conditions of achromaticity when several prisms of different media refract a ray of white light, supposing all their refracting angles very small, and the ray to pass nearly at right angles to the principal section of each.

The refracting angles being A, A', A'', &c., and the refractive indices μ, μ', &c., the several partial deviations will be $D = (\mu - 1)\,A$; $D' = (\mu' - 1)\,A'$, &c.; and their sum, or the total deviation, will be $(\mu - 1)\,A + (\mu' - 1)\,A' + (\mu'' - 1)\,A'' + \&c.$ In order that the emergent ray may be colourless, this must be the same for rays of all colours; and its variation, when μ, μ', &c. are made to vary, must vanish, or

$$A\,\delta\mu + A'\,\delta\mu' + A''\,\delta\mu'' + \&c. = 0.$$

Now, by equation (d) of Art. 439, we have $\delta\mu$, (or, in the notation of that article, $\mu - \mu_o$)

$$= (\mu_o - 1)\left\{a \,.\, \frac{\delta x}{x_o - 1} + b\left(\frac{\delta x}{x_o - 1}\right)^2 + \&c.\right\}$$

Therefore the above equation gives, when arranged according to powers of δx,

$$0 = \left\{A\,(\mu_o - 1)\,a + A'\,(\mu'_o - 1)\,a' + A''\,(\mu''_o - 1)\,a'' + \&c.\right\} \,.\, \frac{\delta x}{x_o - 1}$$
$$+ \left\{A\,(\mu_o - 1)\,b + A'\,(\mu'_o - 1)\,b' + A''\,(\mu''_o - 1)\,b'' + \&c.\right\}\left(\frac{\delta x}{x_o - 1}\right)^2$$
$$+ \&c.$$

taking a', b', &c. to represent the dispersive powers of the various orders for the second prism, a'', b'', &c. for the third, and so on. Hence, in order that this may vanish for all the rays in the spectrum, we must have (putting, for brevity, μ for μ_o, μ' for μ'_o, &c.)

Light. Part II.

$$\left.\begin{array}{l}(\mu-1)\,.\,\mathrm{A}\,a+(\mu'-1)\,\mathrm{A}'\,a'+(\mu''-1)\,\mathrm{A}''\,a''+\&c.=0\\(\mu-1)\,.\,\mathrm{A}\,b+(\mu'-1)\,\mathrm{A}'\,b'+(\mu''-1)\,\mathrm{A}''\,b''+\&c.=0\\(\mu-1)\,.\,\mathrm{A}\,c+(\mu'-1)\,\mathrm{A}'\,c'+(\mu''-1)\,\mathrm{A}''\,c''+\&c.=0\\\quad\&c.\qquad\quad\&c.\qquad\quad\&c.\qquad\quad\&c.\end{array}\right\}\qquad(i)$$

and so on. Generally speaking, the number of these equations being infinite, no finite number of prisms can satisfy them all; but if we attempt only to unite as many rays in the spectrum as there are prisms, which is the greatest approach to achromaticity we can attain, we have as many equations as unknown quantities, *minus* one, and the ratios of the angles to each other become known. Thus, to unite two rays two media suffice, and we can only take into consideration the first order of dispersions, which give

$$(\mu-1)\,\mathrm{A}\,a+(\mu'-1)\,\mathrm{A}'\,a'=0\,;\ \frac{\mathrm{A}'}{\mathrm{A}}=-\frac{\mu-1}{\mu'-1}\cdot\frac{a}{a'}.\qquad(j)$$

To unite three rays we have

$$(\mu-1)\,\mathrm{A}\,a+(\mu'-1)\,\mathrm{A}'\,a'+(\mu''-1)\,\mathrm{A}''\,a''=0$$
$$(\mu-1)\,\mathrm{A}\,b+(\mu'-1)\,\mathrm{A}'\,b'+(\mu''-1)\,\mathrm{A}''\,b''=0$$

whence by elimination

$$\frac{\mathrm{A}'}{\mathrm{A}}=-\frac{\mu-1}{\mu'-1}\cdot\frac{a\,b''-b\,a''}{a'\,b''-b'\,a''};\ \frac{\mathrm{A}''}{\mathrm{A}}=-\frac{\mu-1}{\mu''-1}\cdot\frac{a\,b'-b\,a'}{a''\,b'-b''\,a'};\qquad(k)$$

and so on for any number.

453. Case of two media. In the case of two media, if the quantities b, c, &c. be not known, the dispersive powers of the first order, a, a', should be determined, not by comparison of the extreme red and violet rays, which are too little luminous to render their strict union a matter of importance; we should rather endeavour to unite those rays which are at once powerfully illuminating, and differing much in colour, such as the rays D and F. The exact union of these will insure the approximate union of all the rest better, on the whole, than if we aimed at uniting the extremes of the spectrum, and a far greater concentration of light will be produced. This should be carefully borne in mind in all experiments on the dispersions of glass to be used in the construction of telescopes. Best rays to unite.

454. Best rays to unite in case of three media. If we would produce the greatest possible achromaticity by three prisms, the rays to be selected for determining the values of a, b, a', b', should be C, E, and G; or, which would, perhaps, be still better, C, F, and a ray half way between D and E; but the want of a sufficiently well marked line in that part of the spectrum throws some slight difficulty in the way of this latter combination, when solar light is used, and would oblige us to have recourse to some other method of measurement, of which a variety might be suggested.

455. Cases in which the formulæ become inapplicable to practice. In the case of three media, if the numerators and denominators of the expressions (k) vanish, or nearly so, the solutions become illusory, or at least inapplicable in practice. This happens whenever either of the fractions $\frac{a}{a'}$, $\frac{a}{a''}$, $\frac{a'}{a''}$ becomes equal to either of the corresponding fractions $\frac{b}{b'}$, $\frac{b}{b''}$, or $\frac{b'}{b''}$. Hence, to obtain practicable combinations, it is necessary to employ media which differ as much as possible in their *scales* of dispersive powers, *i. e.* in which the coloured spaces differ as far as possible from proportionality; such, for instance, as flint glass, crown glass, and muriatic acid; or, still better, oil of cassia, crown glass, and sulphuric acid, &c.

§ II. *Of the Achromatic Telescope.*

456. Chromatic aberration explained. In the refracting telescopes described in Art. 380, &c. the different refrangibility of the differently coloured rays presents an obstacle to the extension of their power beyond very moderate limits. The focus of a lens being shorter as the refractive index is greater, it follows, that one and the same lens refracts violet rays to a focus nearer to its surface than red. This is easily seen by exposing a lens to the sun's rays, and receiving the converging cone of rays on a paper placed successively at different distances behind it. At any distance nearer to the lens than its focus for mean rays, the circle on the paper will have a red border, but beyond it a blue one; for the cone of red rays whose base is the lens, envelopes that of violet *within* the focus, its vertex lying beyond the other, but is enveloped by it *without*, for the converse reason. Hence, if the paper be held in the focus for mean rays, or between the vertices of the red and violet cones, *these* will then form a distinct image, being collected in a point: but the extreme, and all the other intermediate rays, will be diffused over circles of a sensible magnitude, and form coloured borders, rendering the image indistinct and hazy. This deviation of the several coloured rays from one focus is called the "*chromatic aberration.*"

457. Least circle of chromatic aberration. Fig. 107. The diameter of the least circle within which all the coloured rays are concentrated by a lens supposed free from spherical aberration is easily found. Thus, in fig. 107, if v be the focus for violet, and r for red rays, $n\,m\,o$ will be the diameter of this circle. Now, by similar triangles, $n\,o=\mathrm{A\,B}\,.\,\frac{m\,v}{\mathrm{C}\,v}$, and also $n\,o=\mathrm{A\,B}\,.\,\frac{m\,r}{\mathrm{C}\,r}$;

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therefore equating these $\frac{mv}{Cv} = \frac{mr}{Cr}$, and $mv = mr \cdot \frac{Cv}{Cr}$, $mv + mr = mr \cdot \frac{Cr + Cv}{Cr} = rv$, consequently $mr = rv \cdot \frac{Cr}{Cr + Cv} = rv \cdot \frac{Cr}{2Cr - rv} = \frac{rv}{2}$ very nearly, since the dispersion is small in comparison with the whole refraction. Therefore $no = \frac{AB}{2} \cdot \frac{rv}{Cr}$. Now, f being the reciprocal focal distance ($= L + D = (\mu - 1)(R' - R'') + D$) we have $rv = -\delta \frac{1}{f} = \frac{\delta f}{f^2} = \frac{\delta \mu (R' - R'')}{f^2}$ $= \frac{\delta \mu}{\mu - 1} \cdot \frac{L}{f^2}$ and $Cr = \frac{1}{f}$, supposing μ to represent the index of refraction for extreme red rays. Hence we get diameter of least circle of chromatic aberration = semi-aperture $\times \frac{L}{f} \cdot \frac{\delta \mu}{\mu - 1}$

$$= \text{semi-aperture} \times \text{dispersive index} \times \frac{L}{f};$$

and for parallel rays, when $L = f$, simply semi-aperture $\times$ dispersive index.

458. Use of very long telescopes. *Corol.* Hence the circle of least colour has the same absolute linear magnitude whatever be the focal length of the lens, provided the aperture be the same. Now, in the telescope, the magnifying power, or the absolute linear magnitude of the image viewed by a given eye-glass, increases in the ratio of the focal length of the object-glass, (382.) Therefore, by increasing the focal length of an object-glass without increasing its aperture, the breadth of the coloured border round the image of any object diminishes in proportion to the image itself, and thus the confusion of vision is diminished, and the telescope will possess a proportionally higher magnifying power. In consequence of this property, before the invention of the achromatic telescope, astronomers were in the habit of using refracting telescopes of enormous length, even so far as 100 or 150 feet; and Huyghens, in particular, distinguished himself by the magnitude and excellent workmanship of his glasses, and by the important astronomical discoveries made with them.

459. Principle of the achromatic telescope. The achromatic object-glass, however, by enabling us to reduce the length of the telescope within more reasonable bounds, has rendered it a vastly more manageable and useful instrument. To conceive its principle, we have only to recur to what has already been said in Art. 451—454, respecting achromatic prisms. A lens is nothing more than a system of infinitely small prisms arranged in circular zones round a centre, with refracting angles increasing as their distance from the centre increases, so as to refract all the rays to one point; and if we can achromatise each elementary prism, the whole system is achromatic. The equations (i) apply at once to this view of the structure of a lens. For, suppose R', R'' to be the curvatures of the two surfaces of the first lens, L' its power, and μ' its refractive index, then, for a given aperture, or at a given distance from the centre, $R' - R''$, the difference of the curvatures, expresses the angle made by tangents to the surfaces, or the refracting angle of the elementary prism; or $R' - R'' = A'$; and similarly for the other lenses, $A'' = R''' - R^{iv}$, and so on, so that the equations become

$$(\mu' - 1)(R' - R'') \cdot \alpha' + (\mu' - 1)(R''' - R^{iv})\, \alpha'' + \&c. = 0 \ \&c.;$$

General equations of achromaticity. that is simply

$$\left.\begin{aligned} L' \cdot a' + L'' \cdot a'' + L''' \cdot a''' + \&c. &= 0 \\ L' \cdot b' + L'' \cdot b'' + L''' \cdot b''' + \&c. &= 0 \\ L' \cdot c' + L'' \cdot c'' + L''' \cdot c''' + \&c. &= 0 \\ \&c. & \end{aligned}\right\}. \qquad (a)$$

460. Otherwise deduced. These equations afford all the relations necessary to insure achromaticity; and when satisfied, since they do not contain D, they show that an object-glass which is achromatic for any one distance of the object is so for all distances. It is evident, that the same system of equations may be obtained directly from the expression in Art. 268 for the joint power of a system of lenses whose individual powers are L', L'', &c. For the condition of achromaticity gives $\delta L = 0$, that is

$$\delta L' + \delta L'' + \delta L''' + \&c. = 0.$$

But since $L' = (\mu' - 1)(R' - R'')$ &c. (according to the system of notation there adopted)

$$\delta L' = (R' - R'')\, \delta \mu' = L' \cdot \frac{\delta \mu'}{\mu' - 1}.$$

But in the equation (d) if we put in succession for μ_0 the values μ', μ'', &c., for $\mu - \mu_0$ respectively, $\delta \mu'$, $\delta \mu''$, &c., and for a, b, &c. the systems of coefficients a', b', &c.; a'', b'', &c.; and suppose $\frac{x - x_0}{x_0 - 1} = p$, we shall have

$$\frac{\delta \mu'}{\mu' - 1} = a' p + b' p^2 + \&c.; \qquad \frac{\delta \mu''}{\mu'' - 1} = a'' p + b'' p^2 + \&c;$$

and therefore

$$0 = L' \{ a' p + b' p^2 + \&c. \} + L'' \{ a'' p + b'' p^2 + \&c. \} + \&c.$$

which, being made to vanish independently of p, gives the very same system of equations as $(a.)$

461. Object glass of two media.

To satisfy all these equations at once with any finite number of lenses being impossible, we must rest content with satisfying as many of the most important as the number of lenses will permit. Thus, if we have two lenses of different media, such as flint and crown glass, for instance, one only of them can be satisfied, and this must of course be the first, viz.

$$L' a' + L'' a'' = 0, \qquad \text{or } \frac{L''}{L'} = -\frac{a'}{a''}; \qquad (b)$$

which shows that the powers of the lenses must oppose each other, and be to each other inversely (and of course their focal lengths directly) as the dispersive powers. In such a combination, the values of a', a'', the dispersive powers, however, ought not to be obtained from the relative refractions for the extreme red and violet rays of the spectrum, (according to the remark in Art. 453,) but rather from the strongest and brightest rays whose colours are in decided contrast; such, for instance, as the rays C and F in Fraunhofer's scale.

462. Object glass of three media.

With three lenses of different media, two of the equations of achromaticity can be satisfied, and the secondary spectrum corrected, thus we have

$$\left.\begin{aligned} 0 &= L' a' + L'' a'' + L''' a''' \\ 0 &= L' b' + L'' b'' + L''' b''' \end{aligned}\right\} \left\{\begin{aligned} \frac{L''}{L'} &= -\frac{a' b''' - b' a'''}{a'' b''' - b'' a'''} \\ \frac{L'''}{L'} &= -\frac{a' b'' - b' a''}{a''' b'' - b''' a''} \end{aligned}\right\}; \qquad (c)$$

and in determining the values of a', b', &c. the rays to be employed should be the brightest yellow for a middle ray, and a pretty strong red and blue for the extremes. The rays B, E, H are perhaps inferior to C, E, G for this purpose.

463.

Hence in a double object-glass having a positive focus the least dispersive lens must be of a convex or positive, and the most so of a negative, or concave character. The order in which they are placed is of no consequence, as far as achromaticity is concerned.

464. Simultaneous destruction of both aberrations.

A single lens, as we have seen, neither admits of the destruction of the spherical, nor chromatic aberration, (Art. 296 and 457;) but if we combine two or more lenses of different media, the equations s, t, u, v of Art. 309, 310, 312, and 313, combined with the equations just derived (a), Art. 459, or so many of them as are not incompatible, afford us the means of annihilating both species of aberration at once; and what is curious, and must be regarded as singularly fortunate, the relations afforded by the destruction of the chromatic aberration, which, at first sight, would appear likely greatly to complicate the inquiry, tend, on the contrary, remarkably to simplify it, being in fact the very relations the analyst would fix upon to limit his symbols, and give his final equations the greatest simplicity their nature admits, if left at his disposal. For, it will be remarked, that in the general equation for the destruction of the spherical aberration, $\Delta f = 0$, or

$$0 = \frac{L'}{\mu'}(\alpha' - \beta' D' + \gamma' D'^2) + \frac{L''}{\mu''}(\alpha''' + \beta'' D'' + \gamma'' D''^2) + \&c.; \qquad (d)$$

the expressions within the parentheses are all of the second degree when expressed in terms of the curvatures of the surfaces, and of $D' = D$ the proximity of the radiant point to the first lens; and as L', L'', &c. are respectively of the first degree, in terms of the curvatures, the whole is, in its general form, of the third degree, and the equation of a cubic form. But the conditions of achromaticity, which assign relations only between L', L'', &c. without involving R', R'', &c. enable us to eliminate these quantities and replace them in the above equation, by giving combinations of a', a'', b', b'', &c., so that it becomes reduced to a quadratic form, and its treatment simplified accordingly.

465. Determination of the powers of the several lenses.

Let us proceed now to develope the equation (d), in which, according to the foregoing remark, when the conditions of achromaticity are introduced, L', L'', &c. may be regarded as given quantities; for, taking $L = L' + L'' + \&c.$ = the power of the compound lens, (which we may suppose given, or, if we please, assume equal to unity,) this, combined with the equations (a), determines the values of L', &c. Thus, in the case of two lenses, if we put π for the ratio of the dispersive powers, or $\varpi = \frac{a'}{a''}$ we have $L' = \frac{L}{1-\varpi}$, $L'' = -\frac{\varpi L}{1-\varpi}$; and similarly for three or more lenses. Suppose then we represent by r', r'', r''', &c. the respective curvatures of the first, or *anterior* surfaces of the first, second, third, &c. lens, in order; the first being that on which the rays first fall. Then we have $L' = (\mu' - 1)(R' - R'') = (\mu' - 1)(r' - R'',)$ so that $R'' = r' - \frac{L'}{\mu' - 1}$; and similarly $R^{iv} = r'' - \frac{L''}{\mu'' - 1}$, &c. We must therefore put in the foregoing expressions

Developement of the general equation.

$$R' = r'; \qquad R'' = r' - \frac{L'}{\mu' - 1}; \qquad R''' = r''; \qquad R^{iv} = r'' - \frac{L''}{\mu'' - 1}, \ \&c.$$

Light. Hence by substitution of these in the values of a, β, &c. (Art. 293) we get Part II.

$$a' = (2+\mu')\,r'^2 - (2\mu'+1)\cdot\frac{\mu'}{\mu'-1}\cdot L' r' + \mu'\cdot\left(\frac{\mu'}{\mu'-1}\right)^2 L^2$$

$$\beta' = (4+4\mu')\,r' - (3\mu'+1)\cdot\frac{\mu'}{\mu'-1}\,L'$$

$$\gamma' = 2+3\mu',$$

and similarly for α'', β'', γ'', &c. So that, substituting again these expressions, and putting for D'' its equal $L'+D'$, for D''' its equal $L'+L''+D'$, and so on, we have, finally, for the general equation $\Delta f = 0$, as follows:

$$\begin{aligned}
0 = {} & \left\{\left(\frac{2}{\mu'}+1\right)L' r'^2 + \left(\frac{2}{\mu''}+1\right)L'' r''^2 + \left(\frac{2}{\mu'''}+1\right)L''' r'''^2 + \&c.\right\} \\
& - \left\{\frac{2\mu'+1}{\mu'-1}L'^2 r' + \frac{2\mu''+1}{\mu''-1}L''^2 r'' + \frac{2\mu'''+1}{\mu'''-1}L'''^2 r''' + \&c.\right\} \\
& - 4\left\{\left(1+\frac{1}{\mu''}\right)L'L'' r'' + \left(1+\frac{1}{\mu'''}\right)(L'+L'')L''' r''' + \&c.\right\} \\
& + \left\{\left(\frac{\mu'}{\mu'-1}\right)^2 L'^3 + \left(\frac{\mu''}{\mu''-1}\right)^2 L''^3 + \left(\frac{\mu'''}{\mu'''-1}\right)^2 L'''^3 + \&c.\right\} \\
& + \left\{\frac{3\mu''+1}{\mu''-1}L'L''^2 + \frac{3\mu'''+1}{\mu'''-1}(L'+L'')L'''^2 + \&c.\right\} \\
& + \left\{\left(\frac{2}{\mu''}+3\right)L'^2 L''^2 + \left(\frac{2}{\mu'''}+3\right)(L'+L'')^2 L''' + \&c.\right\} \\
+ D.\Bigg\{ & - 4\left\{\left(1+\frac{1}{\mu'}\right)L' r' + \left(1+\frac{1}{\mu''}\right)L'' r'' + \left(1+\frac{1}{\mu'''}\right)L''' r''' + \&c.\right\} \\
& + \left\{\frac{3\mu'+1}{\mu'-1}L'^2 + \frac{3\mu''+1}{\mu''-1}L''^2 + \frac{3\mu'''+1}{\mu'''-1}L'''^2 + \&c.\right\} \\
& + 2\left\{\left(\frac{2}{\mu''}+3\right)L'L'' + \left(\frac{2}{\mu'''}+3\right)(L'+L'')L''' + \&c.\right\}
\end{aligned}\Bigg\}$$

$$+ D^2\left\{\left(\frac{2}{\mu'}+3\right)L' + \left(\frac{2}{\mu''}+3\right)L'' + \left(\frac{2}{\mu'''}+3\right)L''' + \&c.\right\}.$$

466. For brevity, let us represent by X, the terms of this expression, independent of the quantity D; by Y, the assemblage of terms multiplied by D'; and by Z, those multiplied by D'^2, and we shall have

$$\Delta f = \frac{y^2}{2}\{X + Y\cdot D + Z\cdot D^2\};$$

and if this vanish the aberration is destroyed. Now, first, if we regard only parallel rays, or suppose $D = 0$, this reduces itself to $X = 0$, so that the condition $X = 0$ being satisfied, the telescope will be perfect when used for astronomical purposes, or for viewing objects so distant that D' may be disregarded.

467. The distinction of aberration an indeterminate problem.

Conditions limiting it. Clairaut's.

The equation $X = 0$ is of the second degree in each of the quantities r' r'', &c., whose number is that of the lenses. Consequently, this condition alone is not sufficient to fix their values; and, without assuming some further relations between them, or some other limitations, the problem is indeterminate, and the aberration may be destroyed in an infinite variety of ways. Confining ourselves at present to the consideration of two lenses only, since $X = 0$ contains only two unknown quantities, one other equation only is required, and we have only to consider what other condition will be attended with the greatest practical advantages. Clairaut has proposed to adjust the two lenses so as to have their adjacent surfaces in contact throughout their whole extent, to allow of their being cemented together, and thus avoid the loss of light by reflection at these surfaces. This certainly would be a great advantage were it possible so to cement two glasses of large size together, as to bring neither of them into a state of strain as the cement cools, or otherwise fixes; and were it not for the further inconvenience, that the media being of course differently expansible by heat, every subsequent change of temperature would necessarily distort their figure, as well as strain their parts, when thus forcibly held together, just as we see a compound lamina of two differently expansible metals assume a greater or less curvature, according to the temperature it is exposed to. Meanwhile the condition in question is algebraically expressed by $L' = (\mu'-1)(r'-r'')$; for in this case $R' = r'$, and $R'' = R''' = r''$, and this being of the first degree only in r', r'', affords a final equation of a quadratic form by elimination with $X = 0$, which latter, in the case before us of two lenses, is the same as the equation (v), Art. 312, writing only r' for R', and r'' for R'''.

Light. Part II.

But this condition of Clairaut's has another and much greater inconvenience, which is, that the resulting quadratic has its roots imaginary, when the refractive and dispersive powers of the glasses are such as are by no means unlikely to occur in practice; and without the limits of refraction and dispersion, for which they are real, the resulting curvatures change so rapidly on slight variations of the data, as to make their computation delicate, and interpolation between them, so as to form a table, very troublesome. D'Alembert, in his *Opuscules*, tom. iii., has proposed a variety of other limitations, such, for instance, as annihilating the spherical aberration for rays of all colours, (which comes to the same as supposing at once $X = 0$ and $\frac{\delta X}{\delta \mu'} \delta \mu' + \frac{\delta X}{\delta \mu''} \delta \mu'' = 0$, and which leads to biquadratic equations, and affords no practical advantage,) &c. But, without going into useless refinements of this kind, the very form of the general equation $X + Y . D' + Z . D'^2 = 0$ points out a condition combining every advantage the case is susceptible of. This consists in putting $Y = 0$. By this supposition, the term depending on D' is destroyed, without assuming $D' = 0$; so that the telescope is not only perfect for parallel rays, but admits of as considerable a proximity of the object without losing its aplanatic character, as the nature of the case will allow. The term $Z . D'^2$ indeed, or

468. D'Alembert's.

Another proposed.

$$D'^2 . \left\{ \left(\frac{2}{\mu'} + 3 \right) L' + \left(\frac{2}{\mu''} + 3 \right) L'' + \&c. \right\},$$

cannot vanish when two lenses only are used, being composed wholly of given functions of the refractive and dispersive powers, unless by D' itself vanishing, or by an accidental adjustment of the values of μ', μ'', L', &c. But except the object be brought within a comparatively small distance from the telescope, (such as ten times its own length,) the square of D' is always so small as to allow of our disregarding this term, and considering the instrument as perfectly aplanatic when $Y = 0$. Now this equation, being of the first degree in r', r'', adds no new algebraic difficulty to the problem, but leads by elimination to a final quadratic; and, what is of most consequence, for such values of μ', μ'', and the dispersive ratio ϖ as occur in practice, the roots of this quadratic are always real, and the resulting curvatures of all the surfaces are moderate, and well adapted for practice; more so, indeed, than in any construction hitherto proposed. They are, moreover, such as to afford remarkable and peculiar facilities for interpolation, as we shall presently see. These reasons seem to leave no room for hesitation in fixing on the condition $Y = 0$, as that which ought to be introduced to limit the problem of the construction of a double object-glass, and to render it, so far as it can be rendered, *aplanatic*.

This equation, in the case in question, is 469

$$0 = 4 \left(1 + \frac{1}{\mu'} \right) L' r' + 4 \left(1 + \frac{1}{\mu''} \right) L'' r''$$

$$- \frac{3\mu' + 1}{\mu' - 1} L'^2 - \left(6 + \frac{4}{\mu''} \right) L' L'' - \frac{3\mu'' + 1}{\mu'' - 1} L''^2; \qquad (f)$$

which is to be combined with (v), Art. 412, in which $R' = r'$ and $R''' = r''$. To reduce these to numbers, μ', μ'' and the dispersive ratio ϖ must first be known. The readiest and most certain way in practice, for the use of the optician, is to form small object-glasses from specimens of the glasses intended to be employed, and by trial work them till the combination is as free from colour as possible, by the test usually had recourse to in practice. This is, to examine with a high magnifying power the image of a well defined white circle, or circular annulus on a black ground. If its edges are totally free from colour, the adjustment is perfect, but (owing to the secondary spectrum) this will seldom be the case; and there will generally be seen on the interior edge of the annulus a faint green, and on the exterior a purplish border, when the telescope is thrown a little out of focus by bringing the eye-glass too near the object-glass, and *vice versâ*. The reason is, that while the great mass of orange and blue rays is collected in one focus, the red and violet are converged to a focus farther from, and the green to one nearer to the object-glass; the refraction of the green rays being in favour of the *convex* or *crown* glass, and of the red and violet (which united form purple) in favour of the flint (see table, Art. 443) or concave lens. The focal lengths of the lenses are then to be accurately determined, and the ratio of the dispersions (ϖ) will then be known, being the same with that of the focal lengths (454). The refractive indices will be best ascertained by direct observation, forming portions of each medium into small prisms. Now, ϖ being known, if we take unity for the power of the compound lens, we have $L' = \frac{1}{1 - \varpi}$ and $L'' = -\frac{\varpi}{1 - \varpi}$, so that L' and L'' are known, and we have therefore only to substitute their values and those of μ', μ'', in the algebraic expressions, and proceed to eliminate by the usual rules. The following compendious table contains the result of such calculations for the values of μ', μ'' and ϖ therein stated, together with the amount of variation produced by varying either of the refractive indices independently of the other, for the sake of interpolation by proportional parts. Fig. 108 is a representation of the resulting object-glass.

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Dimensions of an aplanatic object-glass.

Table for finding the Dimensions of an Aplanatic Object-glass.

Refractive index of crown, or convex lens $= \mu' = 1.524$.
Refractive index of flint, or concave lens $= \mu'' = 1.585$.
Compound focal length $= 10.000$.

	CROWN LENS.					FLINT LENS.				
	First surface, convex.			Second surface, convex.		Third surface, concave.	Fourth surface, convex.			
Dispersive ratio $\varpi =$	Radius for the above refractive indices.	Variation of radius for a change of + 0.010 in ref. index of crown glass.	Variation of radius for a change of + 0.010 in ref. index of flint glass.	Radius of convexity.	Focal length of crown lens.	Radius of concavity.	Radius for the above refractive indices.	Variation of radius for a change of + 0.010 in ref. index of crown glass.	Variation of radius for a change of + 0.010 in ref. index of flint glass.	Focal length of flint lens.
0.50	6.7485	+ 0.0500	− 0.0030	4.2827	5.0	4.1575	14.3697	+ 0.9921	− 0.3962	10.0000
0.55	6.7184	+ 0.0740	− 0.0011	3.6332	4.5	3.6006	14.5353	+ 1.0080	− 0.5033	8.1818
0.60	6.7069	+ 0.0676	+ 0.0037	3.0488	4.0	3.0640	14.2937	+ 1.1049	− 0.5659	6.6667
0.65	6.7316	+ 0.0563	+ 0.0125	2.5208	3.5	2.5566	13.5709	+ 1.1614	− 0.6323	5.3846
0.70	6.8279	+ 0.0335	+ 0.0312	2.0422	3.0	2.0831	12.3154	+ 1.1613	− 0.7570	4.2858
0.75	7.0816	− 0.0174	+ 0.0568	1.6073	2.5	1.6450	10.5186	+ 1.0847	− 0.7207	3.3333

471. Example of the use of the table.

To apply this table to any other proposed state of the data, we have only to consider that to compute the radius of any one of the surfaces, as the first or fourth, we have only to regard each element as varying separately, and take proportional parts for each. The following example will elucidate the process: Required the dimensions for an object-glass of 30 inches focus, the refractive index of the crown glass being 1.519, and that of the flint 1.589; the dispersive powers being as 0.567 : 1, or 0.567 being the dispersive ratio. Here $\mu' = 1.519$, $\mu'' = 1.589$, and $\varpi = 0.567$. The computation must first be instituted for a compound focus $= 10.000$, as in the table, and we proceed thus:

1st. Subtract the decimal (0.567) representing the dispersive ratio from 1.000, and 10 times the remainder ($= 10 \times 0.433 = 4.330$) is the focal length of the crown lens.

2nd. Divide unity by the decimal above mentioned, (0.567,) subtract 1.000 from the quotient $\left(\frac{1}{0.567} = 1.7635\right.$, minus $1 = 0.7635$) and the remainder multiplied by 10 (or 7.635) is the focal length of the flint lens. We must next determine by the tables the radii of the first and fourth surfaces for the dispersive ratios there set down (0.55 and 0.60) next less and next greater than the given one. For this purpose we have

Refractive powers given.....	1.519	and	1.589
Refractive powers in table	1.524	..	1.585
Differences	− 0.005		+ 0.004

The given refraction of the crown being less, and of the flint greater, than their average values on which the table is founded. Looking out now opposite to 0.55 in the first column for the variations in the two radii corresponding to a change of + 0.010 in the two refractions, we find as follows:

	First surface.	Fourth surface.
For a change $=$ + 0.010 in the crown	+ 0.0740	+ 1.0080
For a change $=$ + 0.010 in the flint	− 0.0011	− 0.5033

But the actual variation in the crown instead of + 0.010 being − 0.005, and of the flint + 0.004, we must take the proportional parts of these, changing the sign in the former case; thus we find the variations in the first and last radii to be

	First surface.	Fourth surface.
For — 0.005 variation in the crown	— 0.0370	— 0.5040
For + 0.004 variation in the flint ..	— 0.0004	— 0.2013
Total variation from both causes	— 0.0374	— 0.7053
But the radii in the table are	6.7184	14.5353
Hence the radii interpolated are	6.6810	13.8300

If we interpolate, by a process exactly similar, the same two radii for a dispersive ratio 0.60, we shall find, respectively,

	First surface.	Fourth surface.
For a variation of — 0.005 in the crown	— 0.0338	— 0.5524
For a variation of + 0.004 in the flint	+ 0.0015	— 0.2264
Total variation	— 0.0323	— 0.7788
Radii in table	6.7069	14.2937
Interpolated radii	6.6746	13.5149

Having thus got the radii corresponding to the actual refractions for the two dispersive ratios 0.55 and 0.60, it only remains to determine their values for the intermediate ratios 0.567 by proportional parts; thus

		First radius.	Fourth radius.
For	0.600	6.6746	13.5149
For	0.550	6.6810	13.8300
Differences	+ 0.050	— 0.0064	— 0.3151

0.050 : 0.567 — 0.050 = 0.017 : : — 0.0064 : — 0.0022

0.050 : 0.017 : : — 0.3151 : — 0.1071

So that 6.6810 — 0.0022 = 6.6788, and 13.8300 — 0.1071 = 13.7229, are the true radii corresponding to the given data. Thus we have, for the crown lens, focal length = 4.330 = $\frac{1}{L'}$, radius of first surface = 6.6788 = $\frac{1}{R'}$, index of refraction = 1.519 = μ', whence by the formula $L' = (\mu' - 1)(R' - R'')\frac{1}{R''}$ radius of the other surface is — 3.3868. Again, for the flint lens, the focal length = $\frac{1}{L''}$ = — 7.635, radius of the posterior surface = $\frac{1}{R^{iv}}$ = — 13.7729, index of refraction $\mu'' = 1.589$, whence we find $\frac{1}{R'''}$ = — 3.3871 for the radius of the other surface. The four radii are thus obtained for a focal length of 10 inches, and multiplying by 3 we have for the telescope proposed

radius of first surface = + 20.0364 in.; of second, — 10.1604 in.; of third, — 10.1613 in.; of fourth, — 41.1687 in.

472. Here, then, we see that the radii of the two interior surfaces of the double lens (fig. 108) differ by scarcely more than a thousandth part of an inch; so that, should it be thought desirable, they may be cemented together. This is not merely a casual coincidence, for the particular state of the data; if we cast our eyes down the table we shall find this approximate equality of the interior curvatures (those of the second and third surfaces) maintained in a singular manner throughout the whole extent of the variation of ϖ. Thus the construction, here proposed in reality for glasses of the ordinary materials, approaches considerably to that of Clairaut already mentioned.

473. In order to put these results to the test of experience, Mr. South procured an achromatic telescope to be executed on this construction by Mr. Tulley, one of the most eminent of our British artists, which is now in the possession of J. Moore, Esq. of Lincoln. Its focal length was 45 inches, and aperture $3\frac{1}{4}$, and its performance was found to be fully adequate to the expectation entertained of it, bearing a magnifying power of 300 with perfect distinctness, and separating easily a variety of double stars, &c. A more minute account of its performance will be found in the *Journal of the Royal Institution*, No. 26. Should the splendid example set by Fraunhofer be followed up, and the practice of the optician be in future directed by a rigorous adherence to theory, grounded on exact measurements of the refractive powers of his glasses on the several coloured rays, it will become necessary to develope the above table more in detail.

474. Object-glasses of three media

When three media are employed in the construction of object-glasses, it should be our object to obtain as great a difference as possible in their scales of action on the differently coloured rays. Dr. Blair, to whom we are indebted for the first extensive examination of the dispersive powers of media as a physical character, and who first perceived the necessity of destroying the secondary spectrum, and pointed out the means of doing it, is the only one hitherto who has bestowed much pains on this important part of practical optics; which, considering the extraordinary success he obtained, and the perfection of the telescopes constructed on his principles, is to be regretted. We have no idea, indeed, for the reasons already mentioned, that very large object-

Light. glasses, enclosing fluids, can ever be rendered available; but to render glasses of moderate dimensions more perfect, and capable of bearing a higher degree of magnifying power, is hardly less important as an object of practical utility. His experiments are to be found in the *Transactions of the Royal Society of Edinburgh*, 1791. We can here do little more than present a brief abstract of them.

475. Dr. Blair's construction of an object-glass of three media.

Dr. Blair having first discovered that the secondary fringes are of unequal breadths, when binary achromatic combinations, having equal total refractions, are formed of different dispersive media, was immediately led to consider, that by employing two such different combinations to act in opposition to each other, if the total refractions were *equal*, the ray would emerge of course undeviated, and with its primary spectrum destroyed; but a secondary spectrum would remain, equal to the difference of the secondary spectra in the two combinations. Therefore, by a reasoning precisely similar to that which led to the correction of the primary spectrum itself, (Art. 426 and 427,) if we increase the total refraction of that combination A which, *cæteris paribus*, gives the *least* secondary spectrum, its secondary colour will be increased accordingly, till it becomes equal to that of the other B; so that the emergent beam will be free from the secondary spectra altogether, and will be deviated on the whole in favour of the combination A. Reasoning on these grounds, Dr. Blair formed a compound, or binary achromatic convex lens A, (fig. 109,) of two fluids *a* and *b*, (two essential oils, such as naphtha and oil of turpentine, differing considerably in dispersion,) which, when examined alone, was found to have a greater refractive power on the green rays than on the united red and violet. He also formed a second binary lens B, of a *concave* character, and also achromatic, (*i. e.* having the primary spectrum destroyed,) consisting of the more dispersive oil (*b*) and glass, and in which the green rays are also more refracted than the united red and violet, but *in a greater degree* in proportion to the whole deviation, than in the other combination; and in precisely the same degree was the focal length of this lens increased or its refraction diminished, when compared with that of the combination A. When, therefore, these two lenses were placed together, as in fig. 109, an excess of refraction remained in favour of the convex combination; but the secondary spectra of each being equal and opposite (by reason of the opposite character of the lenses) were totally destroyed. In fact, he states, that in a compound lens so constructed, he could discover no colour by the most rigid test; and thence concluded, not only the red, violet, and green to be united, but also all the rest of the rays, no outstanding colour of blue or yellow being discernible. In placing the lenses together, the intermediate plane glasses may be suppressed altogether, as in fig. 110.

476. Remarkable property of the muriatic acid.

It was in the course of these researches that Dr. Blair was led to the knowledge of the possibility of forming binary combinations, having secondary spectra of opposite characters; that is, in which (the total refraction lying the same way) the order of the colours in the secondary spectra should be inverted. In other words, that while in some combinations the green rays are more refracted than the united red and violet, in others they are less so. He found, for instance, that while in most of the highly dispersive media, including metallic solutions, the green lay among the less refrangible rays of the spectrum, there yet exist media considerably dispersive, in which the reverse holds good. The muriatic acid, among others, is in this predicament. Hence, in binary combinations of glass with this acid, the secondary spectrum consists of colours oppositely disposed from that formed by glass and the oils, or by crown and flint glass. In consequence of this, to form an object-glass of two binary combinations, as described in the last article, they must both be of convex characters. But this affords no particular advantage. Dr. Blair, however, considered the matter in another and much more important light, as offering the means of dispensing with a third medium altogether, and producing by a single binary combination a refraction absolutely free from secondary colour. To this end he considered, that it appears to depend entirely on the chemical nature of the refracting medium, what shall be the order and distribution of the colours in the spectrum, as well as what shall be the total refraction and dispersive powers of the medium; and that therefore by varying properly the ingredients of a medium, it may be practicable, without greatly varying the total refraction and dispersion, still to produce a considerable change in the internal arrangement (if we may use the phrase) of the spectrum; and therefore, perhaps, to form a compound medium in which the seven colours shall occupy spaces regulated by any proposed law, (within certain limits.) Now if a medium could be so compounded as to have the same scale of dispersions, or the same law of distribution of the colours as crown glass with a different absolute dispersion, as we have already seen, nothing more would be required for the perfection of the *double* object-glass. The property of the muriatic acid just mentioned puts this in our power.

Dr. Blair's discovery of media having the same scale of dispersion as glass.

His double object-glasses formed of such media.

It is observed, that the presence of a metal (antimony, for instance) in a fluid, while it gives it a high refractive and dispersive power, at the same time tends to dilate the more refrangible part of the spectrum beyond its due proportion to the less. On the other hand, the presence of muriatic acid tends to produce a contrary effect, contracting the more refrangible part and dilating the less, beyond that proportion which they have in glass. Hence, Dr. Blair was led to conclude, that by mixing muriatic acid with metallic solutions, in proportions to be determined by experience, a fluid might be obtained with the wished for property; and this on trial he found to be the case. The metals he used were antimony and mercury; and to ensure the presence of a sufficient quantity of muriatic acid, he employed them in the state of muriates, in aqueous solution; or, in the case of mercury, in a solution of sal ammoniac, which is a compound of ammonia and muriatic acid, and which is capable of dissolving a considerably greater quantity of corrosive sublimate (muriate, or chloride of mercury) than water alone. By adding liquid muriatic acid to the compound known by the name of butter of antimony, (chloride of antimony,) or sal ammoniac to the mercurial solution, he succeeded completely in obtaining a spectrum in which the rays followed the same law of dispersion as in crown glass, and even in *over-correcting* the secondary spectrum, so as to place its exact destruction completely in his power. It only remained to form an object-glass on these principles. Fig. 111 is such an one, in which, though there are two refractions at the confines of the glass and fluid, yet the chromatic aberration, as Dr. Blair assures us, was totally destroyed, and the rays of different colours were bent from their rectilinear course with the same equality as in reflexion.

Light. Part II.

To such an extent has Dr. Blair carried these interesting experiments, that he assures us he has found it practicable to construct an object-glass of nine inches focal length, capable of bearing an aperture of three inches, a thing which assuredly no artist would ever dream of attempting with glass lenses; and we cannot close this account of his labours without joining in a wish expressed on a similar occasion by Dr. Brewster, whose researches on dispersive powers have so worthily filled up the outline sketched by his predecessor, that this branch of practical optics may be resumed with the attention it deserves, by artists who have the ready means of executing the experiments it would require. Could solid media of such properties be discovered, the telescope would become a new instrument. 477.

These experiments of Dr. Blair lead to the remarkable conclusion, that at the common surface of two media a white ray may be refracted without separation into its coloured elements. In fact, μ and μ' being the refractive indices of the media for any ray as the extreme red, $\frac{\mu'}{\mu}$ will be their relative refractive index for that ray, and $\frac{\mu' + \delta\mu'}{\mu + \delta\mu}$ will be the relative index for any other ray. If, then, the refractive and dispersive powers of the media be such that $\frac{\mu' + \delta\mu'}{\mu + \delta\mu} = \frac{\mu'}{\mu}$, or $\mu\,\delta\mu' = \mu'\,\delta\mu$, that is, if $\frac{\delta\mu}{\delta\mu'} = \frac{\mu}{\mu'}$; and if, moreover, this relation hold good throughout the spectrum, *i. e.* if the increments of the refractive indices, in proceeding from the red to the violet end of the spectrum, be proportional to the refractive indices themselves, then the relative index is the same for all rays, and no dispersion will take place. Now this gives a relation between the dispersive and refractive indices of the two media, viz. $\frac{p'}{p} = \frac{\mu'}{\mu} \cdot \frac{\mu - 1}{\mu' - 1} = \frac{1 - \frac{1}{\mu}}{1 - \frac{1}{\mu}}$; and, in addition to this condition, the scale of dispersions must be the same in both media. According as the dispersions differ one way or the other from this precise adjustment, the violet ray may be either more or less refracted than the red at the common surface of the two media. 478. Case of colourless single refraction at common surface of two media.

We shall terminate the theory of achromatic object-glasses with a problem of considerable practical importance, as it puts it in our power, having obtained an approximate degree of achromaticity in an object-glass, to complete the destruction of the colour without making any alteration in the focal lengths or curvatures of the lenses, by merely placing them at a greater or less distance from one another. 479. Achromatic object-glass with separated lenses.

Problem. To express the condition of achromaticity, when the two lenses of a double object-glass are placed at a distance from each other, ($= t$.)

Resuming the notation of Art. 251 and 268, we have

$$f'' = \mathrm{L}' + \mathrm{D}; \qquad f^{\mathrm{iv}} = \mathrm{L}'' + \frac{f''}{1 - f''t}; \qquad \delta f'' = \delta \mathrm{L}';$$

and

$$\delta f^{\mathrm{iv}} = \delta \mathrm{L}'' + \frac{\delta f''}{(1 - f''t)^2} = \delta \mathrm{L}'' + \frac{\delta \mathrm{L}'}{\{1 - t(\mathrm{L}' + \mathrm{D})\}^2}.$$

Now, that the combination may be achromatic, we must have $\delta f^{\mathrm{iv}} = 0$; and, since t and D are constant, and L' and L'' only vary by the variations of μ', μ'' the refractive indices, we have $\delta \mathrm{L}' = (\mathrm{R}' - \mathrm{R}')\,\delta\mu' = \frac{\delta\mu'}{\mu' - 1}\mathrm{L}' = p'\mathrm{L}'$, and similarly $\delta \mathrm{L}'' = p''\mathrm{L}''$, so that substituting we get

$$\{1 - t(\mathrm{L}' + \mathrm{D})\}^2 + \frac{p'}{p''} \cdot \frac{\mathrm{L}'}{\mathrm{L}''} = 0.$$

Such is the condition of achromaticity. Since it depends on D, it appears that if the lenses of an object-glass be not close together, it will cease to be achromatic for near objects, however perfectly the colour be corrected for distant ones. The eye therefore cannot be achromatic for objects at all distances, its lenses being of great thickness compared to their focal lengths; and, therefore, although in contact at their adjacent surfaces, yet having considerable intervals between others. 480.

For parallel rays the equation becomes 481.

$$p''\,\mathrm{L}''(1 - t\,\mathrm{L}')^2 = -\,p'\,\mathrm{L}';$$

hence, the dispersions and powers of the lenses being given their interval t may be found by the expression

$$t = \frac{1}{\mathrm{L}'}\left\{1 - \sqrt{-\frac{p'}{p''} \cdot \frac{\mathrm{L}'}{\mathrm{L}''}}\right\}.$$

The condition of achromaticity, were the lenses placed close together, would be, as we have already shown, 482.

Light. $-\frac{p'}{p''}\cdot\frac{L'}{L''}=1$. Hence, whenever this fraction is less than unity, that is whenever L'', the power of the Part II. concave or flint lens (which we here suppose to be the second) is *too great*; or when, as the opticians call it, the colour is over-corrected, the object-glass may be made achromatic, or their over-correction remedied, without re-grinding the glasses, merely by separating the lenses; for in this case the quantity under the radical is less than unity, and therefore t is positive, a condition without which the rays could not be refracted as we have supposed them.

483. Moreover, this affords a practical and very easy means of ascertaining, with the greatest precision, the dispersive ratio of the two media. Let a convex lens of crown be purposely a little over-corrected by a concave of flint, and then let the colour be destroyed by separating the lenses. Measure their focal lengths $\left(\frac{1}{L'}\text{ and }\frac{1}{L''}\right)$ and the interval t between them in this state, and we have at once for the value of ϖ the dispersive ratio,

$$\varpi=\frac{p'}{p''}=-\frac{L''}{L'}(1-tL')^2.$$

§ III. *Of the Absorption or Extinction of Light by uncrystallized Media.*

484. All media absorb light. Transparency is the quality by which media allow rays of light freely to pass through their substance, or, it may be, between their molecules; and is said to be more or less perfect, according as a more or less considerable part of the whole light which enters them finds its way through. Among media, consisting of ponderable matter, we know of none whose transparency is perfect. Whether it be that some of the rays in their passage encounter bodily the molecules of the media, and are thereby reflected; or, if this supposition be thought too coarse and unrefined for the present state of science, be stopped or turned aside by the forces which reside in the ultimate atoms of bodies, without actual encounter, or otherwise detained or neutralized by them; certain it is, that even in the most rare and transparent media, such as air, water, and glass, a beam of light intromitted, is gradually extinguished, and becomes more and more feeble as it penetrates to a greater depth within them, and ultimately becomes too faint to affect our organs. Thus, at the tops of very high mountains, a much greater multitude of stars is visible to the naked eye than on the plains at their feet; the weak light of the smallest of them being too much reduced in its passage through the lower atmospheric strata to affect the sight. Thus, too, objects cease to be visible at great depths below water, however free from visible impurities, &c. Dr. Olbers has even supposed the same to hold good with the imponderable media (if any) of the celestial spaces, and conceives this to be the cause why *so few* stars (not more than about five or ten millions) can be seen with the most powerful telescopes. It is probable that we shall be long without means of confirming or refuting this singular doctrine.

485. On the other hand, though no body in nature be perfectly, all are to a certain degree, transparent. One of the densest of metals, gold, may actually be beaten so thin as to allow light to pass through it; and that it passes through the substance of the metal, not through cracks or holes too small to be detected by the eye, is evident from the colour of the transmitted light, which is green, even when the incident light is white. The most opaque of bodies, charcoal, in a different state of aggregation, (as diamond,) is one of the most perfectly transparent; and all coloured bodies, however deep their hues, and however seemingly opaque, must necessarily be rendered visible by rays which have entered their substance; for if reflected at their surfaces, they would all appear white alike. Were the colours of bodies strictly superficial, no variation in their thickness could affect their hue; but, so far is this from being the case, that all coloured bodies, however intense their tint, become paler by diminution of thickness. Thus the powders of all coloured bodies, or the streak they leave when rubbed on substances harder than themselves, have much paler colours than the same bodies in mass.

486. And all absorb the different colours unequally. This gradual diminution in the intensity of a transmitted ray in its progress through imperfectly transparent media, is termed its *absorption*. It is never found to affect equally rays of all colours, some being always absorbed in preference to others; and it is on this preference that the colours of all such media, as seen by transmitted light, depend. A white ray transmitted through a perfectly transparent medium, ought to contain at its emergence the same proportional quantity of all the coloured rays, because the part reflected at its anterior and posterior surfaces is colourless; but, in point of fact, such perfect want of colour in the transmitted beam is never observed. Media, then, are unequally transparent for the differently coloured rays. Each ray of the spectrum has, for every different medium in nature, its own peculiar *index of transparency*, just as the index of refraction differs for different rays and different media.

487. Experiment. The most striking way in which this different absorptive power of one and the same medium on differently coloured rays can be exhibited, is to look through a plain and polished piece of smalt-blue glass, (a rich deep blue, very common in the arts—such as sugar-basins, finger-glasses, &c. are often made of,) at the image of any narrow line of light (as the crack in a window-shutter of a darkened room) refracted through a prism whose edge is parallel to the line, and placed in its situation of minimum deviation. If the glass be extremely thin, all the colours are seen; but if of moderate thickness (as $\frac{1}{40}$ inch) the spectrum will put on a very singular and striking appearance. It will appear composed of several detached portions separated by broad and perfectly black

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intervals, the rays which correspond to those points in the perfect spectrum being entirely extinguished. If a less thickness be employed, the intervals, instead of being perfectly dark, are feebly and irregularly illuminated, some parts of them being less enfeebled than others. If the thickness, on the other hand, be increased, the black spaces become broader, till at length all the colours intermediate between the extreme red and extreme violet are totally destroyed.

488. Law of transmission

The simplest hypothesis we can form of the extinction of a beam of homogeneous light in passing through a homogeneous medium, is, that for every equal thickness of the medium passed through, an equal aliquot part of the rays, which, up to that depth had escaped absorption, is extinguished. Thus, if 1000 red rays fall on and enter into a certain green glass, and if 100 be extinguished in traversing the first tenth of an inch, there will remain 900 which have penetrated so far; and of these one-tenth, or 90, will be extinguished in the next tenth of an inch, leaving 810, out of which again a tenth, or 81, will be extinguished in traversing the third tenth, leaving 729, and so on. In other words, the quantity unabsorbed, after the beam has traversed any thickness of the medium, will diminish in geometrical progression, as t increases in arithmetical. So that if 1 be taken for the whole number of intromitted rays, and y for the number that escape absorption in traversing an unit of thickness, y^t will represent the number escaping, after traversing any other thickness, $= t$. This only supposes that the rays in the act of traversing one stratum of a medium acquire no additional facility to penetrate the remainder. In this doctrine, y is necessarily a fraction smaller than unity, and depending on the nature both of the ray and the medium. Hence, if C represent the number of equally illuminating rays of the extreme red in a beam of white light, C′ that of the next degree of refrangibility, and so on; the beam of white light will be represented by $C + C' + C'' +$ &c.; and the transmitted beam, after traversing the thickness t, will be properly expressed by

$$C \cdot y^t + C' \cdot y'^t + C'' \cdot y''^t + \text{\&c.}$$

Each term representing the intensity of the particular ray to which it corresponds, or its ratio to what it is in the original white beam.

489.

It is evident from this, that, strictly speaking, total extinction can never take place by any finite thickness of the medium; but if the fraction y for any ray be at all small, a moderate increase in the thickness, (which enters as an exponent,) will reduce the fraction y^t to a quantity perfectly insensible. Thus, in the case taken above, where a tenth of an inch of green glass destroys one-tenth only of the red rays, a whole inch will allow to pass only $\left(\frac{9}{10}\right)^{10}$, or 304 rays out of a thousand, while ten times that thickness, or 10 inches, will suffer only $\left(\frac{9}{10}\right)^{100} = 0.0000266$, or less than three rays out of 100,000 to pass, which amounts to almost absolute opacity.

490. Law of absorption of any medium expressed by a curve. Fig. 112.

If x be the index of refraction of any ray in the water spectrum, we may regard y as a function of x; and if on the line R V, (fig. 112,) representing the whole length of the water spectrum, we erect ordinates, Rr, M N, Vv equal to unity and to each other; and also other ordinates Rr, M P, Vv representing the values of y for the rays at the corresponding points; the curve r P v, the locus of P, will be, as it were, a type, or geometrical picture of the action of the medium on the spectrum, and the straight line R N V will be a similar type of a perfectly transparent medium. Now if this be supposed the case when the thickness of the medium is 1, if we take always M P′ : M P :: M P : M N, and M P″ : M P′ :: M P′ : M P, &c. and so on, the loci of P′ P″, &c. will be curves representing the quantities of the rays transmitted by the thicknesses 2, 3, &c. of the medium, and so for intermediate thicknesses, or for a thickness less than 1, as in the curve $\rho \pi v$.

491.

Hence, whatever be the colour of a medium, if its thickness be infinitely diminished, it will transmit all the rays indifferently; for when $t = 0$, $y^t = 1$, whatever be y; and the curve $\rho \pi v$ approaches infinitely near to the line R′ N V′. Thus all coloured glasses blown into excessively thin bubbles are colourless, and so is the foam of coloured liquids.

492.

Again, if there be any, the least, preference given by the medium to the transmission of certain rays beyond others, the thickness of the medium may be so far increased as to give it any assignable depth of tint; for if y be ever so little less than unity, and if between the values of y for different rays there be ever so little difference, t may be so increased as to make y^t as small as we please, and the ratio of y^t to y'^t as different from unity as we please.

493.

In very deep coloured media all the values of y are small. If they were equal, the medium would merely stop light, without colouring the transmitted beam, but no such media are at present known.

494. Ultimate tint of an absorptive medium. Fig. 113.

If the curve rPv, or the type of an absorbent medium have a maximum in any part of the spectrum, as in the green, for instance, (fig. 113;) then, whatever be the proportion in which the other rays enter, by a sufficient increase of thickness, that colour will be rendered predominant; and the *ultimate tint* of the medium, or the last ray it is capable of transmitting, will be a pure homogeneous light of that particular refrangibility to which the maximum ordinate corresponds. Thus green glasses, by an increase of thickness, become greener and greener, their type being as in fig. 113; while yellow ones, whose type is as in fig. 114, change their tint by reduplication, and pass through brown to red.

495. Tint changes by increase of thickness.

This change of tint by increase of thickness is no uncommon phenomenon; and though at first sight paradoxical, yet is a necessary consequence of the doctrine here laid down. If we enclose a pretty strong solution of sap-green, or, still better, of muriate of chromium in a thin hollow glass wedge, and if we look through the edge where it is thinnest, at white paper, or at the white light of the clouds, it appears of a fine green; but if we slide the wedge before the eye gradually so as to look successively through a greater and greater thickness

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of the liquid, the green tint grows livid, and passes through a sort of neutral, brownish hue, to a deep blood-red. To understand this, we must observe, that the curves expressing the types of different absorbent media admit the most capricious variety of form, and very frequently have several maxima and minima corresponding to as many different colours. The green liquids in question have two distinct maxima, as in fig. 115; the one corresponding to the extreme red, the other to the green, but the absolute lengths of the maximum ordinates are unequal, the red being the greater. But as the extreme red is a very feebly illuminating ray, while on the other hand the green is vivid, and affects the eye powerfully, the latter at first predominates over the former, and entirely prevents its becoming sensible; and it is not till the thickness is so far increased as to leave a very great preponderance of those obscure red rays, and subdue their rivals, as in the case represented by the lowest of the dotted curves in the figure, that we become sensible of their influence on the tint. Suppose, for instance, to illustrate this by a numerical example, the index of transparency, or value of y, in muriate of chromium, to be for extreme red rays, 0.9; for the mean red, orange, and yellow, 0.1; for green, 0.5; and for blue, indigo, and violet, 0.1 each; and suppose, moreover, in a beam of white light, consisting of 10,000 rays, *all equally illuminative*, the proportions corresponding to the different colours to be as follows:

Case of a green-red medium. Fig. 115.

Numerical illustration.

Extreme red.	Red and orange.	Yellow.	Green.	Blue.	Indigo.	Violet.
200	1300	3000	2800	1200	1000	500.

Then, after passing through a thickness equal to 1 of the medium, the proportions in the transmitted beam would be

Extreme red.	Red and orange.	Yellow.	Green.	Blue.	Indigo.	Violet.
180	130	300	1400	120	100	50.

After traversing a second unit of thickness, they would be

Extreme red.	Red and orange.	Yellow.	Green.	Blue.	Indigo.	Violet.
162	13	30	700	12	10	5.

and after a third, a fourth, a fifth, and sixth respectively,

Extreme red.	Red and orange.	Yellow.	Green.	Blue.	Indigo.	Violet.
146	1	3	350	1	1	0
131	0	0	175	0	0	0
118	0	0	87	0	0	0
106	0	0	43	0	0	0.

Thus we see, that in the first of these transmitted beams the green greatly preponderates, after the second transmission, it is still the distinguishing colour; but after the third, the red bears a proportion to it large enough to impair materially the purity of its tint. The fourth transmission may be regarded as totally extinguishing all the other colours, and leaving a neutral tint between red and green; while, in all the tints produced by further successive transmissions, the red preponderates continually more and more, till at length the tint becomes no way distinguishable from the homogeneous red of the extremity of the spectrum.

496. Relative illuminative power of the several prismatic rays. Fig. 116.

Whether we suppose the obscurer parts of the spectrum to consist of fewer rays equally illuminative, or of the same number of rays of less intrinsic illuminating power with the brighter, obviously makes no difference in the conclusion, but the former supposition has the advantage of affording a hold to numerical estimation which the latter does not. In the instance here taken, the numbers are assumed at random. But Fraunhofer has made a series of experiments expressly to determine numerically the illuminating power of the different rays of the spectrum. According to which, he has constructed the curve fig. 116, whose ordinate represents the illuminative power of the ray in that part of the spectrum on which it is supposed erected, or the proportional number of equally illuminative rays of that refrangibility in white light. If we would take this into consideration in our geometrical construction, we must suppose the type of white light, instead of being a straight line, as in fig. 112.... 114, to be a curve similar to fig. 116, and the other derivative curves to be derived from it by the same rules as above. But as the only use of such representations is to express concisely to the eye the general scale of action of a medium on the spectrum, this is rather a disadvantageous than a useful refinement.

497. Fig. 117.

To take another instance. If we examine various thicknesses of the smalt-blue glass above noticed, it will be found to appear purely blue in small thicknesses. As the thickness increases, a purple tinge comes on, which becomes more and more ruddy, and finally passes to a deep red; a great thickness being, however, required to produce this effect. If we examine the tints by a prism, we shall find the type of this medium to be as in fig. 117, having four maximum ordinates, the greatest corresponding to a ray at the very farthest extremity of the red, and diminishing with such rapidity as to cause an almost perfect insulation of this ray; the next corresponds to a red of mean refrangibility, the next to the mean yellow, and the last to the violet, the ordinate increasing continually to the end of the spectrum. Thus, when a piece of such glass of the thickness 0.042 inch was used, the red portion of the spectrum was separated into two, the least refracted being a well defined band of perfectly homogeneous and purely red light, separated from the other red by a band of considerable breadth, and totally black. This red was nearly homogeneous; its tint, however, differing in no respect from the former, and being free from the slightest shade of orange. Its most refracted limit came very nearly up to the dark line D in the spectrum. A small, sharp, black line separated this red from the yellow, which was a pretty well defined band of great brilliancy and purity of colour, of a breadth exceeding that of the first red, and bounded on the

green side by an obscure but not quite black interval. The green was dull and ill defined, but the violet was transmitted with very little loss. A double thickness (0.084 inch) obliterated the second red, greatly enfeebled the yellow, leaving it now sharply divided from the green, which was also extremely enfeebled. The extreme red, however, retained nearly its whole light, and the violet was very little weakened. When a great many thicknesses were laid together, the extreme red and extreme violet only passed.

498. Red media.

Among transparent media of most ordinary occurrence, we may distinguish, first, those whose type has its ordinate decreasing regularly, with more or less rapidity from the red to the violet end of the spectrum, or which absorb the rays with an energy more or less nearly in some direct ratio of their refrangibility. In red and scarlet media the absorbent power increases very rapidly, as we proceed from the red to the violet. In yellow, orange, and brown ones, less so; but all of them act with great energy on the violet rays, and produce a total obliteration of them. In consequence of this, by an increase of thickness, all these media finally become red. Examples: red, scarlet, brown, and yellow glasses; port wine, infusion of saffron, permuriate of iron, muriate of gold, brandy, India soy, &c.

499. Simple green media. Dichromatic media.

Among green media, the generality have a single maximum of transmission corresponding to some part of the green rays, and their hue in consequence becomes more purely green by increase of thickness. Of this kind are green glasses, green solutions of copper, nickel, &c. They absorb both ends of the spectrum with great energy; the red, however, more so, if the tint verges to blue; the violet, if to yellow. Besides these, however, are to be remarked media in which the type has two maxima; such may be termed dichromatic, having really two distinct colours. In most of these, the green maximum is less than the red; and the green tint, in consequence, loses purity by increase of thickness, and passes through a livid neutral hue to red, though this is not always the case. Examples: muriate of chrome, solution of sap-green, manganesiate of potash, alkaline infusion of the petals of the peonia officinalis and many other red flowers, and mixtures of red and blue or green media.

500. Blue media. Insulation of the extreme violet.

Blue media admit of great variety, and are generally dichromatic, having two or even a great many maxima and minima in their types; but their distinguishing character is a powerful absorption of the more luminous red rays and the green, and a feeble action on the more refrangible part of the spectrum. Among those whose energy of absorption appears to increase regularly and rapidly from the violet to the red end of the spectrum, we may place the blue solutions of copper. The best example is the magnificent blue liquid formed by supersaturating sulphate of copper with carbonate of ammonia. The extreme violet ray seems capable of passing through almost any thickness of this medium; and this property, joined to the unalterable nature of the solution, and the facility of its preparation, render it of great value in optical researches. A vessel, or tube, of some inches in length, closed at two ends with glass plates, and filled with this liquid, is the best resource for experiments on the violet rays. Ammonio-oxalate of nickel transmits the *blue* and *extreme red*, but stops the *violet*.

501. Purple media.

Purple media act by absorbing the middle of the spectrum, and are therefore necessarily always dichromatic, some of them having red and others violet for their *ultimate* or *terminal tint*. Example: solution of archil; purple, plum-coloured, and crimson glasses; acid and alkaline solutions of cobalt, &c. They may be termed red-purple and violet-purple, according to their terminal tint.

502. Combinations of media.

In combinations of media, the ray finally transmitted is the residuum of the action of each. If x, y, z be the indices of transmissibility of a given ray C in the spectrum for the several media, and r, s, t their thicknesses, the transmitted portion of this ray will be $C \cdot x^r y^s z^t$; and the residuum of a beam of white light (supposing none lost by reflexion at the surfaces) after undergoing the absorptive action of all the media, will be

$$C \cdot x^r y^s z^t + C' \cdot x'^r y'^s z'^t + \&c.$$

An expression which shows that it is indifferent in what order the media are placed. They may therefore be mixed, unless a chemical action take place. Thus also, by the same construction as that by which the type 1 of the first medium is derived from the straight line representing white light, may another type 2 be derived from 1, and so on; and thus an endless variety of types will originate, having so many tints corresponding to them.

503. Insulation of an extreme homogeneous red ray. Insulation of the yellow rays.

This circumstance enables us to insulate, in a state of considerable homogeneity, various rays. Thus, by combining with the smalt-blue glass, already mentioned, any brown or red glass of tolerable fulness and purity of colour, a combination will be formed absolutely impermeable to any but the extreme red ray, and the refrangibility of this is so strictly definite as to allow of its being used as a standard ray in all optical inquiries, which is the more valuable, as the coloured glasses by which it is insulated are the most common of any which occur in the shops, and may be had at any glazier's. If to such a combination a green glass be added, a total stoppage of all light takes place. The same kind of glass, too, enables us to insulate the yellow ray, corresponding to the maximum Y in the type fig. 117, by combining it with a brown glass to stop out the more, and a green to destroy the less, refrangible rays, and by their means the existence of a considerable breadth of yellow light, evidently not depending on a *mixture*, or mutual encroachment of red and green, may be exhibited in the solar spectrum.

504. Alteration of absorptive power by heat.

It has been found by Dr. Brewster, that the proportions of the different coloured rays absorbed by media depend on their temperature. The tints of bodies generally deepen by the application of heat, as is known to all who are familiar with the use of the blow-pipe; thus minium and red oxide of mercury deepen in their hues by heat till they become almost black, but recover their red colours on cooling. Dr. Brewster has, however, produced instances, not merely among artificial glasses, but among transparent minerals, where a transition takes place from red to green on the application of a high temperature; the original tint being, however, restored on cooling, and no chemical alteration having been produced in the medium.

505.

The analysis of the spectrum by coloured media presents several circumstances worthy of remark. First, the irregular and singular distribution in the dark bands which cross the spectrum, when viewed through such

Light. media as have several maxima of transmission, obviously leads us to refer Fraunhofer's *Fixed lines*, and the analogous phenomena to be noticed in the light from other sources, to the same cause, whatever it may be, which determines the absorption of some ray in preference to others. It is no impossible supposition, that the deficient rays in the light of the sun and stars may be absorbed in passing through their own atmospheres, or, to approach still nearer to the origin of the light, we may conceive a ray stifled in the very act of emanation from a luminous molecule by an intense absorbent power residing in the molecule itself; or, in a word, the same indisposition in the molecules of an absorbent body to permit the propagation of any particular coloured ray through, or near them, may constitute an obstacle *in limine* to the production of the ray from them. At all events, the phenomena are obviously related, though we may not yet be able to trace the particular nature of their connection. Part II.

506. The next circumstance to be observed is, that when examined through absorbent media all idea of regular gradation of colour from one end to the other of the spectrum is destroyed. Rays of widely different refrangibility, as the two reds noticed in Art. 497, have absolutely the same colour, and cannot be distinguished. On the other hand, the transition from pure red to pure yellow, in the case there described, is quite sudden, and the contrast of colours most striking, while the dark interval which separates them, by properly adjusting the thickness of the glass, may be rendered very small without any tinge of orange becoming perceptible. What then, we may ask, is become of the orange; and how is it, that its place is partly supplied with red on one side, and yellow on the other? These phenomena certainly lead us very strongly to believe that the analysis of white light by the prism is not the only analysis of which it admits, and that the connection between the refrangibility and colour of a ray is not so absolute as Newton supposed. Colour is a sensation excited by the rays of light, and since two rays of different refrangibilities are found to excite absolutely the same sensation of colour, there is no *primâ facie* absurdity in supposing the converse,—that two rays capable of exciting sensations of different colours may have identical indices of refraction It is evident, that if this be the case, no mere change of direction by refractions through prisms, &c. could ever separate them; but should they be differently absorbable by a medium through which they pass, an analysis of the compound ray would take place by the destruction of one of its parts. This idea has been advocated by Dr. Brewster, in a Paper published in the *Edinburgh Philosophical Transactions*, vol. ix., and the same consequence appears to follow from experiments, published in the same volume of that collection. According to this doctrine, the spectrum would consist of at least three distinct spectra of different colours, red, yellow, and blue, over-lapping each other, and each having a maximum of intensity at those points where the compound spectrum has the strongest and brightest tint of that colour.

507. Cases of persons who see only two colours. It must be confessed, however, that this doctrine is not without its objections; one of the most formidable of which may be drawn from the curious affection of vision occasionally (and not very rarely) met with in certain individuals, who distinguish only two colours, which (when carefully questioned and examined by presenting to them, not the ordinary compound colours of painters, but optical tints of known composition) are generally found to be yellow and blue. We have examined with some attention a very eminent *optician*, whose eyes (or rather eye, having lost the sight of one by an accident) have this curious peculiarity, and have satisfied ourselves, contrary to the received opinion, that all the prismatic rays have the power of exciting and affecting them with the sensation of *light*, and producing distinct vision, so that the defect arises from no insensibility of the retina to rays of any particular *refrangibility*, nor to any colouring matter in the humours of the eye, preventing certain rays from reaching the retina, (as has been ingeniously supposed,) but from a defect in the sensorium, by which it is rendered incapable of appreciating exactly those differences between rays on which their colour depends. The following is the result of a series of trials, in which a succession of optical tints produced by polarized light, passing through an inclined plate of mica, in a manner hereafter to be described, was submitted to his judgment. In each case, two uniformly coloured circular spaces placed side by side, and having *complementary tints* (i. e. such that the sum of their light shall be white) were presented, and the result of his judgment is here given in his own words.

Colours according to the judgment of an ordinary eye.		Colours as named by the individual in question.		Inclination of the plate of mica to eye.
Circle to the left.	Circle to the right.	Circle to the left.	Circle to the right.	
Pale green.	Pale pink.	Both alike, no more colour	in them than in the cloudy sky out of window.	89.5°
Dirty white.	Ditto, both alike.	Both darker than before, but	no colour.	85.0
Fine bright pink.	Fine green, a little verging on bluish.	Very pale tinge of blue.	Very pale tinge of blue.	81.1
White.	White.	Yellow.	Blue.	76.3
The limit of	pink and red.	Both more coloured	than before.	
Rich grass green.	Rich crimson.	Yellow.	Blue.	74.9
		Better, but neither	full colours.	
Dull greenish blue.	Pale brick red.	Blue.	Yellow.	72.8
		Neither so rich	colours as the last.	
Purple (rather pale.)	Pale yellow.	Blue.	Yellow.	71.7
		Coming up to good colours,	than a gilt picture-frame.	
Fine pink.	Fine green.	Yellow, but has got a good deal of blue in it.	Blue, but has a good deal of yellow in it.	69.7
Fine yellow.	Purple.	Good yellow.	Good blue.	68.2
		Better colours than	any yet seen.	
Yellowish green.	Fine crimson.	Yellow, but has a good deal of blue.	Blue, but has a good deal of yellow.	67.0
Good blue, verging to indigo.	Yellow, verging to orange.	Blue.	Yellow.	65.5
		Both gay colours, particularly	the yellow to the right.	
Red, or very ruddy pink.	Very pale greenish blue, almost white.	Yellow.	Blue.	63.8
Rich yellow.	Full blue.	Fine bright yellow.	Pretty good blue.	62.7
White.	Fiery orange.	Has very little colour.	Yellow, but a different yellow, it is a blood-looking yellow.	61.2
Dark purple.	White.	A dim blue, wants light.	White, with a dash of yellow and blue.	59.5
Dull orange red.	White.	Yellow	White, with blue and yellow in it.	59.0
White.	Dull dirty olive.	White.	Dark.	57.1
Very dark purple.	White.	Dark.	White.	55.0

Instead of presenting the colours for his judgment, he was now desired to arrange the apparatus so as to make the strongest possible succession of contrasts of colour in the two circles. The results were as follow: 508.

Colours according to the judgment of an ordinary eye.		Colours as named by the individual in question.		Inclination of the plate of mica to eye.
Circle to the left.	Circle to the right.	Circle to the left.	Circle to the right.	
Pale ruddy pink.	Blue green.	Yellow.	Blue.	59.1°
Blue green.	Pale ruddy pink.	Blue.	Yellow.	65.3
Yellow.	Blue.	Yellow.	Blue.	63.1
White.	Fiery orange.	Blue.	Yellow.	61.1
Pale brick-red.	White.	Yellow.	Blue.	58.5
Indigo.	Pale yellow.	Blue.	Yellow.	54.2
Yellow.	Indigo.	Yellow.	Blue.	52.1

It appears by this, that the eyes of the individual in question are only capable of fully appreciating blue and yellow tints, and that these names uniformly correspond, in his nomenclature, to the more and less refrangible rays, generally; all which belong to the former, indifferently, exciting a sense of "blueness," and to the latter of "yellowness." Mention has been made of individuals seeing well in other respects, but devoid altogether of the sense of colour, distinguishing different tints only as brighter or darker one than another; but the case is, probably, one of extremely rare occurrence.

509. Mayer's hypothesis of three primary colours.

Mayer, in an Essay *De Affinitate Colorum*, (Opera inedita, 1775,) regards all colours as arising from three primary ones, red, yellow, and blue; regarding white as a neutral mixture of rays of all colours, and black as a mere negation of light. According to this idea, were we acquainted with any mode of mixing colours in simple numerical ratios, a scale might be formed to which any proposed colour might be at once referred. He proposes to establish such a scale in which the degrees of intensity of each simple colour shall be represented by the natural numbers 1, 2, 3. . . . 12; 1 denoting the lowest degree of it capable of sensibly affecting a tint, and 12 the full intensity of which the colour is capable, or the total amount of it existing in white light. Thus r^{12} denotes a full red of the brightest and purest tint, y^{12} the brightest yellow, and b^{12} the brightest blue. To represent mixed tints, he combines the symbols of the separate ingredients. Thus $r^{12}\ y^{4}$, or, more conveniently, $12\ r + 4\ y$, represents a red verging strongly to orange, such as that of a coal fire.

510. Modification of Mayer's scale.

The scale proposed is convenient and complete, so far as regards what he calls perfect colours, which arise from white light by the *subtraction* of one or more proportions of its elementary rays; but a very slight modifi-

cation of his system will render it equally applicable to all, and it may be presented as follows. Suppose we fix on 100 as a standard intensity of each primary colour; or the number of rays of that colour (all supposed equally effective) which falling on a sheet of white paper, or other surface perfectly neutral, (*i. e.* equally disposed to reflect all rays) shall produce a full tint of that particular kind, and let us denote by such an expression as $x\,R + y\,Y + z\,B$, the tint produced by the incidence of x such rays of primary red, y such rays of yellow, and z such rays of blue on the same surface together. It is obvious then, that the different numerical values assigned to x, y, z, from 1 to 100, will give different symbols of tints, whose number will be $100 \times 100 \times 100 = 1000000$, and therefore quite sufficient in point of extent to embrace all the variety of colours the eye can distinguish. The number of tints recognised as distinct by the Roman artists in Mosaic is said to exceed 30,000; but if we suppose ten times this amount to occur in nature (and it is obvious that these must be greatly more numerous than the purposes of the painter admit) we are still much within the limits of our scale. It only remains to examine how far the tints themselves are expressible by the members of the scale proposed.

511. Whites, greys, and neutral tints.

And first, then, of whites, greys, and neutral tints. The most perfectly neutral tints, which are, in fact, only greater and less intensities of whiteness, are those we observe in the clouds in an ordinary cloudy day, with occasional gleams of sunshine. From the most sombre shadows to the snowy whiteness of those cumulus-shaped clouds on which the sun immediately shines, we have nothing but a series of whites, or greys, represented by such combinations as $R + Y + B$, $2R + 2Y + 2B$, &c.; or $n\,(R + Y + B)$ which, for brevity, we may represent by $n\,W$. To be satisfied of this we need only look through a tube blackened on the inside to prevent surrounding objects influencing our judgments; and any small portion thus insulated of the darkest clouds will appear to differ in no respect from a portion similarly insulated of a sheet of white paper more or less shaded.

512. Reds, yellows, and blues.

The various intensities of pure reds, yellows, and blues are represented by $n\,R$, $n\,Y$, and $n\,B$ respectively. They are rare in nature; but blood, fresh gilding, or gamboge moistened, and ultramarine may be cited as examples of them. Scarlets and vivid reds, such as vermilion and minium, are not free from a mixture of yellow, and even of blue; for all the primary colours are greatly increased in splendour by a certain mixture of white, and whenever any primary colour is peculiarly glaring and vivid, we may be sure that it is in some degree diluted with white. The blue of the sky is white, with a very moderate addition of blue.

513. The mixture of red and yellow produces all the shades of scarlet, orange, and the deeper browns, when of feeble intensity. When diluted with white, we have lemon colour, straw colour, clay colour, and all the brighter browns; the last-mentioned tints growing duskier and dingier as the coefficients are smaller.

514. Browns.

The browns, however, are essentially sombre tints, and produce their effects chiefly by contrast with other brighter hues in their neighbourhood. To produce a brown, the painter mixes black and yellow, or black and red, (that is, such impure reds as the generality of red pigments,) or all three; his object is to stifle light, and leave only a residuum of colour. There is a brown glass very common in modern ornamental windows. If examined with a prism, it is found to transmit the red, orange, and yellow rays abundantly, little green, and no pure blue. The small quantity of blue, then, that its tint does involve, must be that which enters as a component part of its green, (in this view of the composition of colours,) and its characteristic symbol may thus be, perhaps, of some such form as $10\,R + 9\,Y + 1\,B$; that is to say, $(9\,R + 8\,Y) + 1\,(R + Y + B)$, or an orange of the character $9\,R + 8\,Y$ diluted with one ray of white. It must be confessed, however, that the composition of brown tints is the least satisfactory of all the applications of Mayer's doctrine. He himself has passed it unnoticed.

515. Purples.

Combinations of red and blue, and their dilutions with white, form all the varieties of crimson, purple, violet, rose colour, pink, &c. The richer purples are entirely free from yellow. The prismatic violet, when compared with the indigo, produces a sensible impression of redness, and must therefore be regarded on this hypothesis as consisting of a mixture of blue and red rays.

516. Greens.

Blue and yellow, combined, produce green. The green thus arising is vivid and rich; and, when proper proportions of the elementary colours are used, no way to be distinguished from the prismatic green. Nothing can be more striking, and even surprising, than the effect of mixing together a blue and a yellow powder, or of covering a paper with blue and yellow lines, drawn close together, and alternating with each other. The elementary tints totally disappear, and cannot even be recalled by the imagination. One of the most marked facts in favour of the idea of the existence of three primary colours, and of the possibility of an analysis of white light distinct from that afforded by the prism, is to see the prismatic green thus completely imitated by a mixture of adjacent rays totally distinct from it, both in refrangibility and colour.

517. The same colour produced by different prismatic combinations.

The hypothesis of three primary colours, of which, in different proportions, all the colours of the spectrum are composed, affords an easy explanation of a phenomenon observed by Newton, viz. that tints no way distinguishable from each other may be compounded by very different mixtures of the seven colours into which he divided it. Thus we may regard white light, indifferently, as composed of

$$\left.\begin{aligned} R &= a + b + c \text{ rays of pure red} \\ Y &= d + e + f + g \text{ rays of pure yellow} \\ B &= h + i + k + l \text{ rays of pure blue} \end{aligned}\right\} \text{ or of } \begin{cases} b \text{ rays of pure red} = R' \\ c + d \text{ rays of orange } (c \text{ red} + d \text{ yellow}) = O' \\ e \text{ rays of pure yellow} = Y' \\ f + h \text{ rays of green } (f \text{ yellow} + h \text{ blue}) = G' \\ g + i \text{ rays of prismatic blue } (g \text{ yellow} + i \text{ blue}) = B' \\ k \text{ rays of indigo, or pure blue} = I' \\ l + a \text{ rays of violet } (l \text{ blue} + a \text{ red}) = V' \end{cases}$$

and any tint capable of being represented by $x \,.\, \mathrm{R} + y \,.\, \mathrm{Y} + z\,\mathrm{B}$, may be represented equally well by

$$m \,.\, \mathrm{R}' + n \,.\, \mathrm{O}' + p \,.\, \mathrm{Y}' + q \,.\, \mathrm{G}' + r \,.\, \mathrm{B}' + s \,.\, \mathrm{I}' + t \,.\, \mathrm{V}',$$

provided we assume m, n, p, &c., such as to satisfy the equations

$$m\,b + n\,c + t\,a = x; \qquad n\,d + p\,e + q\,f + r\,g = y; \qquad q\,h + r\,i + s\,k + t\,l = z.$$

518. Dr. Young's hypothesis of three other primary colours.

From what has been said we shall now proceed to show, that, without departing from Mayer's doctrine, any other three prismatic rays may still be equally assumed as fundamental colours, and all the rest compounded from them, provided we attend only to the predominant tint resulting, and disregard its dilution with white. For instance, Dr. Young has assumed red, green, and violet as his fundamental colours; and states, as an experimental fact in support of this doctrine, that the perfect sensations of yellow and blue may be produced, the former by a mixture of red and green, and the latter by green and violet. (*Lectures on Natural Philosophy*, p. 439.) Now, if we mix together yellow and white in the proportion of m yellow + n white, the compound will produce a perfect sensation of yellow, unless m be small compared to n; but, assuming white to be composed as above, this compound is equivalent to

$$n\,\mathrm{R}\ \text{red} + (m + n)\,\mathrm{Y}\ \text{yellow} + n\,\mathrm{B}\ \text{blue}.$$

On the other hand, if we mix together P such red rays (each of the intensity b) and Q such green rays (each consisting of yellow, of the intensity f, and blue of the intensity h) as are supposed in the foregoing article to exist in the spectrum, we have a compound of

$$\mathrm{P} \,.\, b\ \text{red} + \mathrm{Q} \,.\, f\ \text{yellow} + \mathrm{Q} \,.\, h\ \text{blue},$$

and these will be identical with the former, if we take

$$n\,\mathrm{R} = \mathrm{P}\,b; \qquad (m + n)\,\mathrm{Y} = \mathrm{Q}\,f; \qquad n\,\mathrm{B} = \mathrm{Q}\,h.$$

Eliminating Q from the two last of these, we get

$$\frac{m}{n} = \frac{f}{h} \,.\, \frac{\mathrm{B}}{\mathrm{Y}} - 1$$

for the relation between M and N. Now the only conditions to be satisfied are that M shall be positive, and not much less than N; and it is evident that these conditions may be fulfilled an infinite number of ways by a proper assumption of the ratio of f to h. In the same manner, if we suppose a mixture of M rays primary blue = B with N rays of white (= R + Y + B) to be equivalent to P rays of prismatic green mixed with Q of violet, we get the equation

$$\frac{m}{n} = \frac{l}{a} \,.\, \frac{\mathrm{R}}{\mathrm{B}} + \frac{h}{f} \quad \frac{\mathrm{Y}}{\mathrm{B}} - 1.$$

519. Numerical Illustration.

Suppose, for example, we regard white light as consisting of 20 rays of primary red, 30 of yellow, and 50 of blue, and the several prismatic rays to consist as follows:

Red	8	rays primary red = h.
Orange	7	..... red + 7 primary yellow = $c + d$.
Yellow	8	..... yellow = e.
Green	10	..... yellow + 10 primary blue = $f + h$.
Blue	6	..... yellow + 12 primary blue = $g + i$.
Indigo	12	..... blue = k.
Violet	16	..... blue + 5 primary red = $l + a$.

Then will the union of 15 rays of such red with 30 of such green, produce a compound ray containing $15 \times 8 = 120$ of primary red, $30 \times 10 = 300$ of primary yellow, and $30 \times 10 = 300$ of primary blue; which are the same as exist in a yellow, consisting of 6 rays of white combined with 4 of primary yellow. In like manner, if 75 such green rays be combined with 100 such violet, the result will be $100 \times 5 = 500$ rays of primary red, $+ 75 \times 10 = 750$ of primary yellow, $+ 75 \times 10 + 100 \times 16 = 2350$ of primary blue, which together compose a tint identical with that which would result from the union of 25 rays of white with 22 of primary blue; that is to say, a fine lively blue. The numbers assumed above, it must be understood, are merely taken for the sake of illustration, and are no way intended to represent the true ratios of the differently coloured rays in the spectrum.

520. Phenomena of coloured flames

The analogy of the fixed lines in the solar spectrum might lead us to look for similar phenomena in other sources of light. Accordingly, Fraunhofer has found, that each fixed star has its own particular system of dark and bright spaces in its spectrum; but the most curious phenomena are those presented by coloured flames, which produce spectra (when transmitted through a colourless prism) hardly less capricious than those afforded by solar light transmitted through coloured glasses. Dr. Brewster, Mr. Talbot, and others, have examined these

Light. phenomena with attention; but the subject is not exhausted, and promises a wide field of curious research. Part II.

The following facts may be easily verified:

521. Flames of combustibles burning feebly. 1. Most combustible bodies consisting of hydrogen and carbon, as tallow, oil, paper, alcohol, &c. when first lighted and in a state of feeble and imperfect combustion, give blue flames. These, when examined by the prism, by letting them shine through very narrow slits parallel to its edge, as described in Art. 487, all give interrupted spectra, consisting, for the most part, of narrow lines of very definite refrangibility, either separated by broad spaces entirely dark, or much more obscure than the rest. The more prominent rays are, a very narrow definite yellow, a yellowish green, a vivid emerald green, a faint blue, and a strong and copious violet.

522. Burning strongly. 2. In certain cases when the combustion is violent, as in the case of an oil lamp urged by a blow-pipe, (according to Fraunhofer,) or in the upper part of the flame of a spirit lamp, or when sulphur is thrown into a white-hot crucible, a very large quantity of a definite and purely homogeneous yellow light is produced; and in the latter case forms nearly the whole of the light. Dr. Brewster has also found the same yellow light to be produced when spirit of wine, diluted with water and heated, is set on fire; and has proposed this as a means of obtaining a supply of homogeneous yellow light for optical experiments.

523. Flames coloured by saline bodies. 3. Most saline bodies have the power of imparting a peculiar colour to flames in which they are present, either in a solid or vaporous state. This may be shown in a manner at once the most familiar and most efficacious, by the following simple process: Take a piece of packthread, or a cotton thread, which (to free it from saline particles should have been boiled in clean water,) and having wetted it, take up on it a little of the salt to be examined in fine powder, or in solution. Then dip the wetted end of it into the cup of a burning wax candle, and apply it to the exterior of the flame, not quite in contact with the luminous part, but so as to be immersed in the cone of invisible but intensely-heated air which envelopes it. Immediately an irregular sputtering combustion of the wax on the thread will take place, and the invisible cone of heat will be rendered luminous, with that particular coloured light which characterises the saline matter employed.

524. The colour depends chiefly on the base. Thus it will be found that, in general,

Salts of soda give a copious and purely homogeneous yellow.
Salts of potash give a beautiful pale violet.
Salts of lime give a brick red, in whose spectrum a yellow and a bright green line are seen.
Salts of strontia give a magnificent crimson. If analyzed by the prism two definite yellows are seen, one of which verges strongly to orange.
Salts of magnesia give no colour.
Salts of lithia give a red, (on the authority of Dr. Turner's experiments with the blow-pipe.)
Salts of baryta give a fine pale apple-green. This contrast between the flames of baryta and strontia is extremely remarkable.
Salts of copper give a superb green, or blue green.
Salt of iron (protoxide) gave white, where the sulphate was used.

Of all salts, the muriates succeed best, from their volatility. The same colours are exhibited also when any of the salts in question are put (in powder) into the wick of a spirit lamp. If common salt be used, Mr. Talbot has shown that the light of the flame is an absolutely homogeneous yellow; and, being at the same time very copious, this property affords an invaluable resource in optical experiments, from the great ease with which it is obtained, and its identity at all times. The colours thus communicated by the different bases to flame, afford in many cases a ready and neat way of detecting extremely minute quantities of them; but this rather belongs to Chemistry than to our present subject. The pure earths, when violently heated, as has recently been practised by Lieutenant Drummond, by directing on small spheres of them the flames of several spirit lamps urged by oxygen gas, yield from their surfaces lights of extraordinary splendour, which, when examined by prismatic analysis, are found to possess the peculiar definite rays in excess, which characterise the tints of flames coloured by them; so that there can be no doubt that these tints arise from the molecules of the colouring matter reduced to vapour, and held in a state of violent ignition.

PART III.

OF THE THEORIES OF LIGHT.

Light. Part III.

AMONG the theories which philosophers have imagined to account for the phenomena of light, two principally have commanded attention; the one conceived by Newton, and called from his illustrious name, in which light is conceived to consist of excessively minute molecules of matter projected from luminous bodies with the immense velocity due to light, and acted on by attractive and repulsive forces residing in the bodies on which they impinge, which turn them aside from their rectilinear course, and reflect and refract them according to the laws observed. The other hypothesis is that of Huygens, and also called after his name; which supposes light to consist, like sound, in undulations, or pulses, propagated through an elastic medium. This medium is conceived to be of extreme elasticity and tenuity; such, indeed, that though filling all space, it shall offer no appreciable resistance to the motions of the planets, comets, &c. capable of disturbing them in their orbits. It is, moreover, imagined to penetrate all bodies; but in their interior to exist in a different state of density and elasticity from those which belong to it in a disengaged state, and hence the refraction and reflexion of light. These are the only *mechanical* theories which have been advanced. Others, indeed, have not been wanting; such as Professor Oersted's, who, in one of his works, considers light as a succession of electric sparks, or a series of decompositions and recompositions of an electric fluid filling all space in a neutral or balanced state, &c. &c. In this part, however, we propose only to give an account of the Newtonian and Huygenian theories, so far as they apply to the phenomena already described; and thus prepare ourselves for the remaining more complex branches of the History of the Properties of Light, which can hardly be understood, or even described, without a reference to some theoretical views. 525.

§ I. *Of the Newtonian or Corpuscular Theory of Light.*

Postulata. 1. That light consists of particles of matter possessed of inertia and endowed with attractive and repulsive forces, and projected or emitted from all luminous bodies with nearly the same velocity, about 200,000 miles per second. 526

2. That these particles differ from each other in the intensity of the attractive and repulsive forces which reside in them, and in their relations to the other bodies of the material world, and also in their actual masses, or inertia.

3. That these particles, impinging on the retina, stimulate it and excite vision. The particles whose inertia is greatest producing the sensation of red, those of least inertia of violet, and those in which it is intermediate the intermediate colours.

4. That the molecules of material bodies, and those of light, exert a mutual action on each other, which consists in attraction and repulsion, according to some law or function of the distance between them; that this law is such as to admit, perhaps, of several alternations, or changes from repulsive to attractive force; but that when the distance is below a certain very small limit, it is always attractive up to actual contact; and that beyond this limit resides at least one sphere of repulsion. This repulsive force is that which causes the reflexion of light at the external surfaces of dense media; and the interior attraction that which produces the refraction and interior reflexion of light.

5. That these forces have different absolute values, or intensities, not only for all different material bodies, but for every different species of the luminous molecules, being of a nature analogous to chemical affinities, or ~~electric~~ attractions; and that hence arises the different refrangibility of the rays of light.

electric

6. That the motion of a particle of light under the influence of these forces and its own velocity is regulated by the same mechanical laws which govern the motions of ordinary matter, and that therefore each particle describes a trajectory capable of strict calculation so soon as the forces which act on it are assigned.

7. That the distance between the molecules of material bodies is exceedingly small in comparison with the extent of their spheres of attraction and repulsion on the particles of light. And

8. That the forces which produce the reflexion and refraction of light are, nevertheless, absolutely insensible at all measurable or appreciable distances from the molecules which exert them.

9. That every luminous molecule, during the whole of its progress through space, is continually passing through certain periodically recurring states, called by Newton fits of easy reflexion and easy transmission, in virtue of which (from whatever cause arising, whether from a rotation of the molecules on their axes, and the consequent alternate presentation of attractive and repulsive poles, or from any other conceivable cause) they are more disposed, when in the former states or phases of their periods, to obey the influence of the repulsive or reflective forces of the molecules of a medium; and when in the latter, of the attractive. This curious and delicate part of the Newtonian doctrine will be developed more at large hereafter.

Light. 527. It is the 7th and 8th of these assumptions only which render the course pursued by a luminous molecule, under the influence of the reflective or refractive forces, capable of being reduced to mathematical calculation; for it follows immediately from the 8th, that, up to the very moment when such a molecule arrives in physical contact with the surface of any medium, it is acted on by no sensible force, and therefore not sensibly deviated from its rectilinear path; and, on the other hand, as soon as it has penetrated to any sensible depth within the surface, or among the molecules, by reason of the 7th of the above postulates, it must be equally attracted and repelled by them in all directions, and therefore will continue to move in a right line, as if under the influence of no force. It is only, therefore, within that insensible distance on either side the surface, which is measured by the diameter of the sphere of action of each molecule, that the whole flexure of the ray takes place. Its trajectory then may be regarded as a kind of hyperbolic curve, in which the right lines described by it, previous and subsequent to its arrival at the surface, are the infinite branches, and are confounded with the asymptotes, and the curvilinear portion is concentered as it were in a physical point. Now, in explaining the phenomena of reflexion and refraction, it is not the nature of this curve that we are called on to investigate. This will depend on the laws of corpuscular action, and must necessarily be of great complexity. All we have to inquire, is the *direction* the ray will ultimately take after incidence, and the final change, if any, in its velocity. Part III.

528. Motion of a luminous molecule under the influence of any forces. Let us, then, consider the motion of a molecule urged to or from the surface of a medium by the united attractions or repulsions of all its particles acting according to any conceivable mathematical law. And, first, it is evident, that supposing the surface mathematically smooth, and the number of attractive or repulsive particles of which it consists, infinite, their total resultant force on the luminous molecule will act in a direction perpendicular to the surface; and will be insensible at all sensible distances from the surface, provided the elementary forces of each molecule decrease with sufficiently great rapidity as the distances increase. This condition being supposed, let x and y be the coordinates of the molecule at any assigned instant; the plane of the x and y being supposed to coincide with that of its trajectory, out of which plane there is evidently no force to turn it, and which must of course be perpendicular to the surface of the medium in which x is supposed to lie: y then will be the perpendicular distance of the luminous molecule from this surface, and Y (some function of y decreasing with extreme rapidity) will represent the force urging it inwards, or *towards* the surface when the molecule is without, *from* when within the medium. Therefore, by the principles of Dynamics, supposing $d\,t$ to denote the element of the time, we shall have for the equations of the motion

$$\frac{d^2\,x}{d\,t^2} = 0\,; \qquad \frac{d^2\,y}{d\,t^2} + \mathrm{Y} = 0\,; \qquad (a)$$

and hence, multiplying the first by $d\,x$, the second by $d\,y$, adding and integrating, we get

$$\frac{d\,x^2 + d\,y^2}{d\,t^2} + 2\int \mathrm{Y}\,d\,y = \text{constant}.$$

Now, v being the velocity of the molecule, we have $v^2 = \frac{d\,x^2 + d\,y^2}{d\,t^2}$, and therefore this equation becomes

$$v^2 = \text{constant} - 2\int \mathrm{Y}\,d\,y.$$

It is, however, only with the terminal velocity, or that attained by the light after undergoing the total action of the medium, that we are concerned, and therefore if we put V for its primitive, or initial, and V′ for its terminal velocity, we shall have, by extending the integral from the value of y at the commencement of the ray's motion (y_o) to its value at the end (y_i),

$$\mathrm{V}'^2 - \mathrm{V}^2 = -2\int \mathrm{Y}\,d\,y. \qquad (b)$$

Since y_o and y_i are supposed infinite, and since the function Y decreases by hypothesis with such rapidity as to become absolutely insensible for all sensible values of y, it is clear that we may take $y_o = +\infty$ for the first limit of the integral in all cases. With regard to the other, we have now to distinguish two principal cases:

529. Case of reflexion. The first is that of *reflexion*, where the ray, no matter whether before its arrival at the surface, or at reaching it, or even after passing some small distance into the medium, is turned back by the prevalence of the repulsive force, and pursues the whole of its course afterwards without the medium. Now in this case if we resolve the integral $\int \mathrm{Y}\,d\,y$ into its elements, these, in the approach of the molecule to the surface, may be represented as follows,

$$\&c. + \mathrm{Y}' \times -d\,y + \mathrm{Y}'' \times -d\,y + \mathrm{Y}''' \times -d\,y + \&c.$$

But in the recess of the molecule, the values of y increase again by the same steps as they before diminished and become identical with the former ones; and Y′, Y″, &c., the values of Y corresponding to the successive values of y, remain therefore the same, both in size and magnitude; the corresponding elements of the integral generated during the recess of the molecule will be then

$$\&c. + \mathrm{Y}' \times +d\,y + \mathrm{Y}'' \times +d\,y + \mathrm{Y}''' \times +d\,y + \&c.$$

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So that, combining both, the latter exactly destroy the former, and give $\int Y\,dy = 0$ when extended from one end to the other of the trajectory. Thus we have, in the case of reflexion,

$$V'^2 - V^2 = 0, \qquad \text{or } V' = V.$$

530. Case of refraction.

The second case is that in which the whole course of the ray after incidence lies within the medium, or the case of refraction. Here the values of y before incidence are all positive, and after, all negative; and, moreover, the change of sign in dy which happened in the case of reflexion, does not here take place. Hence $\int Y\,dy$ must be extended from $+\infty$ to $-\infty$, and its value will not vanish, but (on account of the rapid decrease of the function Y) will have some finite value. Now this can only be dependent on the arbitrary quantities which enter into the composition of Y; in other words, on the nature of the medium and the ray, and not at all on the constants which determine the direction of the ray with respect to the surface, (as its inclination or the position of the plane of incidence.) Hence we may suppose $\int Y\,dy = -\frac{1}{2}kV^2$, where k is a constant independent of the direction of the ray, and determined only by its nature and that of the medium, and we shall have

$$V'^2 = V^2(1+k); \qquad V' = V.\sqrt{1+k} = \mu V, \qquad (c)$$

putting $\sqrt{1+k} = \mu$.

531. Law of velocities.

Hence we see that both in refraction and reflexion, on this hypothesis, the velocity of the ray after deviation is the same in whatever direction the ray be incident, viz. in a given ratio to the velocity before incidence, this ratio being one of equality in the case of reflexion.

552. Direction of the ray after flexure.

Let us next consider the direction of the ray after flexure. To this end let θ = the angle made by its path at any moment with the perpendicular to the surface, then will $\sin\theta = \frac{dx}{ds}$, putting ds for $\sqrt{dx^2+dy^2}$, the element of the arc. Now if we integrate the equation $\frac{d^2x}{dt^2} = 0$ once we get $\frac{dx}{dt}$ = constant = c, and $dx = c\,dt$, wherefore $\sin\theta = \frac{c\,dt}{ds}$. But $v = \frac{ds}{dt}$, therefore $\sin\theta = \frac{c}{v}$. Let therefore θ_0 and θ_1 represent the initial and terminal values of θ, or the angles of incidence and reflexion, or refraction of the rectilinear portions of the ray, and we get

Constancy of ratio of sines of incidence and refraction.

$$\sin\theta_0 = \frac{c}{V}, \qquad \text{and } \sin\theta_1 = \frac{c}{V'},$$

and dividing one by the other

$$\frac{\sin\theta_0}{\sin\theta_1} = \frac{V'}{V} = \mu.$$

That is to say, the sines of incidence and refraction, or reflexion, are to each other in a constant ratio, viz. the inverse ratio of the velocities of the ray before and after incidence.

533. Thus we see the Newtonian hypothesis satisfies the fundamental conditions of refraction and reflexion without entering into any consideration respecting the laws of the refracting and reflecting forces, or even the order of their superposition. There may be as many alternations of attraction and repulsion as we please, and the reflected or refracted ray may therefore, prior to its final recess from the surface, make any variety of undulations; all that is required is the extremely rapid decrease of the function Y expressing the total force before the distance attains a sensible magnitude.

534. Hence also, V and V′ being the velocities before and after incidence, and μ the index of refraction, we have

$$V' : V :: \mu : 1,$$

which shows, that when a ray passes from a rarer medium to a denser, its velocity is increased, and *vice versâ*.

535. Refractive power of a medium.

Moreover, we have

$$k = \frac{V'^2 - V^2}{V^2} = \left(\frac{V'}{V}\right)^2 - 1 = \mu^2 - 1 = \frac{2\int -Y\,dy}{V^2}.$$

Now if we suppose the *form* of the function Y to be the same for all media, and that they differ in the energy of action only by reason, first, of a greater density, owing to which more molecules are brought within the sphere of activity; and, secondly, by reason of a greater or less affinity, or intensity of action of each molecule, we may suppose Y to be represented by $S.n.\phi(y)$, where S is the specific gravity, or density, n the intrinsic refractive energy of the medium, and $\phi(y)$ a function absolutely independent of the peculiarities of the medium, and the same for all natural bodies. Hence $\int -Y\,dy = S.n.\int -\phi(y)\,dy = S.n.$ constant because $\int -\phi(y)\,dy$ taken from $y = +\infty$ to $y = -\infty$ will now be an absolute numerical constant. We have then, according to this doctrine,

$$n = \frac{\mu^2 - 1}{S} \times \frac{V^2}{2.\text{constant}}.$$

If μ be the refractive index of a given standard ray out of a vacuum, V the velocity of that ray in vacuo is known, and is also an absolute constant; so that n, the *intrinsic refractive power* of the medium is proportional to

Light. $\frac{(\text{refractive index})^2 - 1}{\text{specific gravity}}$. Such is Newton's idea of the *refractive power* of a medium as differing from its refractive *index*. It rests, however, on a purely hypothetical assumption, that of the similarity of *form* of the law of force for all media, respecting which we can be said to know nothing whatever. For a table of its values for different media, see the Collection of Tables at the end of this Essay. Part III.

536. Principle of least action employed. The constancy of the ratio of the sines of incidence and refraction has here been derived by direct integration of the fundamental equations. There is, however, another mode of deducing this important law, much more circuitous, it is true, in this simple case, but which offers peculiar advantages in the more complicated ones of double refraction; and which, therefore we shall here explain, to familiarize the reader beforehand with its principle and mode of application. It consists in the employment of what is called, in Dynamics, the *principle of least action*, in virtue of which the sum of each element of the trajectory described by any moving molecule multiplied by the velocity of its description (or the integral $\int v\,ds$) is a minimum when taken between any two fixed points in the trajectory. The trajectory described by any luminous molecule may be regarded as consisting of two rectilineal portions, or hyperbolic branches, confounded with their asymptotes, and one curvilinear one concentrated in a space of insensible magnitude, a physical point. Within this *point* the whole operation of the flexure of the ray, however complicated, is performed; and *here* the velocity is variable. In the branches it is uniform. Suppose, then, A and B to be any two fixed points in these, taken as points of departure and arrival of a ray, and let C be the point in the surface of a reflecting or refracting medium where the flexure takes place, and suppose A C = S, B C = S′ and let σ be the excessively minute curvilinear portion of the ray at C, and v the variable velocity with which it is described, V and V′ being those with which S and S′ are described. Then may the integral $\int v\,ds$ be resolved into the three portions $\int V\,dS + \int v\,d\sigma + \int V'\,dS'$. Of these the second is utterly insensible, by reason of the minuteness of σ, and the other two, since V and V′ are constant, become merely $V.S + V'.S'$.

537. The position of C, then, with respect to A and B, will be determined by the condition $V.S + V'.S' =$ a minimum, A and B being supposed fixed, and C any how variable on the surface. Now, in the case before us, V the velocity of the light before, and V′ that after incidence, are *both*, as we showed in Article 529 and 530, independent of the direction of the incident and reflected or refracted rays, or of the position of C; and, therefore, are to be considered as absolute constants in this problem of minima, which is thus reduced to a simple geometrical question. Given A and B to find C, a point in a given plane, such that V (= constant) × A C + V′ (= constant) × B C shall be a minimum. Nothing is easier than the solution. Put a, b, c, a', b', c' for the respective coordinates of A and B, and x, y, o for that of C, taking the given plane for that of the x, y. Then

Solution of the geometrical problem of the minimum

$$V.S + V'.S'. = V.\sqrt{(x-a)^2 + (y-b)^2 + c^2} + V'.\sqrt{(x-a')^2 + (y-b')^2 + c^2}$$

is to be a minimum by the variation of x and y, independent of each other. This gives, by differentiation,

$$\frac{V}{S}\left\{(a-x)\,dx + (b-y)\,dy\right\} + \frac{V'}{S'}\left\{(a'-x)\,dx + (b'-y)\,dy\right\} = 0;$$

and this, since x and y are independent, must vanish, whatever values are assigned to dx and dy, therefore we must have separately

$$\frac{V}{S}(a-x) + \frac{V'}{S'}(a'-x) = 0; \qquad \frac{V}{S}(b-y) + \frac{V'}{S'}(b'-y) = 0. \qquad (d)$$

These give, respectively,

$$\frac{S'}{S} = -\frac{V'}{V}\cdot\frac{a'-x}{a-x}; \qquad \frac{S'}{S} = -\frac{V'}{V}\cdot\frac{b'-y}{b-y};$$

by equating which we get

$$(a'-x)(b-y) = (b'-y)(a-x);$$

or multiplying out and reducing

$$y = x\cdot\frac{b-b'}{a-a'} + \frac{ab' - ba'}{a-a'};$$

and, consequently,

$$b'-y = \frac{b-b'}{a-a'}(a'-x)$$

This equation expresses, that the two portions S and S′ of the ray before and after incidence on the surface at C both lie in one plane, and that this plane is perpendicular to the surface, or to the plane of the coordinates x, y.

538. Constancy of the ratio of the sines deduced. Again, if we resume the equations (d) and putting them under the form

$$S'(a-x) = -\frac{V'}{V}S(a'-x); \qquad S'(b-y) = -\frac{V'}{V}(b'-y).S.$$

Square and add them we get

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$$S'^2\left\{(a-x)^2+(b-y)^2\right\}=\left(\frac{V'}{V}\right)^2.\left\{(a'-x)^2+(b'-y)^2\right\}.S^2.$$

Now if we put θ for the angle made by the portion S with a perpendicular to the surface, or the angle of *incidence* of the ray, and θ' for that made by the other S′ with the same perpendicular, or the angle of emergence, we shall have

$$\sin\theta=\frac{\sqrt{(a-x)^2+(b-y)^2}}{S}\quad\text{and}\quad\sin\theta'=\frac{\sqrt{(a'-x)^2+(b'-y)^2}}{S'}.$$

So that the above equation is equivalent simply to

$$\sin\theta=\frac{V'}{V}.\sin\theta',$$

which is the same with the result before obtained.

539. Advantages afforded by the principle of least action.

The principle of least action, then, in the case before us, has enabled us to dispense with one integration of the differential equations expressing the motion of the luminous molecule; and its applicability to this purpose depends, as we have seen, on the relation between V and V′; the velocities of the light, before and after incidence, being known. This relation has here been deduced *à priori;* but had it been merely known, as a matter of fact, a conclusion established by experiment, it would not be on that account the less applicable to the same purpose, and the laws of refraction and reflexion would be derivable from it by the same process. There would, however, be this main difference; that, in the latter case, we should have no occasion to employ the differential equations at all, and therefore none to enter into any consideration of the *forces* acting on the luminous molecule, or their mode of action. The principle of least action establishes, independent of, and anterior to, all particular suppositions as to the forces which operate the flexure of the ray, (further than that they are functions of the distances from their origins or centres,) an analytical relation between the velocities before and after incidence, and the directions of its direct and deviated branches; a relation nearly as general as the laws of dynamics themselves, and expressive, in fact, of only the one condition above mentioned. And this relation, from its form, enables us, whenever the relation of the velocities is known, to determine that of the directions of the two portions of the ray, and *vice versâ*, without having recourse to the differential equations at all. In the simple case before us this may seem a needless refinement, the equations being so simple. It is otherwise, however, in the theory of double refraction. There the forces in action are altogether unknown, not only in respect of their intensity, but of their directions; and so far, therefore, from being able in that theory to *integrate* the equations of the ray's motion, we cannot even express them analytically. The principle we are now considering is, in such a case, all the ground we have to stand upon; and has been ingeniously and elegantly applied by Laplace, in that theory, to reduce the complicated laws of double refraction under the dominion of analysis.

Applicable to other cases.

540. Mode of its application in general.

In fact, suppose that the velocities of the incident and deviated portions of the rays, instead of being the same in every direction, varied with the positions of these portions with respect to the surface of the medium, or to any fixed lines or axes in space. Then will V and V′, instead of being invariable, be represented by functions of the three coordinates of the point C, either rectangular, as x, y, z; or polar, as ϕ, θ, and γ; and the portions S and S′ of the rays intercepted between A and B respectively, and the surface at C, will, in like manner, be functions of the same coordinates. So that the condition

$$V.S+V'.S'=\text{a minimum}$$

will afford, by differentiation and putting the differential equal to zero, an equation of the form $L\,dx+M\,dy+N\,dz=0$, or $L\,d\phi+M\,d\theta+N\,d\gamma=0$, as the case may be. The equation of the surface also being differentiated affords another relation of the same kind; and these being the only conditions to which the differentials dx, dy, dz are subject, we may eliminate one, and put the coefficients of the remaining ones separately equal to zero. Thus we get two equations between the coordinates, which, combined with that of the surface, suffice to determine them, i. e. to fix the point C at which the ray A C must meet the surface, in order that, being there deviated by the action of the medium, it may, after flexure, proceed to B; and thus the problem of reflexion or refraction may be resolved in all its generality, so soon as the nature of the functions V, V′ is known. But to return to the case of ordinary reflexion and refraction, from which this is a digression.

541. Course of a ray in the confines of a reflecting and refracting medium traced.

Let us consider, a little more in detail, what may be conceived to happen to a ray at the confines of the surface of a medium. We may suppose, then, that there exist a series of laminar spaces, or strata, within which the attractive and repulsive action of the molecules of the medium alternately predominate. Of these there may be any number, and either may be exterior to the rest. It is, in fact, the assemblage of these laminæ which is to be regarded as *the surface* of the medium. Suppose now a ray A a (fig. 119) to be moving towards the medium. Its course will be rectilinear up to a, where it first comes within the action of the medium. If the first stratum into which it enters be one of attraction, its course will be bent as $a\,b$ into a curve concave towards the surface, and its velocity in the direction perpendicular to the surface will be increased. Arrived at b let the force change to repulsive; the trajectory will have at b a point of contrary flexure, the portion $b\,c$ within this lamina will be convex to the surface, and the velocity towards the surface will be diminished in the whole progress of the ray through it, and so for any number of alternations. Let us now suppose, that in passing through any repulsive lamina, as C, the repulsion should be so strong, or the original velocity of approach to the surface so small, as that the whole of it shall be destroyed. In this case the ray for a moment will be moving as at C, parallel to the surface, but the repulsive force continuing its action will turn it back; and the forces

Fig. 119.

Light. now being all equal to what they were before, but acting in a contrary direction with respect to the motion of the molecule, it will describe a portion C d' c' b' a' B similar, and equal to the portion on the other side of C. This is the case of reflexion. But suppose, as in fig. 120, the ray to have such an initial velocity of approach, or the repulsive forces to be so feeble, compared to the attractive, that before its whole velocity perpendicular to the surface is destroyed, it shall have passed through all the strata of attraction and repulsion, and entered the region where the forces of all the molecules are in equilibrium, as at *e*. In this case the remainder of its course will be rectilinear, and wholly within the medium. This is the case of refraction. In both cases, it is the final course it takes, or the direction of the asymptotic branches a' B or *e* B, about which only we have any knowledge; of the number and nature of the undulations of its course between a and a', or *e*, we know nothing. Part III.

542. Motion of a ray at common surface of two media. The whole of this reasoning applies equally to the motion of a luminous molecule at the confines of two media, as at the surface separating one medium from a vacuum. The molecules of either medium being supposed *uniformly distributed*, and *acting equally in all directions around them*, the resultant of all their forces on the luminous particle must be perpendicular to the common surface, which is all that is required in the foregoing theory.

543. Newtonian idea of a ray of light as composed of a succession of molecules. In the Corpuscular doctrine, a ray of light is understood to mean a continued succession or stream of molecules, all moving with the same velocity along one right line, and following each other close enough to keep the retina in a constant state of stimulus, i. e. so fast, that before the impression produced by one can have time to subside another shall arrive. It appears, by experiment, that to produce a continued sensation of light, it is sufficient to repeat a momentary flash about 8 or 10 times in a second. If a red-hot coal on the point of a burning stick be whirled round, so as to describe a circle, and the velocity of rotation be greater than 8 or 10 circumferences per second, the eye can no longer distinguish the place of the luminous point at any instant, and the whole circle appears equally bright and entire. This shows, evidently, that the sensation excited by the light falling on any one point of the retina, must remain almost without diminution till the impression is repeated during the subsequent revolution of the luminary. Now, if uninterrupted vision can be produced by momentary impressions, repeated at intervals so distant as a tenth of a second, it is easy to conceive that the individual molecules of light in a ray will not require to follow close on each other to affect our organs with a continued sense of light. As their velocity is nearly 200,000 miles per second, if they follow each other at intervals of 1000 miles apart, 200 of them would still reach our retina per second, in every ray. This consideration frees us from all difficulties on the score of their jostling, or disturbing each other in space, and allows of infinite rays crossing at once through the same point of space without at all interfering with each other, especially when we consider the minuteness which must be attributed to them, that (moving with such swiftness) they should not injure our organs. If a molecule of light weighed but a single grain, its inertia would equal that of a cannon ball of upwards of 150 pounds weight, moving at the rate of 1000 feet per second. What then must be their tenuity, when the concentration of millions upon millions of them, by lenses or mirrors, has never been found to produce the slightest mechanical effect on the most delicately contrived mechanism, in experiments made expressly to detect it. (See Mr. Bennet's *Experiments*, *Phil Trans.* 1792, vol. lxxxii. p. 87.)

Their distance *inter se.*

Their extreme tenuity illustrated.

544. Partial reflexion explained on Newton's principles. When a ray of light falls on a reflecting or refracting surface, since all its molecules move with equal velocity and are incident in the same line, it would seem that whatever took place with one should equally happen to all; and that, if the first underwent reflexion, all should do so; while, on the other hand, if one could penetrate the surface, and pursue its course entirely within the medium, all ought to do the same. This, however, is contrary to experience; as whenever a ray of light is incident on the exterior surface of any medium, a part only is reflected, and the rest enters the medium. No theory can be satisfactory which does not render a good account of so principal a fact. The Newtonian doctrine accounts for it by the fits of easy reflexion and transmission. To understand this explanation we must recur to the ninth postulate, (Art. 526,) and suppose two molecules to arrive at the surface under the same incidence, the one in a fit of easy reflexion, the other in one of easy transmission. The former will then be more affected by the repulsive forces, the latter by the attractive of the molecules of the medium; and hence it is evident, that the one may be reflected under circumstances of incidence, &c. in which the other will penetrate the surface and be refracted. Now it will depend entirely on the nature of the medium, and the initial velocity of a luminous molecule *towards* it, (which is as the cosine of the angle of incidence,) whether it will require the whole exertion of its repulsive forces, in their most energetic manner, to destroy that velocity and produce reflexion, or only a part of them. In the former case only such molecules as arrive *in the most favourable* disposition to be reflected, or in the most intense phase of a fit of easy reflexion, can be reflected. In the latter, such as arrive in less favourable dispositions, or *in less intense phases* of fits of reflexion, may be reflected; and if the repulsive forces of the medium be very intense, in comparison with the attractive ones, or if the obliquity of incidence be so great as to give the molecule a very small velocity perpendicular to the surface, even those molecules which arrive in the less energetic phases of fits of easy transmission may still be unable to penetrate the strata of repulsion.

545. Reflexion more copious at great obliquities. Hence, then, we see that the proportion of the molecules of a ray falling on the surface of a medium in every possible state or phase of their fits, which undergo reflexion, will depend, first, on the nature of the medium on whose surface they fall, or if it be the common surface of two, then on both; secondly, on the angle of incidence. At great obliquities, the reflexion will be more copious; but even at the greatest, when the incident ray just grazes the surface, it by no means follows that every molecule, or even the greater part, *must* be reflected. Those which arrive in the most favourable phases of their fits of transmission, will obey the influence of small attractive forces, in preference to strong repulsive ones; but it will depend entirely on the nature of the media whether the former or the latter shall prevail, the fits in the Newtonian doctrine being conceived only to *dispose* the luminous molecules, other circumstances being favourable, to reflexion or transmission; to exalt the forces which

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tend to produce the one and to depress those which act in favour of the other, but not to determine, absolutely, its reflexion or transmission under all circumstances.

546. Confirmed by experiment.

These conclusions are verified by experience. It is observed, that the reflexion from the surfaces of transparent (or indeed any) media, becomes sensibly more copious as the angle of incidence increases; but at the *external* surface of a single medium is never total, or nearly total. In glass, for instance, even at extreme obliquities, a very large portion of the light still enters the glass and undergoes refraction. In opaque media, such as polished metals, the same holds good; the reflexion increases in vividness as the incidence increases, but never becomes total, or nearly so. The only difference is, that here the portion which penetrates the surface is instantly absorbed and stifled.

547. Reflexion at common surface of two media.

The phenomena which take place when light is reflected at the common surface of two media, are such as from the above theory we might be led to expect,—with the addition, however, of some circumstances which lead us to limit the generality of our assumptions, and tend to establish a relation between the attractive and repulsive forces, to which the refraction and reflexion of light are supposed to be owing. For it is found, that when two media are placed in perfect contact, (such as that of a fluid with a solid, or of two fluids with one another,) the intensity of reflexion at their common surface is always less, the nearer the refractive indices of the media approach to equality; and when they are exactly equal, reflexion ceases altogether, and the ray pursues its course in the second medium, unchanged either in direction, velocity, or intensity. It is evident, from this fact, which is general, that the reflective or refractive forces, in all media of equal refractive densities, follow exactly the same laws, and are similarly related to one another; and that in media unequally refractive, the relation between the reflecting and refracting forces is not arbitrary, but that the one is dependent on the other, and increases and diminishes with it. This remarkable circumstance renders the supposition made in Art. 535, of the identity of form of the function Y, or $\phi\ (y)$ expressing the law of action of the molecules of all bodies on light indifferently, less improbable.

548. Phenomena exhibited experimentally.

To show experimentally the phenomena in question, take a glass prism, or thin wedge of very small refracting angle (half a degree, for instance: almost any fragment of plate glass, indeed, will do, as it is seldom the two sides are parallel,) and placing it conveniently with the eye close to it, view the image of a candle reflected from the exterior of the face next the eye. This will be seen accompanied at a little distance by another image, reflected internally from the other face, and the two images will be nearly of equal brightness, if the incidence be not very great. Now, apply a little water, or a wet finger, or, still better, any black substance wetted, to the posterior face, at the spot where the internal reflexion takes place, and the second image will immediately lose great part of its brightness. If olive oil be applied instead of water, the defalcation of light will be much greater; and if the substance applied be pitch, softened by heat, so as to make it adhere, the second image will be totally obliterated. On the other hand, if we apply substances of a higher refractive power than glass, the second image again appears. Thus, with oil of cassia it is considerably bright; with sulphur, it cannot be distinguished from that reflected at the first surface; and if we apply mercury, or amalgam, (as in a silvered looking-glass,) the reflexion at the common surface of the glass and metal is much more vivid than that reflected from the glass alone.

549. Phenomena depending on the foregoing principles.

The destruction of reflexion at the common surface of two media of equal refractive powers explains many curious phenomena. If we immerse an irregular fragment of a colourless transparent body (as crown glass) in a colourless fluid of precisely equal refractive power, it disappears altogether. In fact, the surface being only visible by the rays reflected from it; destroy this reflexion, and the object must cease to be seen, unless from any opacity in its substance reflecting rays from its interior, which is not here contemplated. Hence, if the powder of any such substance be moistened with a fluid of the same refractive density, all the internal and external reflexions at the surfaces of the small fragments of which it consists, which, blended and confused, present the general appearance of a white opaque mass, will be destroyed, and the powder will be rendered perfectly transparent. A familiar instance of this nature is the transparency given to paper by moistening it with water, or, still better, with oil; paper is composed of an infinity of minute transparent, or nearly transparent fibres of a ligneous substance, having a refractive power probably not very different from some of the more refractive oils. Its whiteness is caused by the confused reflexion of the incident rays at all possible angles, both internally and externally, those which have escaped reflexion at one fibre, undergoing it among those beneath. If moistened with any liquid, the intensity of these reflexions is weakened, and the more the more nearly its refractive power approaches to that of the paper itself; so that a considerable number of rays find their way through it, and emerge at the posterior surface. The transparency acquired by the hydrophane, by immersion in water, is, no doubt, owing to this cause; the water filling up the minute pores, and enfeebling the internal reflexion; and Dr. Brewster, in a very curious and interesting Paper on the tabasheer, (a siliceous concretion found in sugar-canes, and the lowest in the scale of refracting powers among solids,) has explained on this principle a number of extraordinary phenomena exhibited on moistening that substance with various liquids, (see *Philosophical Transactions*, 1819.)

Transparency of oiled paper.

550. Total internal reflexion.

The reasoning of Art. 529 applies, it is evident, equally to the case when a ray is reflected from the interior surface of a dense medium placed in air, and when from the exterior. The only difference is, that in the latter case the reflexion is performed by the action of repulsive, and in the former by that of attractive forces. The course of a ray internally reflected may be conceived, as in fig. 121 and 122; and the reflexion may take place in any of the attractive regions, or laminæ, whether within or without the true surface, i. e. the last layer of molecules which constitute the medium. There is one case of internal reflexion, however, too remarkable to be passed without more particular notice. It is, that when the interior angle of incidence exceeds the *limiting angle* whose sine is $\frac{1}{\mu}$, (see Art. 183 *et seq.*;) and when, as we there stated, as a result of experiment, the

Light. internal reflexion is *total*. To see how this happens, let us consider a ray incident exactly at this angle, and in the most intense phase of its fit of transmission. Then will it be refracted; and, since the angle of refraction must be just 90°, (by reason of the generality of the demonstration of the law of refraction in Art. 529,) it will emerge, grazing the surface, exactly at the extreme boundary of the outermost region C B, (fig. 123,) where all sensible action ceases. Its initial velocity under these circumstances in the direction perpendicularly to the surface, is barely sufficient to carry it up to this extreme limit, where it is quite annihilated. If, then, we conceive another ray, also incident in the most intense phase of its fit of transmission, but at an angle more oblique by an infinitely small quantity, then, since its initial velocity at right angles to the surface is less, it will be destroyed before it has quite reached this limit, and the ray will therefore begin to move parallel to the surface, just *within* the last limit to the sphere of its action. Part III.

551. The outermost sphere of action necessarily attractive. Now the last action which the surface exerts, or that force which extends to the greatest distance from it, cannot be otherwise than attractive; for, first, were it repulsive, it is evident that no ray incident externally at an extreme incidence, (i. e. approaching indefinitely to 90°,) could by possibility escape reflexion; and, secondly, no ray on that supposition could emerge from within the medium, without having at its emergence an obliquity to the surface greater than some finite angle, the last action of the medium being in this case to bend it *outwards*, both which consequences are contrary to fact. Or we may consider the point thus, Since a ray incident within, at the limiting angle, emerges, if it emerge at all, parallel to the surface; and since every point in the curve described by it previous to the instant of emergence is nearer to the medium than the line of its ultimate direction, it is geometrically impossible that the curvature immediately adjacent to the point of emergence should be otherwise than *concave* towards the medium; and must, therefore, of necessity be produced by a force directed *to* it, i. e. an attractive one.

552. Hence, the luminous molecule we have been considering, will be within the attractive region at the moment when its perpendicular motion is destroyed; it will, therefore, be turned inwards, as at the dotted line fig. 123, and be reflected. *A fortiori*, therefore, will every molecule incident in a less intense phase of a fit of transmission, or in one of reflexion, as well as every one incident at a more oblique incidence, i. e. with a less initial perpendicular velocity, be reflected. Those in which the circumstances are most favourable to transmission will reach the exterior attractive region, as in fig. 123. Others in which they are less so will be reflected in some intermediate region, as in fig. 122, while those which are incident at extreme internal obliquities, or in the most intense phases of fits of reflexion, will have their courses as represented in fig. 121.

553. Fact respecting oblique reflexion from water. The conclusion at which we have arrived in the last Art. that the attractive force of a medium on the molecules of light extends to a greater distance than the repulsive, is, as we have seen, a necessary consequence of dynamical principles; and so far from being in opposition to Newton's doctrine of reflexion, as has been said, is in perfect accordance with it. Dr. Brewster has been led to the same conclusion by peculiar considerations grounded on his experiments on the law of polarization, (*Phil. Trans.*, 1815, p. 133,) and has applied it to explain a curious fact noticed by Bouguer, viz. that although water be much less reflective than glass at small incidences, yet at great ones (as $87°\frac{1}{2}$) it is much more so. Now, supposing the light to have undergone *the whole effect* of the refracting forces, in both cases before it suffers reflexion, its incidence, when it reaches the region of the repulsive forces, will have been diminished in the case of glass, to 57° 44′, but in that of water only to 61° 5′, and therefore being incident more obliquely on the water it ought to be more copiously reflected. Whatever we may think of the validity of this explanation, it is certainly ingenious, and the fact extremely remarkable, and deserving of all attention.

554. Experiment showing the phenomena of total reflexion. To see the phenomena of total reflexion to the best advantage, lay down a right-angled glass prism on a black substance close to a window, with its base horizontal, as in fig. 124, and apply the eye close to the side, looking downwards. The base will be seen divided into two portions, by a beautiful coloured arch like a rainbow concave to the eye, the portion above the arch being extremely brilliant and vivid, and giving a reflexion of all external objects no way to be distinguished from reality. On the other hand, the space within the concavity of the bow is comparatively sombre, the reflexion of the clouds, &c. on that part of the base being much less vivid. If, instead of placing it on a black body, we hold it in the hand, and place a candle *below* it, this will be visible; but (wherever placed) will always appear in some part of the base within the concavity of the bow. Fig. 124 represents the course of the rays in this experiment, E being the eye, N G, O F, P D rays incident through the farther side at various angles of obliquity on the base, and reflected to the eye at E, of which O F is incident precisely at the limiting angle. It is obvious, that all the rays towards N incident on that part of the base beyond F being too oblique for transmission will be totally reflected, while those incident between F and A, being less oblique than is required for total reflexion, will be only partially so, a portion escaping through the base in the direction D Q. Again, if we place a luminary at any point as L below the base, it is manifest that to reach the eye, a ray from it must fall between A and F, as L D, and that no ray falling on any part of the base between B and F can be refracted to E.

555. Reflected prismatic bow. The coloured arch separating the region of total from that of partial reflexion, is thus explained. For, simplicity, let us suppose the eye within the medium, (to avoid considering the reflexion at the inclined surface A C of the prism;) and, first, considering only the extreme red rays, if we drop a perpendicular from the eye on the base of the prism, and make this the axis of a cone, the side of which is inclined to the axis at the angle whose sine is $\frac{1}{\mu}$, (or the limiting angle for extreme red rays;) and if we conceive such rays to emanate in all directions *from the eye*, then all which fall *without* the circular base of this cone will be totally, but those within only partially reflected. Thus, were there no other than such red rays of this precise refrangibility, the

region of partial reflexion would be a circle whose radius = height of the eye above the base × tangent of the angle whose sine is $\frac{1}{\mu} = \frac{H}{\sqrt{\mu^2 - 1}}$. In like manner, the radius of the circular space, within which only a partial reflexion of violet rays takes place, is $\frac{H}{\sqrt{\mu'^2 - 1}}$, or $\frac{H}{\sqrt{(\mu + \delta\mu)^2 - 1}}$, being less than the value of the same radius for the red rays. Hence, in the space between the two circles, the violet rays will be totally, and the red only partially reflected; and, therefore, the whole of this space will have an excess of violet light. A similar reasoning holds good for the intermediate rays; and the shading away from the bright space without, to the comparatively dark one within, will, in consequence, be performed by the abstraction first of the red, next of the orange rays, and so on through the spectrum, leaving a residual light, which continually deviates more and more from white, and verges to blue. If now we suppose each ray to be incident in the contrary direction so as to be reflected to the eye instead of emanating from it, every thing will equally hold good, and the eye will see a bright space without; separated from an obscure space within the base of the cone, the transition from one to the other being not sudden, but marked by a blue border, the colour of which is more lively towards the interior. Now such is the fact, with one difference, however, that the coloured arch appears slightly tinged with pink on its convex side. This, as it is incompatible with theory, can be owing, it should seem, to no cause but contrast; a most powerful source of illusion in all the phenomena of colours, and of which this is, perhaps, one of the most striking and curious instances. Newton (*Optics*, part ii. exp. 16) takes no notice of this part of the phenomenon, (which was first observed and described by Sir W. Herschel,) though he gives the same explanation of the rest with that here set down. The effect of refraction at the side B A of the prism will somewhat modify the figure of the bow, giving it a tendency to a conchoidal form at great obliquities of the emergent rays.

556. Transmitted prismatic bow.

If the side B C of the prism be covered with black paper, and a bright scattered light be thrown on the base from below, (as from an emeried glass applied with its rough side close to the base,) the converse of the above described phenomena will be seen. A totally black space will be seen beyond F, and a bright one within it. The separation being marked by a bow of a vivid red colour, graduating through orange and pale yellow into white, the red being outwards. It is evident that this phenomenon is, in all its parts, complementary to that of the blue bow seen by reflexion, and therefore requires no more particular explanation. It should be noticed, however, that in this bow no appearance of blue or violet within its concavity is ever seen; so that the effect which we have above attributed to contrast in the reflected bow has nothing corresponding to it in the transmitted one.

557. Reflexion at surfaces artificially polished explained.

The intensity and regularity of reflexion at the external surface of a medium, is found to depend not merely on the nature of the medium, but very essentially on the degree of smoothness and polish of its surface. But it may reasonably be asked, how any regular reflexion can take place on a surface polished by art, when we recollect that the process of polishing is, in fact, nothing more than grinding down large asperities into smaller ones by the use of hard gritty powders, which, whatever degree of mechanical comminution we may give them, are yet vast masses, in comparison with the ultimate molecules of matter, and their action can only be considered as an irregular tearing up by the roots of every projection which may occur in the surface. So that, in fact, a surface artificially polished must bear somewhat of the same kind of relation to the surface of a liquid, or a crystal, that a ploughed field does to the most delicately polished mirror, the work of human hands. Now to this question the Newtonian doctrine furnishes an answer quite satisfactory. Were the reflexion of light performed by actual impact of its molecules upon those of the reflecting medium, no regular ordinary reflexion could ever take place at all, as it would depend entirely on the shape of the molecules, or asperities of the surface, and the inclinations of their surfaces to the general surface of the medium at the point of incidence, what should be the direction ultimately taken by each particular ray. Now these must vary in every possible manner in uncrystallized bodies, so that in reflexion from the surfaces of *these* the light would be uniformly scattered in every direction. On the other hand, in crystallized media, each molecule presenting only a limited number of strictly plane surfaces, and the corresponding faces of all being mathematically parallel, reflexion would indeed be regular; but its direction would be regulated only by that of the incident ray and the position of certain fixed lines within the crystal; and would be quite independent of either the smoothness or the inclination of the polished surfaces of it, whether natural or artificial; add to which, that instead of the reflected pencil of rays being *single*, it would in most cases be multiple. All these consequences are so contrary to fact, that it is evident we must suppose the distance to which the forces producing reflexion extend much greater not only than the size of, or interval between individual molecules, but even greater than the minute inequalities or furrows in the artificially polished surfaces of media. Granting this, the difficulty vanishes; for the average action of many molecules, or many corrugations, will present an uniformity, while individually they may offer the greatest diversity. To illustrate this, we need only cast our eyes on fig. 125, where A B represents the rough surface of a medium, and A C the radius of one of the spheres of attraction, or repulsive activity of a single molecule A. Conceiving now the summits of all the elevations *a*, *b*, *c*, *d* to lie in a plane, let spheres be described with their centres equal to A C. Then their intersections will generate a kind of mamillated surface $\alpha\ \beta\ \gamma\ \delta$, which, however, if the radii of the spheres be at all considerable with respect to the distances of their centres, will approach exceedingly near to a mathematical plane, infinitely more so than the surface A B need be supposed. Hence, a ray of light impinging on the medium will come within the sphere of its action not at an irregular surface, but nearly at a plane one; and the resultant action of all the molecules in action, supposing them distributed with uniformity over A B, will be perpendicular to this surface. The same will hold good of the layer of molecules (however interrupted) immediately under the summits *b*, *c*, *d*, &c., and of all the other

Light not reflected by direct impact on bodies.

But by forces at a distance.

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Oblique regular reflexion from rough surfaces.

558. It is evident that the inequalities in the mamillary surfaces above described will become more considerable as their radii are diminished, or as the interval of their centres is greater, and in proportion will the regularity of reflexion and refraction be interrupted. Hence too it follows, that the more oblique the incidence of the ray, the greater may be the roughness of the surface which will give a regular reflexion; and this is perfectly consonant to fact, as may be easily tried with a piece of emeried glass, which, although so rough as to give no regular image at a perpendicular incidence, will yet give a pretty distinct one at great obliquities. The reasons are, first, that a very oblique ray will not require to penetrate so far within the sphere of repulsion, to have its motion perpendicular to the surface destroyed; and, secondly, that it cannot pass between two contiguous elevations or depressions of the imaginary surface $\alpha\,\beta\,\gamma\,\delta$, but by reason of its obliquity must traverse several of them, and thus undergo a more regular *average* exertion of the forces of the medium.

Regular refraction at surfaces artificially polished.

559. Thus the reflexion of light is explained on the Newtonian doctrine. But it may still be asked, how refraction at a surface artificially polished can ever be regular. In reflexion, the ray never reaches the asperities of the surface; it undergoes their average action, equalized by distance, and mutually compensated. In refraction, it is otherwise. Here the rays must actually traverse the surface, and must therefore actually pass through all its inequalities at every possible angle of obliquity. The answer to this is equally plain. Neither refraction nor reflexion are performed *close* to the surface, either wholly, or in great part. The greater part by far of the flexure of the ray is performed (either internally or externally) at a distance, out of the reach of these irregularities, and by the action of a much more considerable thickness of the medium than is occupied by them. Their action must be compared to the effect of mountains on the earth's surface in disturbing the general force of gravity. A stone let fall close to one of them, from a moderate height, follows not the true vertical but the direction of the deviated plumbline, which is sensibly different. Whereas, if let fall from the moon to the earth's centre, it would pass among them, were they greater a thousand fold than they are, without experiencing any sensible perturbation or change of direction in their neighbourhood.

560. In fact, however, no regular refraction can be obtained from surfaces sensibly rough, at all comparable to the regularity of their reflexion. This may arise from the impossibility of a refracted ray penetrating the surface at a sufficient degree of obliquity. It is, however, a remarkable fact, that the *regular* internal reflexion from a roughened surface, even at extreme obliquities, is scarcely sensible, even in cases where the external reflexion at the same obliquities is perfectly regular and copious. This would seem to indicate, that the forces which operate the external reflexion of a ray exert their energy wholly without the medium.

Intensity of the forces producing refraction.

561. Whatever be the forces by which bodies reflect and refract light, one thing is certain, that they must be incomparably more energetic than the force of gravity. The attraction of the earth on a particle near its surface produces a deflexion of only about 16 feet in a second; and, therefore, in a molecule moving with the velocity of light, would cause a curvature, or change of direction, absolutely insensible in that time. In fact, we must consider, first, that the time during which the whole action of the medium takes place, is only that within which light traverses the diameter of the sphere of sensible action of its molecules at the surface. To allow so much as a thousandth of an inch for this space is beyond all probability, and this interval is traversed by light in the $\frac{1}{12{,}672{,}000{,}000{,}000}$ part of a second. Now, if we suppose the deviation produced by refraction to be 30°, (a case which frequently happens,) and to be produced by a uniform force acting during a whole second; since this is equivalent to a linear deflexion of 200,000 miles × sin 30°, or of 100,000 miles = 33,000,000 × 16 feet, such a force must exceed gravity on the earth's surface 33,000,000 times. But, in fact, the whole effect being produced not in one second, but in the small fraction of it above mentioned, the intensity of the force operating it (see MECHANICS) must be greater in the ratio of the square of one second to the square of that fraction; so that the least improbable supposition we can make gives a mean force equal to $4{,}969{,}126{,}272 \times 10^{24}$ times that of terrestrial gravity. But in addition to this estimate already so enormous, we have to consider that gravity on the earth's surface is the resultant attraction of its whole mass, whereas the force deflecting light is that of only those molecules immediately adjoining to it, and within the sphere of the deflecting forces. Now a sphere of $\frac{1}{1000}$ of an inch diameter, and of the mean density of the earth, would exert at its surface a *gravitating* force only $\frac{1}{1000} \times \frac{1 \text{ inch}}{\text{diameter of the earth}}$ of ordinary gravity, so that the actual *intensity* of the force exerted by the molecules concerned cannot be less than $\frac{1000 \times \text{earth's diameter}}{1 \text{ inch}}$ (= 46,352,000,000) times the above enormous number, or upwards of 2×10^{44} when compared with the ordinary intensity of the gravitating power of matter. Such are the energies concerned in the phenomena of light on the Newtonian doctrine. In the undulatory hypothesis, numbers not less immense will occur; nor is there any mode of conceiving the subject which does not call upon us to admit the exertion of mechanical forces which may well be termed infinite.

562. Dr. Wollaston has proposed the observation of the angle at which total reflexion first takes place at the common surface of two media, the index of refraction of one of which is known, as a means of determining that of the other; and, in the *Philosophical Transactions for* 1802, has described an ingenious apparatus which gives a measure of the index required almost by inspection. If we lay any object under the base of a prism

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of flint glass with air alone interposed, the internal angle of incidence at which the visual ray begins to be totally reflected, and at which of course the object ceases to be seen by refraction is about 39° 10′; but when the object has been dipped in water, and brought into contact with the glass, it continues visible (while the eye is depressed) by means of the greater refractive power of the water, as far as $57\frac{1}{2}°$ of incidence. When any kind of oil, or any resinous cement, is interposed, this angle is still greater, according to the refractive power of the medium employed; and by cements that refract more strongly than the glass, the object may be seen through the prism at whatever angle of incidence it is viewed. All that is requisite, then, to determine the refractive index of any body less refractive than glass, is to bring the substance to be examined in optical contact with the base of a prism, and to depress the eye (or increase the angle of incidence) till it ceases to be seen as a dark spot on the silvery reflexion of the sky on the rest of the base. With fluids and soft solids, or fusible ones, the requisite contact is easily obtained; but with solids, they must be brought to smooth surfaces, and applied to the base by the intervention of some fluid or cement of higher refractive power than the glass, which (since the surfaces of the interposed stratum are parallel) will produce no change in the total deviation of a ray passing through it, and therefore no error in the result. By this method, opaque as well as transparent substances may be examined, or bodies of unhomogeneous density, as the crystalline lens of the eye. It may seem paradoxical to speak of the refractive power of an opaque body; but it will be remembered, that opacity is merely a consequence of intense absorbent power, and that before a ray can be absorbed, it must enter the medium, and of course obey the laws of refraction at its surface. By this method, Dr. Wollaston has determined the refractions of a great variety of bodies; but Dr. Brewster remarks, that the method must be liable to some source of inaccuracy, which renders it unsafe to trust entirely to it in practice. Dr. Young has remarked, that the index of refraction given by it, belongs in strictness to the extreme red rays.

Dr. Wollaston's method of determining refractive powers.

§ II. *General Statement of the Undulatory Theory of Light.*

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The undulatory theory, among whose chief supporters we have to number Huygens, Descartes, Hooke, and Euler, and, in later times, the illustrious names of Young and Fresnel, who have applied it with singular success and ingenuity to the explanation of those classes of phenomena which present the greatest difficulties to the Corpuscular doctrine, requires the admission of the following hypotheses or postulata:

Postulata in the system of undulations

1. That an excessively rare, subtle, and elastic medium, or *ether*, as it is called, fills all space, and pervades all material bodies, occupying the intervals between their molecules; and, either by passing freely among them, or, by its extreme rarity, offering no resistance to the motions of the earth, the planets, or comets in their orbits, appreciable by the most delicate astronomical observations; and having inertia, but not gravity.

2. That the molecules of the ether are susceptible of being set in motion by the agitation of the particles of ponderable matter, and that when any one is thus set in motion it communicates a similar motion to those adjacent to it; and thus the motion is propagated further and further in all directions, according to the same mechanical laws which regulate the propagation of undulations in other elastic media, as air, water, or solids, according to their respective constitutions.

3. That in the interior of refracting media the ether exists in a state of less elasticity, compared with its density, than in vacuo, (i. e. in space empty of all other matter;) and that the more refractive the medium, the less, relatively speaking, is the elasticity of the ether in its interior.

4. That vibrations communicated to the ether in free space are propagated through refractive media by means of the ether in their interior, but with a velocity corresponding to its inferior degree of elasticity.

5. That when regular vibratory motions of a proper kind are propagated through the ether, and, passing through our eyes, reach and agitate the nerves of our retina, they produce in us the sensation of light, in a manner bearing a more or less close analogy to that in which the vibrations of the air affect our auditory nerves with that of sound.

6. That as, in the doctrine of sound, the frequency of the aerial pulses, or the number of excursions to and fro from its point of rest made by each molecule of the air, determines the pitch, or note, so, in the theory of light, the frequency of the pulses, or number of impulses made on our nerves in a given time by the ethereal molecules next in contact with them, determines the *colour* of the light; and that as the absolute extent of the motion to and fro of the particles of air determine the *loudness* of the sound, so the *amplitude*, or extent of the excursions of the ethereal molecules from their points of rest, determine the brightness or intensity of the light.

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The velocity of all undulations equal.

The application of these postulates to the explanation of the phenomena of light, presumes an acquaintance with the theory of the propagation of motion through elastic media. This we shall assume, referring to our article on sound for the demonstration of all the properties and laws of motions so propagated, as we shall have occasion to employ. One of the principal of these is, that supposing the elastic medium uniform and homogeneous, all motions of whatever kind are propagated through it in all directions with one and the same uniform velocity, a velocity depending solely on the elasticity of the medium as compared with its inertia, and bearing no relation to the greatness or smallness, regularity or irregularity of the original disturbance. Thus, while the intensity of light, like that of sound, diminishes as the distance from its origin increases, its velocity remains invariable, and thus, too, as sounds of every pitch, so light of every colour, travels with one and the same velocity, either in vacuo, or in a homogeneous medium.

Now here arises, *in limine*, a great difficulty; and it must not be dissembled, that it is impossible to look on

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Objection from the phenomena of dispersion.

it in any other light than as a most formidable objection to the undulatory doctrine. It will be shown presently that the deviation of light by refraction is a consequence of the difference of its velocities within and without the refracting medium, and that when these velocities are given the amount of deviation is also given. Hence it would appear to follow unavoidably, that rays of all colours must be in all cases equally refracted; and that, therefore, there could exist no such phenomenon as dispersion. Dr. Young has attempted to gloss over this difficulty, by calling in to his assistance the vibrations of the ponderable matter of the refracting medium itself, as modifying the velocity of the ethereal undulations within it, and that differently according to their frequency, and thus producing a difference in the velocity of propagation of the different colours; but to us it appears with more ingenuity than success. We hold it better to state it at once in its broadest terms, and call on the reader to suspend his condemnation of the doctrine for what it *apparently* will not explain, till he has become acquainted with the immense variety and complication of the phenomena which it will. The fact is, that neither the corpuscular nor the undulatory, nor any other system which has yet been devised, will furnish that complete and satisfactory explanation of *all* the phenomena of light which is desirable. Certain admissions must be made at every step, as to modes of mechanical action, where we are in total ignorance of the acting forces; and we are called on, where reasoning fails us, occasionally for an exercise of faith. Still, if we regard hypotheses and theories as no other way valuable than as means of classifying and grouping together phenomena, and of referring facts to laws which, though possibly empirical, are yet, so far as they are so, correct representations of nature, and as such must be deducible from real primary laws, whenever they shall be discovered, we cannot but admit their importance. The undulatory system especially is necessarily liable to considerable obscurities; as the doctrine of the propagation of motion through elastic media is one of the most abstruse and difficult branches of mathematical inquiry, and we are therefore perpetually driven to indirect and analogical reasoning, from the utter hopelessness of overcoming the mere mathematical difficulties inherent in the subject when attacked directly.

566. Objection from the rectilinear propagation of light answered.

It is thus that we are encountered at the very outset of its application with another objection, which, in the eyes of Newton, appeared decisive against its admission, but which has since been, in a considerable degree, overcome. How is it that *shadows* exist. Sounds make their way freely round a corner,—why does not light do so? A vibration propagated from a centre in an elastic medium, and intercepted by an immovable obstacle having a small orifice, ought to spread itself, it is said, from this orifice beyond the screen as from a new centre, and fill the space beyond with undulations propagated from it in every direction. Thus, as in Acoustics, the orifice is heard as a new source of sound; so, in Optics, it ought to be seen in all directions as a new luminary. To this the answer is, first, that it is not demonstrable that a vibratory motion communicated to one particle of an elastic medium is propagated with equal *intensity* to every surrounding molecule in whatever direction situated with respect to the line of its motion, though it is with equal *rapidity*; and therefore that we have no reason to presume, *à priori*, but rather the contrary, that the motions of the vibrating particles at the orifice should be propagated laterally with *equal* intensity in all directions; secondly, that it is not true, in fact, that sounds are propagated round the corner of an obstacle *with the same intensity* as in their original direction, as any one may convince himself by the following simple experiment. Take a common tuning fork, and, holding it (when set in vibration) about three or four inches from the ear, with its flat side towards it, when its sound is distinctly heard, let a strip of card, somewhat longer than the flat of the tuning fork, be interposed, at about half an inch from the fork. The sound will be almost entirely intercepted by it; and if the card be alternately removed and replaced in pretty quick succession, alternations of sound and silence will be perceived; proving that the undulations of the air are by no means propagated with equal intensity by the circuitous route round the edge of the card, as by the direct one. Indeed any one has only, to be convinced of the fact, to attend to the sound of a carriage in the act of turning a corner from the street in which he happens to be to an adjoining one; to which we may add, that, even when there is no obstacle in the way, sounds are by no means equally audible in all directions from the sounding body, as any one may convince himself by holding a vibrating tuning fork, or pitchpipe, near his ear, and turning it quickly on its axis. This last phenomenon was first noticed, we believe, by Dr. Young, (*Phil. Trans.*, 1802, p. 25,) and since more fully described (in *Schweiggers Jahrbuch*, 1826) by M. Weber. Now if there be any inequality *at all* in the intensity of the direct and lateral propagation of undulations in a medium, it must arise from the constitution of the medium, and the proportion of the amplitude of the excursions of the vibrating particles to their distance from each other; and may therefore easily be conceived to differ in any imaginable degree in different media, and there is, at least, no absurdity in supposing the ether so constituted as to admit of comparatively very feeble lateral propagation. Now, thirdly, in point of fact, light does spread itself in a certain small degree into the shadows of bodies, out of its strict rectilinear course, giving rise to the phenomena of *inflexion* or *diffraction*, of which more presently, and which are completely accountable for on the undulatory doctrine, and form, in fact, its strongest points. For further information on this confessedly abstruse subject, the reader must consult our article on Sound, and the works cited at the end of this Essay. It is enough here to show, that the objection which has been urged by Newton and his followers with such force against the doctrine of undulations, is really not conclusive against it, but founded rather on inadequate conceptions of the nature of elastic fluids, and the laws of their undulations.

567. Mode in which the retina is excited by vibrations of ether.

Although any kind of impulse, or motions regulated by any law, may be transferred from molecule to molecule in an elastic medium, yet in the theory of light it is supposed that only such primary impulses as recur according to regular periodical laws, at equal intervals of time, and repeated many times in succession, can affect our organs with the sensation of light. To put in motion the molecules of the nerves of our retina with sufficient efficacy, it is necessary that the almost infinitely minute impulse of the adjacent ethereal molecules should be often and regularly repeated, so as to multiply, and, as it were, concentrate their effect. Thus, as a great pendulum may be set in swing by a very minute force often applied at intervals exactly equal to its time

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of oscillation, or as one elastic solid body can be set in vibration by the vibration of another at a distance, propagated through the air, if in exact unison, even so may we conceive the gross fibres of the nerves of the retina to be thrown into motion by the continual repetition of the ethereal pulses; and such only will be thus agitated, as from their size, shape, or elasticity are susceptible of vibrating in times exactly equal to those at which the impulses are repeated. Thus it is easy to conceive how the limits of visible colour may be established; for if there be no nervous fibres in unison with vibrations more or less frequent than certain limits, such vibrations, though they reach the retina, will produce no sensation. Thus, too, a single impulse, or an irregularly repeated one, produces no light; and thus also may the vibrations excited in the retina continue a sensible time after the exciting cause has ceased, prolonging the sensation of light (especially of a vivid one) for an instant in the eye in the manner described, (Art. 543.) We may thus conceive the possibility of other animals, such as insects, incapable of being affected with any of our colours, and receiving their whole stock of luminous impressions from a class of vibrations altogether beyond our limits, as Dr. Wollaston has ingeniously imagined (we may almost say proved) to be the case with their perceptions of sound.

568. Motion of a vibrating luminous molecule.

The law of motion of every particle of the ether is regulated by that of the molecule of the luminary from which it takes its origin; and will be regular or irregular, periodical or not, according as that of the original molecule is so or otherwise. But it is only with motions which may be regarded as infinitely small that we are concerned in this theory. The displacement of each particle, either of the ether or of the luminary, is supposed to be so minute as not to detach it from, or change its *order* of situation among the neighbouring ones. Now when we consider only such infinitesimal displacements from the position of equilibrium, it is evident, that the tension arising from them, or the force by which the displaced molecule is urged, must be proportional in quantity to its distance from its point of rest, and must tend directly to that point, provided we suppose the medium equally elastic in all directions. Hence, by the laws of Dynamics, its trajectory must be an ellipse described in one plane about the point of equilibrium as its centre; or, if one of the axes of the ellipse vanish, a straight line having that point in its middle, in which it oscillates to and fro, performing all its excursions in the latter case, or its revolutions in the former, whether great or small, in equal times, and following the law of a vibrating pendulum. We will, for the present, consider the case of rectilinear vibrations as the most simple, and show hereafter how the more general one may be reduced to it.

569. Laws of rectilinear vibrations.

Proposition. To define the motion of a vibrating molecule of a luminary, supposing its excursions to and fro to be performed in straight lines.

Putting x for its distance from its point of rest, t for the time elapsed since a given epoch, and v for its velocity, and E for the absolute elastic force, the force urging the molecule to its point of equilibrium will be $\mathrm{E} \cdot x$, and will tend to diminish x; hence (supposing gravity to be represented by $32\frac{1}{6}$ feet) we must have

$$\frac{dv}{dt} = -\frac{d^2x}{dt^2} = \mathrm{E}x, \text{ and therefore } \frac{2\,d^2x \cdot dx}{dt^2} = -2\,\mathrm{E}x\,dx, \text{ or, integrating, } \frac{dx^2}{dt^2} \text{ or } v^2 = \mathrm{E}(a^2 - x^2)$$

where a is the greatest distance of excursion, or the *semiamplitude* of the vibration. Hence,

$$v = \sqrt{\mathrm{E}} \cdot \sqrt{a^2 - x^2} = \frac{-dx}{dt}, \text{ and therefore } dt = -\frac{dx}{\sqrt{\mathrm{E}} \cdot \sqrt{a^2 - x^2}}; \text{ or, integrating, } t + \mathrm{C} = \frac{1}{\sqrt{\mathrm{E}}} \text{ arc . cos } \frac{x}{a}, \text{ that is}$$

$$x = a \cdot \cos\{\sqrt{\mathrm{E}} \cdot (t + \mathrm{C})\}; \qquad v = a \cdot \sqrt{\mathrm{E}} \cdot \sin\{\sqrt{\mathrm{E}}\,(t + \mathrm{C})\}$$

Such are the velocity and distance from the middle point of its vibration of the molecule at any instant. If we call T the whole period in which the molecule has performed one complete evolution, consisting of a complete excursion to and fro on both sides of its point of equilibrium, we shall have at the commencement of the motion when $v = 0$, or $x = a$, $a \cdot \cos\{\sqrt{\mathrm{E}} \cdot (t + \mathrm{C})\} = a$, or $(t + \mathrm{C})\sqrt{\mathrm{E}} = 0$; and when one quarter of a period has been performed, or the molecule has arrived at its greatest distance $-a$ on the opposite side of the centre $-a = a \cdot \cos\{\sqrt{\mathrm{E}}\,(t + \frac{1}{2}\mathrm{T} + \mathrm{C})\}$, or $\sqrt{\mathrm{E}} \cdot (t + \mathrm{C} + \frac{1}{2}\mathrm{T}) = \pi$, putting π for the semicircumference of a circle whose diameter is 1. Hence we get by subtraction

$$\tfrac{1}{2}\mathrm{T} \cdot \sqrt{\mathrm{E}} = \pi; \qquad \mathrm{T} = \frac{2\pi}{\sqrt{\mathrm{E}}}.$$

Hence we may eliminate E, and introduce T instead of it, which will give the equations $\sqrt{\mathrm{E}} = \frac{2\pi}{\mathrm{T}}$,

$$x = a \cdot \cos 2\pi \cdot \frac{t + \mathrm{C}}{\mathrm{T}}; \qquad v = a\sqrt{\mathrm{E}} \sin 2\pi \cdot \frac{t + \mathrm{C}}{\mathrm{T}};$$

which equations express the laws required, and which if the time t be supposed to commence at the moment when $v = 0$, or when the molecule is at the extremity of one of its excursions, become simply

$$x = a \cdot \cos 2\pi \cdot \frac{t}{\mathrm{T}}; \qquad v = a\sqrt{\mathrm{E}} \sin 2\pi \cdot \frac{t}{\mathrm{T}}.$$

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570. *Corol.* Hence the excursions of the molecule to and fro will consist of four principal phases, in each of which its motion is similar, but in contrary directions, or on contrary sides of the centre. In the first phase the molecule is to the *right* of the centre of motion, and is approaching the centre, or moving from *right* to *left*. In the second, it is to the left of the middle point, and moving from it, or still *from right* to *left*. These two phases we shall term the *positive phases*. In the third phase the molecule lies on the left side, and its motion is towards the centre, and from left to right. In the fourth, it is to the right again, receding from the centre, and moving still from left to right. These we shall term the *negative phases* of its vibration.

571. Laws of rectilinear vibrations of an ethereal molecule. *Proposition.* To define the rectilinear vibrations of any molecule of the ether, propagated from a luminous particle vibrating as in the last proposition.

In the propagation of motions through elastic, uniform media, the same or a similar motion to that of any one molecule is communicated to every other in succession; but this communication occupies time, and the motion of a molecule at a distance from the origin of the vibrations does not commence till after the lapse of an interval of time proportional to that distance, being the time in which the propagated impulse, whether of sound or light, &c. runs over that distance with a certain uniform velocity due to the intrinsic elasticity of the medium, and which in the case of light is about 200,000 miles per second; in that of sound about ~~1200~~ feet. And when the vibration of the original source of motion has ceased, that of the ethereal molecule does not cease on the instant, but continues for a time equal to that which elapsed before its commencement. Hence, if we call V the velocity of light, and D the distance of the molecule from the luminous point, $\frac{D}{V}$ will be the interval between the commencement of the motion of the latter and of the former; hence $-t$ being the time elapsed at any instant since the commencement of the first positive phase of the vibration of the luminous point, $t - \frac{D}{V}$ will be the corresponding time in the case of the ethereal molecule. Thus we have, for the equations of the motions of the former,

$$x = a \,.\, \cos 2\pi \,.\, \frac{t}{T}; \qquad v = b \,.\, \sin 2\pi \,\frac{t}{T}; \text{ where } b = a\sqrt{E}$$

and in that of the latter

$$x = a \,.\, \cos 2\pi \,.\, \left(\frac{t - \frac{D}{V}}{T}\right); \qquad v = \beta \,.\, \sin 2\pi \left(\frac{t - \frac{D}{V}}{T}\right); \text{ where } \beta = a\sqrt{E}$$

a being the semiamplitude of the vibration, or the extent of the excursion of the ethereal molecule from its point of rest.

572. *Corol.* Hence it is evident that the actual velocity of the molecules of ether may be less in any proportion than that of light; for the maximum value of v depends for its numerical magnitude solely on a, or on the amplitude of excursion, and on E, and not at all on V the velocity of propagation of the wave.

573. Waves of light defined. *Corol.* 2. If we suppose the luminous molecule to have made, from the commencement of its motion, any number of vibrations and parts of a vibration in the time t; then if we consider an ethereal molecule at a distance V . t from it in any direction, (i. e. situated in a spherical surface whose radius is V . t,) this molecule will be just beginning to be put in motion. If we suppose another spherical surface concentric with the former, but having its radius less than the former by V . T, which in future we shall call λ, every particle situated in this surface will have just completed one vibration, and be commencing its second, and so on. The interval between these surfaces will comprehend, arranged in spherical, concentric shells, molecules in every phase of their vibrations,—those in each shell being in the same phase. This assemblage of molecules is termed *a wave*, and as the impulse continues to be propagated forwards it is evident that the wave will continue to increase in radius, and will comprehend in succession all the molecules of the medium to infinity.

574. Undulations or pulses. *Definition.* The interval between the internal and external surface of a luminous wave is called an *undulation*, or a pulse, and its length is evidently $= V \,.\, T = \lambda$, or the space run over by light in the time T of one complete period, or vibration of the luminous molecule. It is therefore proportional to that time.

575. Different colours have different lengths of their undulations. Hence the lengths of the undulations of differently coloured rays differ *inter se*. For, by Postulate 6, the number of vibrations made in any given time by the ethereal particles determines the colour. Now the more numerous the vibrations are, *dato tempore*, the shorter their duration; hence T, which represents this duration, is *less*; and therefore λ, or the length of the undulation less for the violet than for the red rays. From experiments to be presently described, it has been found, that the lengths of the undulations in air, or the values of λ for the different rays, as also the number of times they are repeated in one second, are as in the following table:

Colours.	Length of an undulation in parts of an inch in air λ =.	Number of such undulations in an inch or $\frac{1}{\lambda}$	Number of undulations per second.
Extreme	0·0000266	37640	458,000000,000000
Red	0·0000256	39180	477,000000,000000
Intermediate	0·0000246	40720	495,000000,000000
Orange.....	0·0000240	41610	506,000000,000000
Intermediate	0·0000235	42510	517,000000,000000
Yellow	0·0000227	44000	535,000000,000000
Intermediate	0·0000219	45600	555,000000,000000
Green	0·0000211	47460	577,000000,000000
Intermediate	0·0000203	49320	600,000000,000000
Blue.....	0·0000196	51110	622,000000,000000
Intermediate	0·0000189	52910	644,000000,000000
Indigo	0·0000185	54070	658,000000,000000
Intermediate	0·0000181	55240	672,000000,000000
Violet	0·0000174	57490	699,000000,000000
Extreme	0·0000167	59750	727,000000,000000
			Taking the velocity of light at 192000 miles per second.

576. From this table we see, that the sensibility of the eye is confined within much narrower limits than that of the ear, the ratio of the extreme vibrations being nearly 1·58 : 1, and therefore less than an octave, and about equal to a minor sixth. That man should be able to measure, with certainty, such minute portions of space and time, is not a little wonderful; for it may be observed, whatever theory of light we adopt, these periods and these spaces have a *real existence*, being, in fact, deduced by Newton from direct measurements, and involving nothing hypothetical but the names here given them.

577. Direction of a ray. The direction of a ray in the undulatory system is a line perpendicular to the surface of the wave at any point. When, therefore, the vibration is propagated through an uniform ether, the wave being bounded by spherical surfaces, the direction of the ray is constant, and from the centre. Thus in this system a ray of light moves in a right line in an uniform medium.

578. Law of intensity of light. The intensity of a ray is, of course, in some certain determinate ratio of the impulse made on the retina *dato tempore* by the ethereal molecules, and therefore in some certain ratio of their amplitudes of excursion, or their absolute velocities. The principle of the conservation of living forces requires that the amplitude of excursion of a molecule, situated at any distance from the vibrating centre, should be as the distance inversely, (see Acoustics.) If then we suppose the sensation created in the retina to be as the simple *vis inertiæ* of the molecules producing it, light ought to decrease inversely as the distance; if as the *vis viva*, (which is as the square of the velocity,) inversely as the square of the distance. As we know nothing of the mode in which the immediate *sensation* of light or sound is produced in the *sensorium*, we have no reason to prefer one of these ratios to the other *a priori*. But when we consider, that in the division of a beam of light by partial reflexion, or by double refraction, or otherwise, there is neither gain nor loss of light, (supposing the perfect transparency and polish of the medium which operates the division) so that the sum of the intensities remains constant, however the absolute velocities of the vibrating molecules may change, either in quantity, or (as in the case of reflexion, where they must be conceived to rebound from each other, mediately or immediately) in sign, the agreement of this law in all cases with that of the conservation of the *vis viva*, and its opposition in the other mentioned case to that of the uniform motion of the centre of gravity, (which would make not the *sum*, but the *difference* of the intensities constant, were the simple ratio of their velocities assumed for their measure,) (see Dynamics,) leaves us no choice in preferring the square of the absolute velocity, or of the amplitude of excursion of a vibrating molecule, for the measure of the intensity of the ray it propagates; and thus the observed law of the diminution of light is reconciled to the undulatory doctrine.

579. Form of the wave. When the medium through which the vibrations are transmitted is not uniformly elastic, the waves will make unequal progress in different directions, according to the law of elasticity. In this case the figure of the wave will not be spherical. If we suppose the elasticity to vary by insensible gradations, as when light passes through the atmosphere, whose refracting power is variable, the figure of the wave will be flattened towards that part where the elasticity is less. Thus, in fig. 126, if A B be the earth's surface, C D, E F, G H, &c. the atmospheric strata, and S a luminous point, the waves will be less curved as they approach the perpendicular S B; and the line S, 1, 2, 3, 4, 5, &c. drawn so as to intersect them all at right angles, will be a curve convex downwards, so that a ray will appear to be continually bent downwards towards the earth, as we see really happens. Let us now proceed to consider the explanation of the phenomena of reflexion and refraction on the undulatory system.

580. Perpendicular reflexion. The perpendicular reflexion of light may be conceived, by the analogy of an elastic ball in motion impinging directly on another at rest, and in this way it has been illustrated by Dr. Young. If the balls be equal, the whole motion of the impinging ball will be transferred to the other, no reflexion taking place; and thus the impulse may be propagated undiminished along a line of balls as far as we please. So it is with light moving in a uniform medium, or passing from one medium to another of equal elasticity. But if a less ball impinge on

Light. a greater at rest, it will be reflected, and with a momentum which is greater in proportion to the difference in size of the balls. Part III.

581. Principles. But to render an account of oblique reflexion and refraction, and the other phenomena we shall have to speak of, it will be necessary to lay down the following principles, which are either self-evident or follow immediately from the elementary principles of dynamics.

582. Superposition of small motions. 1. When any number of very minute impulses is communicated at once to the particles of any medium, or of any mechanical system under the influence of any forces, the motion of each particle at any instant will be the sum of all the motions which it would have at that instant, had each of the impulses been communicated to the system alone, (the word sum being understood in its algebraical sense.)

583. Principle of secondary waves. 2. Every vibrating molecule in an elastic medium, whether vibrating by an original impulse, or in consequence of an impulse propagated to it from others, may be regarded as a centre of vibration from which a system of secondary waves emanates in all directions, according to the laws of the propagation of waves in the medium.

Proposition. In the reflexion of light on the undulatory doctrine, the angle of incidence is equal to that of reflexion.

584. Law of reflexion at a plane. Let A B be a plane surface separating the two media, and S the luminous point propagating a series of spherical waves, of which let A a be one. So soon as this reaches the surface at A, a partial reflexion will take place; and regarding the point A as a new centre of vibration, spherical waves will begin to be propagated from it as a centre, one of which proceeds forwards *into* the reflecting medium, with a velocity greater or less than that of the incident wave, as the case may be; the other backwards into the medium of incidence, with a velocity equal to that of the incident wave. It is only with the latter we are at present concerned. Conceive now the wave A a to move forward into the position B b; then in the time that it has run over the space P B, the wave propagated from A will have run back over a distance A d = P B, and the hemisphere whose radius is A d will represent this wave. Between A and B take any point X, and describe the hemispheric surface X c. Then regarding X as a centre of vibration, its vibrations will not commence till the wave has reached it. It will, therefore, begin to vibrate later than A, by the whole time the wave A a takes to run over P Q; but when once set in vibration, it propagates backwards a spherical wave with the same velocity, so that when the original wave has advanced into the situation B b, the wave from X will have expanded into a hemisphere, whose radius X c is equal to P B − P Q, or A B. Now this being true of every point X, if we conceive a surface touching all these hemispheres in d, c, B, this surface will mark the points at which the reflected impulse has just arrived, and which just begins to move when the original wave has reached B, and will, therefore, be the surface of the reflected wave. Conceive now the spherical surface b B prolonged below the plane A B, as represented by the dotted line D C B, and the same of the spheres about A and X. Then the spherical surfaces D C B and C c being both perpendicular to S X C, must touch each other in C, hence the surface touching all the hemispheres about A, X, &c. as centres, *below* A B is a segment of a sphere having S for a centre, and therefore the surface B c d or the reflected wave is a segment of a sphere having its centre at s as much below the line A B as S is above it.

Now to an eye placed at X, the luminous point S will appear in the direction S X perpendicular to the incident wave, and the eye placed in c will perceive the reflected image of S at s in the direction c s, perpendicular to the reflected wave; but c s passes through X, because the spheres c C and B b touch at c. Therefore the *ray* by which s is seen at c passes through X. But the surfaces B D, B d being similar and equal, the angle B X c = B X C = A X S, that is, the angle of incidence is equal to that of reflexion. Q. E. D.

585. Reflexion at curved surfaces. Fig. 128. *Cor.* If the reflecting surface be not a plane, the reflected wave will not be spherical; its form is, however, easily determined as follows: Suppose the direct wave to have assumed the position B b. Take any point X in the reflecting surface, and describe the sphere X Q, and with the centre X and radius = B Q, describe another sphere. Do this for every point in the surface A B, and the surface which is a common tangent (as B c d) to all these spheres, is the surface of the reflected wave, because it marks the farthest limit to which the reflected impulse has reached in all directions at the instant when the direct impulse has reached B. Now take Y infinitely near to X, and, making the same construction at Y, let c, e be the points in the reflected wave to which X c and Y e are respectively perpendicular. Draw X r perpendicular to Y e, and X q to S Y q, then, since Y e = S B − S Y, and X c = S B − S X, we have Y e − X c, or Y r = S X − S Y = Y q, and X Y being common to the right angled triangles X Y r, X Y q, the angle r Y X must be equal to X Y q or to S Y A, so that the same law of reflexion holds good in curve as in plane surfaces.

586. Law of refraction. Fig. 129. *Proposition.* To demonstrate the law of refraction in the undulatory system.

Let S, fig. 129, be a luminous point, and let any wave propagated from it reach in succession the points Y, X, B of any curve surface Y X B of a refracting medium, whereof X and Y are supposed infinitely near each other. As the wave strikes Y, X, B, each of these points will become centres of undulation, which will be propagated in the refracting medium with a velocity different from that of light in the medium of incidence, by reason of their different elasticities, (Postulate 3.) Let V : v :: velocity in the first medium to that in the second, (a constant ratio by hypothesis,) and, describing the sphere B Q R, take X c = $\frac{v}{V}$. Q X and Y c = $\frac{v}{V}$. Y R, then will X c and Y e represent the spaces run over by the refracted secondary waves propagated from X and Y respectively, when the direct wave has reached B. Hence, if about X and Y as centres, and with these radii we describe spheres, and suppose e, c to be points in the curve surface which is a tangent to all such spheres, it is clear that X c and Y e will be perpendicular to this surface, that is, to the surface of the refracted primary wave; hence, X c and Y e will be the directions of the refracted rays at X and Y. Draw X q, X r perpendicular

Light. respectively to Y R and Y e, then will $Yq = SX - SY$ and $Yr = Ye - Xc = \frac{v}{V}.YR - \frac{v}{V}.XQ = \frac{v}{V}$ Part III.

$(YR - XQ) = \frac{v}{V}\{(SR - SY) - (SQ - SX)\} = \frac{v}{V}.(SX - SY) = \frac{v}{V}.Yq$. Hence we have Yq : $Yr :: V : v$. But since S X, S Y are direct rays, and X c, Y e the corresponding refracted ones, therefore S X Y is the complement of the angle of incidence of S X, and, consequently, Y X q is equal to the angle of incidence itself, and X Y r will be the complement of the angle of refraction, and therefore Y X r ($= 90° -$ X Y r) = the angle of refraction of S Y, or, (since the points Y, X are infinitely near each other,) of S X, hence we have

Y q : X Y :: sin incidence : 1,
X Y : Y r :: 1 : sin refraction.

And compounding Y q : Y r :: sin incidence : sin refraction. But we proved before, that Y q : Y r in the constant ratio of V : v; therefore the sine of incidence : that of refraction in the same constant ratio. Q. E. D.

Corollary 1. In the cases both of reflexion and refraction, the undulation is propagated from the luminous point to any other point in the least possible time. For the surface both of the reflected and refracted waves mark the extreme limits to which the impulse has been propagated by reflexion or refraction in a given time. The undulation propagated from X (fig. 127) in any other direction than X c, as, for instance, X γ, will fall short of the surface B c d, and the point γ therefore will have been reached, and passed by the reflected or refracted primary wave in the situation $\beta\ \gamma\ \delta$, before it can be reached by the secondary undulation propagated from X in the direction X γ. 587.

Corollary 2. This property in the undulatory system corresponds to the principle of least action in the corpuscular doctrine, and may be thus stated generally: 588. Law of swiftest propagation

A reflected or refracted ray will always pursue such a course as would be described in the least possible time, by a point moving from the point of its departure to that of its arrival, with the velocities corresponding to the media in which it moves, and the direction of its motion.

It is evident that this is general, and applies to cases where the medium is either of variable elasticity, or has different elasticities in different directions; for the *ray* is by definition a perpendicular to the surface of the *wave*, or to a surface, the *locus* of all the molecules in the medium, which are just attained by the undulation, and just commencing their vibration, so that the reasoning of Corol. 1, applies equally to all cases. 589. Applies generally.

The properties of foci and Caustics flow with such elegance and simplicity from this doctrine, that it would be unpardonable not to instance its application to that part of the theory of Optics. 590. Foci in the undulatory system. Defined.

Definition. A focus is a point at which the same wave arrives at the same instant from more than one point in a surface.

It is evident, that when this is the case, the ethereal molecules in the focus will be agitated by the united force of all the undulations which reach them in the same phase at the same instant, and will be proportionally more violent as the focus is common to a greater number of points, and the light in the focus will be proportionally more intense.

Proposition. Required to determine the nature of the surface which shall refract all rays from one point rigorously to one focus. Let F (fig. 129) be the focus, then will every part of a wave propagated from S and refracted at the surface A B, reach F at the same instant; therefore time of describing S X with velocity V + time of describing X F with velocity v is constant for every point in the surface. Or, 591.

$$\frac{SX}{V} + \frac{FX}{v} = \text{constant, or } SX + \mu.FX = \text{constant, } \mu \text{ being the relative index of refraction.}$$

This equation then defines the nature of the curve sought, and it is easy to perceive its identity with that expressed by the equation (n) Art. 232, obtained from a direct consideration of the law of refraction, but by a much more intricate process.

The intensity of the reflected or refracted ray cannot be computed generally in the present very imperfect state of our knowledge of the theory of waves. M. Poisson, however, in the case of perpendicular incidence, and on the particular hypothesis of the luminous vibrations being performed in the direction of the ray itself, has succeeded in investigating the comparative intensities of the incident, reflected, and transmitted rays. His results are as follows: Taking μ, μ' for the absolute refractive indices of the media, he finds (on the supposition that the *intensity* of light is as the *square* of the absolute velocity of the vibrating molecules): 592. Intensity of a ray reflected perpendicularly.

Intensity of reflected ray : that of incident :: $(\mu' - \mu)^2 : (\mu' + \mu)^2$. Intensity of the *intromitted* ray : that of the incident :: $4\mu^2 : (\mu + \mu')^2$. Intensity of the ray intromitted from a medium whose refractive index $= \mu$ into a parallel plate of one whose refractive index $= \mu'$, in contact at its second surface with a third whose refractive index $= \mu''$, reflected at their common surface, and again emergent at the first surface: intensity of the ray originally incident on the first surface :: $16\mu^2\mu'^2(\mu'' - \mu')^2 : (\mu + \mu')^4.(\mu' + \mu'')^2$. And, lastly, the intensity of the ray transmitted through the parallel plate of the second medium into the third : that of the original incident ray :: $16\mu^2\mu'^2 : (\mu + \mu')^2.(\mu' + \mu'')^2$ which (in the case where the third medium is the same as the first, becomes $16\mu^2\mu'^2 : (\mu + \mu')^4$.

These results of M. Poisson, so far as they have been hitherto satisfactorily compared with experiment, manifest at least a general accordance, and the undulatory doctrine thus furnishes a plausible explanation of the connection of the reflecting power of a medium with its refractive index, and of the diminished reflection at the common surfaces of media in contact.—They have been in great measure (it should be observed) anticipated by Dr. Young, in his Paper on Chromatics, (*Encyclop. Brit.*) by reasoning which M. Poisson terms indirect, but which, we confess, appears to us by no means to merit the epithet. 593.

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If photometrical experiments enable us to determine the proportion of the reflected to the incident light, we may thence conclude the index of refraction of the reflecting medium, and that in cases where no other mode will apply. Thus, M. Arago having ascertained that about half the incident light is reflected at a perpendicular incidence from mercury, we have in this case $\left(\frac{\mu' - \mu}{\mu' + \mu}\right)^2 = \frac{1}{2}$; $\frac{\mu'}{\mu} = 5{\cdot}829$ for the refractive index of mercury out of air; and this is perfectly consonant to the general tenor of optico-chemical facts, which assign to the heavy and especially to the white metals (as indicated in their transparent combinations) enormous refractive and dispersive powers. This curious and interesting application has not been overlooked by Dr. Young in the Paper alluded to.

594. Applied to determine refractive indices.

595. To complete the theory of reflexion and refraction on the undulatory hypothesis, it will be necessary to show what becomes of those oblique portions of the secondary waves, diverging in all directions from every point of the reflecting or refracting surfaces (as X γ, fig. 127) which do not conspire to form the principal wave. But to understand this, we must enter on the doctrine of the interference of the rays of light,—a doctrine we owe almost entirely to the ingenuity of Dr. Young, though some of its features may be pretty distinctly traced in the writings of Hooke, (the most ingenious man, perhaps, of *his* age,) and though Newton himself occasionally indulged in speculations bearing a certain relation to it. But the unpursued speculations of Newton, and the *appercus* of Hooke, however distinct, must not be put in competition, and, indeed, ought scarcely to be mentioned with the elegant, simple, and comprehensive theory of Young,—a theory which, if not founded in nature, is certainly one of the happiest fictions that the genius of man has yet invented to group together natural phenomena, as well as the most fortunate in the support it has unexpectedly received from whole classes of new phenomena, which at their first discovery seemed in irreconcileable opposition to it. It is, in fact, in all its applications and details one succession of *felicities*, insomuch that we may almost be induced to say, if it be not true, it deserves to be so. The limits of this Essay, we fear, will hardly allow us to do it justice.

§ III. *Of the Interference of the Rays of Light.*

596. General principles of interference

The principle on which this part of the theory of Light depends, is a consequence of that of the "Superposition of small motions" laid down in Art. 583. If two waves arrive at once at the same molecule of the ether, that molecule will receive at once both the motions it would have had in virtue of each separately, and its resultant motion will, therefore, be the diagonal of a parallelogram whose sides are the separate ones. If, therefore, the two component motions agree in direction or very nearly so, the resultant will be very nearly equal to their sum, and in the same direction. If they very nearly oppose each other, then to their difference. Suppose, now, two vibratory motions consisting of a series of successive undulations in an elastic medium, all similar and equal to each other, and indefinitely repeated, to arrive at the same point from the same original centre of vibration, but by different routes (owing to the interposition of obstacles or other causes) exactly, or very nearly in the same final direction; and suppose, also, that owing either to a difference in the lengths of the routes, or to a difference in the velocities with which they are traversed, the time occupied by a wave in arriving by the first route (A) is less than that of its arriving by the other (B). It is clear, then, that any ethereal molecule placed in any point common to the two routes A, B, will begin to vibrate in virtue of the undulations propagated along A, before the moment when the first wave propagated along B reached it. Up, then, to this moment its motions will be the same as if the waves along B had no existence. But after this moment, its motions will be very nearly the sum or difference of the motions it would have separately in virtue of the two undulations each subsisting alone, and the more nearly, the more nearly the two routes of arrival agree in their final direction.

597. Case of complete discordance

Now it may happen, that the difference of the lengths of the routes ~~or~~ the difference of velocities is such, that the waves propagated along B shall reach the intersection exactly one-half an undulation behind the others, *i. e.* later by exactly half the time of a wave running over a space equal to a complete undulation. In that case, the molecule which in virtue of the vibrations propagated along A would (at any future instant) be in one phase of its excursions from its point of rest, would, in virtue of those propagated along B, if subsisting alone, be at the same instant in exactly the opposite phase, *i. e.* moving with equal velocity in the contrary direction. (See Art. 570.) Hence, when both systems of vibration coexist the motions will constantly destroy each other, and the molecule will remain at rest. The same will hold good if the difference of routes or velocities be such, that the vibrations propagated along B shall reach the intersection of the routes exactly $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. of a complete period of undulation after those propagated along A; for the similar phases of vibration recurring periodically, and being (by hypothesis) continually repeated for an indefinite time, it is no matter whether the first vibration propagated along B be superimposed on, *or interfere with* (as it is called) the first, or any subsequent one propagated along A, provided the difference of their phases be the same.

598. Case of complete accordance.

On the other hand it may happen, that the waves propagated along B do not reach the intersection till exactly one, two, or more whole periods after the corresponding waves propagated along A. In this case, the molecule at the intersection will, at any instant subsequent to the time of arrival of the first wave along B, be agitated at once by both vibrations in the *same* phase, and therefore the velocity and amplitude of its excursions will, instead of being destroyed, be doubled.

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Lastly, it may happen, that the difference of the times of arrival of the corresponding waves is neither an exact even, or odd multiple of half a complete period of undulation. In that case, the molecule will vibrate with a joint motion, less than double what it would have in virtue of either separately. 599.

An apt illustration of the case of interference here described, may be had by considering the analogous case in the interference of waves on the surface of water. Conceive, for instance, two equally broad canals A and B to enter ~~two canals~~ at right angles into the side of a reservoir, at both whose apertures, from an origin at a great distance, a wave arrives at the same instant, and runs along the two canals with equal, uniform velocities. Let their sides be perfectly smooth, and their breadths everywhere equal, but let them be led, by a gentle curvature, to meet in a point at some distance, and, the curvature of B being supposed somewhat greater than that of A, let the distance from their intersection to the reservoir, measured along B, be greater than along A. It is obvious, that (if we consider only a single wave) the portion of it propagated along A will reach the intersection first, and after it that propagated along B, so that the water at that point will be agitated by two waves in succession. But, let the original cause of undulation be continually repeated so as to produce *an indefinite series of equal and similar waves.* Then, if the difference of lengths of the two canals be just equal to half the interval between the summits of two consecutive waves, it is evident that when the summit of any wave propagated along A has reached the intersection, the depression between two consecutive summits (*viz.* that corresponding to the wave propagated along A, and that of the wave immediately preceding it) will arrive at the intersection by the course B. Thus, in virtue of the wave along A the water will be raised as much above its natural level, as it will be depressed below it by that along B. Its level will, therefore, be unchanged.—Now as the wave propagated along A passes the intersection, it subsides, from its maximum, by precisely the same gradations as that along B, passing it with equal velocity, rises, from its minimum, so that the level will be preserved at the point of intersection, undisturbed so long as the original cause of undulation continues to act regularly. So soon as it ceases, however, the last half wave which runs along B will have no corresponding portion of a wave along A to interfere with, and will, therefore, create a single fluctuation at the point of concourse. 600. Illustration from waves propagated along canals

In the theory of the interferences of light we may disregard these commencing and terminal, uncompensated undulations, and parts of undulations, as being so few in number as to excite no impression on the retina, and consider the interfering rays as of indefinite duration, or as destitute of either beginning or end. 601. Initial and terminal vibrations disregarded.

According to the foregoing reasoning then it appears, that if two rays having a common origin, *i. e.* forming parts of one and the same system of luminous waves proceeding from a common centre, be conducted by different routes to one point which we will suppose to be situated on a white screen, or on the retina of the eye, they will there produce a bright point, or the sensation of light, if their difference of routes be an even multiple of the length of half an undulation and a dark one; or the sense of darkness, if an odd multiple of it; and if intermediate, then a feebler or a stronger sense of light, as the difference of routes approximates to one or the other of these limits. That two lights should in any case annihilate each other, and produce darkness, appears a strange paradox, yet experiment confirms it; and the fact was observed, and broadly stated by Grimaldi long before any plausible reason could be given of it. 602. Mutual annihilation of two rays of light in opposite phases.

Having thus obtained a general idea of the nature of interferences, let us now endeavour to subject their effects to a more strict calculation. To this end it will be necessary to fix with precision the sense of some words hitherto used rather loosely. 603.

Definition. The *phase* of an undulation affecting any given molecule of ether at any instant of time, is numerically expressed by an arc of a circle to radius unity, increasing proportionally to the time—commencing at 0 when the molecule is at rest at its greatest positive distance of excursion, and becoming equal to one circumference when the molecule, after completing the whole of a vibration, returns again to the same state of rest at the same point. Thus, in the equation $v = a \cdot \sqrt{E} \sin\left(2\pi \cdot \frac{t+C}{T}\right)$, $2\pi \cdot \frac{t+C}{T}$ is the phase of the undulation at the instant t. 604. Definitions. *Phase.*

Definition. The *amplitude* of vibration of a ray or system of waves is the coefficient a, or the maximum excursion from rest, of each molecule of the ether in its course. 605. Amplitude of a ray.

Corol. The intensity of a ray of light is as the square of the amplitude of the vibrations of the waves of which it consists.

Definition. Similar rays, or systems of luminous waves, are such as have the vibratory motions of the ethereal molecules which compose them regulated by the same laws, and their vibrations performed in equal times, and the curves or straight lines they describe in virtue of them, similar and similarly situated in space, so that the motions of any two corresponding molecules in each, shall at every instant of time be parallel to each other. 606. Similar rays.

Corol. Similar rays have the same colour.

Definition. The *origin* of a ray, or a system of waves, is the vibrating material centre from which the waves begin to be propagated, or more generally, a fixed point in its length, at which an ethereal molecule, at an assumed epoch, was in the phase 0 of its undulation. 607. Origin of a ray.

Corol. Two systems of interfering waves having their origins distant by an exact number of undulations, may be regarded as having a common origin. 608.

Proposition. To find the origin of a ray, having given the expression for the velocity of one of its vibrating molecules. 609. To find the origin of a ray.

Let $a = a \cdot \sqrt{E}$, and let $v = a \cdot \sin\left(2\pi \cdot \frac{t+C}{T}\right)$ be the expression given for the velocity

Light. of any assumed molecule (M) at the instant t. Let V represent the velocity of light, and λ the length of an undulation, and δ the distance run over by light in the time t. Then will $\delta = V\,t$ and $\lambda = V\,T$, Part III.

and consequently $\frac{t}{T} = \frac{\delta}{\lambda}$. Suppose v_0 to represent the velocity of a vibrating molecule at the origin of the ray at the instant t, then will $v_0 = a\,.\,\sin\left(2\,\pi\,.\,\frac{t}{T}\right) = a\,.\,\sin\left(2\,\pi\,\frac{\delta}{\lambda}\right)$. But the molecule M moves only by an impulse communicated to it from the origin, and therefore all its motions are later than those at the origin by a constant interval equal to the time required for light to run over the distance of M from the origin. Call D that distance, then $\frac{D}{V}$ is the interval in question, and $t - \frac{D}{V}$ is the time elapsed at the instant t, since the molecule commenced its periodic motions; therefore its velocity v must $= a\,.\,\sin\left(\frac{2\,\pi\quad t - \frac{D}{V}}{T}\right)$, and consequently $C = -\frac{D}{V}$, or $D = -V\,C$.

Hence we see that the distance of the molecule M from the origin of the ray, is equal to the space described by Light, in a time represented by the arbitrary constant C, and is therefore given when C is so, and *vice versâ*.

610. *Corol.* Since $V\,T = \lambda$ the expression for the velocity becomes

$$v = a\,.\,\sin 2\,\pi\,.\left(\frac{t}{T} - \frac{D}{\lambda}\right) = a\,.\,\sin 2\,\pi\left(\frac{\delta - D}{\lambda}\right) \text{ and similarly } x = a\,.\,\cos 2\,\pi\left(\frac{\delta - D}{\lambda}\right)$$

611. Resultant of two interfering rays investigated

Proposition. To determine the colour, origin, and intensity of a ray resulting from the interference of two similar rays, differing in origin and intensity.

Let a^2 and a'^2 be the intensities of the rays, or a, a' their amplitudes of vibration, and take $a = a\,.\,\sqrt{E}$, $a' = a'\,.\,\sqrt{E}$, then, if we put θ for the phase of vibration of a molecule M at the instant t which it would be in, in virtue of the first system of waves (A), and $\theta + k$ for its phase, in virtue of the other (B), $\frac{k}{2\,\pi}\,.$ T will represent the time taken by light to run over a space equal to the interval of their origin, and the velocities and distances from rest which M would have, separately at the instant t, in virtue of the two rays, will be

$$v = a\,.\,\sin\theta\;;\; v' = a'\,.\,\sin(\theta + k), \text{ and } x = a\,.\,\cos\theta\;;\; x' = a'\,.\,\cos(\theta + k).$$

Therefore, in virtue of the resulting ray, it will have the velocity

$$v + v' = a\,.\,\sin\theta + a'\,.\,\sin(\theta + k), \text{ and } x + x' = a\,.\,\cos\theta + a'\,.\,\cos(\theta + k).$$

Let the former be put equal to A . sin $(\theta + B)$, the possibility of which assumption will be shown by our being able to determine A and B, so as to satisfy this condition. Then we have

$$(a + a'\,.\,\cos k)\,\sin\theta + a'\,.\,\sin k\,.\,\cos\theta = A\,.\,\cos B\,.\,\sin\theta + A\,.\,\sin B\,.\,\cos\theta,$$

and equating like terms,

$$A\,.\,\cos B = a + a'\,.\,\cos k\;;\; A\,.\,\sin B = a'\,.\,\sin k,$$

whence we get, dividing one by the other,

$$\tan B = \frac{a'\,.\,\sin k}{a + a'\,.\,\cos k}\;;\; A = \frac{a'\,.\,\sin k}{\sin B} = \sqrt{a^2 - 2\,a\,a'\,.\,\cos k + a'^2}$$

and these values being determined, A and B are known, and, therefore, $v + v' = A\,.\,\sin(\theta + B)$. Similarly, if we put $x + x' = A'\,.\,\cos(\theta + B')$ we obtain values of A′ and B′ precisely similar, writing only $a\,a'$ for a, a' respectively.

612. *Corol.* 1. Hence we conclude, 1st. that the resultant ray is similar to the component ones, and has the same period, i. e. the same colour.

613. Fresnel's theorem. *Corol.* 2. M. Fresnel has given the following elegant rule for determining the amplitude and origin of the resultant ray, which follows immediately from the value of A and the equation $\sin B = \frac{a'}{A}\,.\,\sin K$ above found.

Construct a parallelogram, having its adjacent sides proportional to the amplitudes a, a' of the component rays, and the angle between them measured by a circular arc to radius unity, equal to the differences of their phases, then will the diagonal of this parallelogram represent on the same scale the amplitude of the resulting ray, and the angle included between it, and either side will represent the difference of phases between it and the ray corresponding; or, which comes to the same thing, the difference of their origins (when reduced to space.)

Light. Part III.

Corol. 3. Thus in the case of complete discordance, the diagonal of the parallelogram vanishes, and the angle becomes 180°, or half a circumference, corresponding to a difference of origins of half an undulation. In that of complete *accordance*, the angle is 0, or 360°, and the origins of the rays coincide, or (which comes to the same thing) differ by an exact undulation, and the diagonal is double of the side, so that the *intensity* of the compound ray is four times that of either ray singly.

614. Cases of complete concord and discord.

Corol. 4. If the origins of two equally intense rays differ by one quarter of an undulation, the resultant ray will have its amplitude to that of either component one, as $\sqrt{2} : 1$, and, therefore, its intensity double, and its origin will differ one-eighth of an undulation from that of either. Thus in this particular care, the *brightness* of the compound ray is the sum of the brightnesses of the components, and its position exactly intermediate between them.

615. Case of discordance by a quarter of an undulation.

Corol. 5. Any ray may be resolved into two, differing in origin and amplitude, by the same rules as govern the resolution of forces in Mechanics.

616. Composition and resolution of rays.

Corol. 6. The sum of the *intensities* of the component rays exceeds that of the resultant, when their origins differ by less than a quarter of an undulation, falls short of it when the difference is between $\frac{1}{4}$ and $\frac{1}{2}$, again exceeds it when between $\frac{1}{2}$ and $\frac{3}{4}$, and so on. For the value of A′, above found, gives

$$a^2 + a'^2 - A^2 = 2\, a\, a' .\, \cos k\,;$$

now a^2, a'^2, and A^2, represent the intensities of the respective rays whose momenta are a, a', and A.

617. Relative intensities of simple and compound rays.

Corol. 7. In the same manner may any number of similar rays be *compounded*, and the resultant ray will be similar to the elementary rays, and *vice versâ*.

Let us now consider the interference of waves having the same period (or colour) but in all other respects dissimilar.

618. General problem of interferences.

The law of vibration of the molecules of the luminous bodies which agitate the ether, restricting their motions to ellipses performed in planes, the same will hold good of the motions of each molecule of the ether. Now every elliptic vibration, or rather revolution, performed under the influence of a force directed to its centre and proportional to the distance, is decomposed into three rectilinear vibrations, lying in any three planes at right angles to each other, each of which separately would be performed by the action of the same force in the same time, and according to the same laws of velocity, time, and space. Hence, every elliptic vibration may be expressed by regarding the place of the vibrating molecule at any instant t as determined by three coordinates x, y, z, such that, θ being an arc proportional to the time, we shall have

$$(1.)\ \left\{ \begin{array}{ll} x = a\,.\,\cos(\theta + p)\,; & -\dfrac{dx}{dt} = u = a\,.\,\sin(\theta + p) \\ y = b\,.\,\cos(\theta + q)\,; & -\dfrac{dy}{dt} = v = \beta\,.\,\sin(\theta + q) \\ z = c\,.\,\cos(\theta + r)\,; & -\dfrac{dz}{dt} = w = \gamma\,.\,\sin(\theta + r) \end{array} \right\}\ (2.)$$

In fact, if we multiply the first of these equations by an indeterminate l, the second by m, and the third by n and add, we get

$$(3)\,; \qquad \begin{aligned} l\,x + m\,y + n\,z = \cos\theta\ &\{ l\,a\,.\,\cos p + m\,b\,.\,\cos q + n\,c\,.\,\cos r \} \\ - \sin\theta\ &\{ l\,a\,.\,\sin p + m\,b\,.\,\sin q + n\,c\,.\,\sin r \} \end{aligned}$$

and, therefore, if we determine l, m, n, so that

$$l\,a\,.\,\cos p + m\,b\,.\,\cos q + n\,c\,.\,\cos r = 0\,;\ l\,a\,.\,\sin p + m\,b\,.\,\sin q + n\,c\,.\,\sin r = 0$$

which (being equations of the first degree only) is always possible, we shall have, independently of θ,

$$l\,x + m\,y + n\,z = 0\,; \qquad (4,)$$

and this, being the equation of a plane, shows that the whole curve represented by the above equations lies in one plane. Again, if we eliminate θ between the equations, involving x and y only, we have

$$\cos^{-1}\frac{x}{a} - \cos^{-1}\frac{y}{b} = p - q,$$

or, taking the cosines on both sides,

$$\frac{x\,y}{a\,b} - \sqrt{1 - \frac{x^2}{a^2}}\,.\,\sqrt{1 - \frac{y^2}{b^2}} = \cos(p - q)$$

and reducing, we get the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\,.\,\frac{x}{a}\,.\,\frac{y}{b}\,.\,\cos(p - q) = \sin(p - q)^2\,; \qquad (5,)$$

which is the equation of an ellipse having the origin of the x and y in its centre, and the same is true *mutatis mutandis* of the equations between x and z, and between y and z. Thus the curve represented by the three equations between x, y, z, θ, has an ellipse about the centre for its projection on each of the planes at right angles to each other, and is, of course, itself an ellipse.

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619. Suppose now two systems of waves, or two rays coincident in direction, to interfere with each other. If we accent the letters of the above expressions to represent corresponding quantities for the second system, we shall have

$$\left.\begin{aligned} \mathrm{X} = x + x' &= a\,.\,\cos(\theta + p) + a'\,.\,\cos(\theta + p') \\ \mathrm{Y} = y + y' &= b\,.\,\cos(\theta + q) + b'\,.\,\cos(\theta + q') \\ \mathrm{Z} = z + z' &= c\,.\,\cos(\theta + r) + c'\,.\,\cos(\theta + r') \end{aligned}\right\} \quad (6)$$

and similarly for the velocities $u + u'$, $v + v'$, $w + w'$. In the same manner, then, as we proceeded in the case of two similar rays, let us suppose

$$a\,.\,\cos(\theta + p) + a'\,.\,\cos(\theta + p') = \mathrm{A}\,.\,\cos(\theta + \mathrm{P})$$

and developing

$$(a\,.\,\cos p + a'\,.\,\cos p')\cos\theta - (a\,.\,\sin p + a'\,.\,\sin p')\sin\theta = \mathrm{A}\,.\,\cos\mathrm{P}\,.\,\cos\theta - \mathrm{A}\,.\,\sin\mathrm{P}\,.\,\sin\theta,$$

whence we get

$$\left.\begin{aligned} \tan\mathrm{P} &= \frac{a\,.\,\sin p + a'\,.\,\sin p'}{a\,.\,\cos p + a'\,.\,\cos p'}\;;\quad \mathrm{A} = \frac{a\,.\,\sin p + a'\,.\,\sin p'}{\sin\mathrm{P}} \\ \text{or,}\qquad \mathrm{A} &= \sqrt{a^2 + 2\,a\,a'\,.\,\cos(p - p') + a'^2} \end{aligned}\right\};\quad (7)$$

Thus we have $\mathrm{X} = \mathrm{A}\,.\,\cos(\theta + \mathrm{P})$, and, similarly, $\mathrm{Y} = \mathrm{B}\,.\,\cos(\theta + \mathrm{Q})$, and $\mathrm{Z} = \mathrm{C}\,.\,\cos(\theta + \mathrm{R})$, and a process exactly similar gives us the corresponding expressions for the velocities.

620. Composition and resolution of vibrations generally.

Thus we see that the same rules of composition and resolution apply to dissimilar as to similar vibrations. Each vibration must first be resolved into three rectilinear vibrations in three fixed planes at right angles to each other. These must be separately compounded to produce new rectilinear vibrations in the coordinate planes, which together represent the resulting elliptic vibration, and will have the same period as the component ones. By inverting the process, a vibration of this kind may be resolved into any number of others we please, having the same period.

621. Case of interference of rectilinear vibrations.

A great variety of particular cases present themselves, of which we shall examine some of the principal. And first, *when the interfering vibrations are both rectilinear.*

Since the choice of our coordinate planes is arbitrary, let us suppose that of the x, y to be that in which both the vibrations are performed. Of course the resulting one will be performed in the same. Therefore we may put $z = 0$, or $c = 0$, $c' = 0$, and content ourselves with making

$$\left.\begin{aligned} x &= a\,.\,\cos(\theta + p)\;;\; y = b\,.\,\cos(\theta + p) \\ x' &= a'\,.\,\cos(\theta + p')\;;\; y' = b'\,.\,\cos(\theta + p') \end{aligned}\right\};\quad (8)$$

The resultant vibration is the general elliptic.

because $\frac{x}{y}$ and $\frac{x'}{y'}$ are constant in this case, and X, Y, A, B, P, Q, denoting as in the general case, we have

$$\mathrm{X} = \mathrm{A}\,.\,\cos(\theta + \mathrm{P})\;;\quad \mathrm{Y} = \mathrm{B}\,.\,\cos(\theta + \mathrm{Q})\;;$$

and, by elimination of θ,

$$\left(\frac{\mathrm{X}}{\mathrm{A}}\right)^2 + \left(\frac{\mathrm{Y}}{\mathrm{B}}\right)^2 - 2\cos(\mathrm{P} - \mathrm{Q})\,.\,\frac{\mathrm{X}\,\mathrm{Y}}{\mathrm{A}\,\mathrm{B}} = \sin(\mathrm{P} - \mathrm{Q})^2;\quad (9)$$

where A, B, P, Q, are determined as in equations, (7.) In the general case, then, the resulting vibration is elliptic.

622. Case when the resultant is rectilinear.

The ellipse degenerates into a straight line by the diminution of its minor axis when $\mathrm{P} = \mathrm{Q}$. Now this gives $\tan\mathrm{P} = \tan\mathrm{Q}$, or

$$\frac{a\,.\,\sin p + a'\,.\,\sin p'}{a\,.\,\cos p + a'\,.\,\cos p'} = \frac{b\,.\,\sin p + b'\,.\,\sin p'}{b\,.\,\cos p + b'\,.\,\cos p'}$$

which, reduced, takes the form

$$\left(\frac{a'}{a} - \frac{b'}{b}\right)\,.\,\sin(p - p') = 0.$$

There are, therefore, two cases, and two only in which the resulting vibration is rectilinear. The first, when $p - p' = 0$, or when the component vibrations have a common origin, or are in complete accordance; the

Case when their directions coincide.

other, when $\frac{a'}{a} = \frac{b'}{b}$, that is, when they are both performed in one plane, and in the same direction. For if we call m and m' the amplitudes, and ψ, ψ' the angles they make with the axis of the x, we have

$$a = m\,.\,\cos\psi\;;\; b = m\,.\,\sin\psi\;;\; a' = m'\,.\,\cos\psi'\;;\; b' = m'\,.\,\sin\psi',$$

so that the above equation is equivalent to $\tan\psi = \tan\psi'$, or $\psi = \psi'$.

623. The latter case we have already fully considered. In the former, we have $\cos(p - p') = 0$, and, therefore,

$$\mathrm{A} = a + a'\;;\; \mathrm{B} = b + b'\;;\; \mathrm{P} = p\;;\; \mathrm{Q} = p,$$

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and, finally, $$\frac{Y}{X} = \frac{b + b'}{a + a'} = \tan \phi \quad ; \qquad (10)$$

which is the tangent of the angle made by the resulting rectilinear vibration with the axis of the x.

If we put M for the amplitude of the resulting vibration, we have $M \cdot \cos \phi = A$; $M \cdot \sin \phi = B$; therefore, $M^2 \cdot (\cos \phi^2 + \sin \phi^2)$ or $M^2 = A^2 + B^2$.

Case of complete accordance of non-coincident vibrations.

624. Amplitude and position of resultant vibration determined.

Now, $$A^2 = (a + a')^2 = (m \cdot \cos \psi + m' \cdot \cos \psi')^2$$
$$B^2 = (b + b')^2 = (m \cdot \sin \psi + m' \cdot \sin \psi')^2$$

and, therefore, adding these values together, and reducing

$$M^2 = m^2 + 2 m m' \cdot \cos (\psi - \psi') + m'^2 ; \qquad (11)$$

Now, $\psi - \psi'$ is the angle between the directions of the component vibrations, so that this equation expresses that the amplitude of the resultant vibration is in this case also the diagonal of a parallelogram, whose sides are the amplitudes of the component ones; and it is easily shown, by substituting in $\tan \phi = \frac{b + b'}{a + a'}$ the above values of $a + a'$, $b + b'$, that the diagonal has also the position of the resultant line of vibration.

625. *Corol.* 1. Any rectilinear vibration may be resolved into two other rectilinear vibrations, whose amplitudes are the sides of any parallelogram, of which the amplitude of the original vibration is the diagonal, and which are in complete accordance, or have a common origin with it.

626. *Corol.* 2. Hence any rectilinear vibration may be readily reduced to the directions of two rectangular coordinates, or, if necessary, into those of three, by the rules of the resolution of forces, and the component vibrations, however numerous, will be in complete accordance with the resultant.

627. The ellipse degenerates into a circle when $\cos (P - Q) = 0$, or $P - Q = 90^\circ$, and, also, $A = B$. Now the former condition gives $\tan P + \cot Q = 0$, that is

Case of circular vibrations.

$$\frac{a \cdot \sin p + a' \cdot \sin p'}{a \cdot \cos p + a' \cdot \cos p'} + \frac{b \cdot \cos p + b' \cdot \cos p'}{b \cdot \sin p + b' \cdot \sin p'} = 0$$

or reducing

$$\cos (p - p') = - \frac{a\,b + a'\,b'}{a\,b' + a'\,b} = - \tfrac{1}{2} \frac{m^2 \cdot \sin 2\psi + m'^2 \sin 2\psi'}{m\,m' \cdot \sin (\psi + \psi')} \qquad (12)$$

The condition $A = B$, or $A^2 = B^2$, gives

$$a^2 + 2\,a\,a' \cdot \cos (p - p') + a'^2 = b^2 + 2\,b\,b' \cdot \cos (p - p') + b'^2$$

whence we, in like manner, obtain

$$\cos (p - p') = - \frac{(a^2 + a'^2) - (b^2 + b'^2)}{2\,a\,a' - 2\,b\,b'} = - \tfrac{1}{2} \frac{m^2 \cdot \cos 2\psi + m'^2 \cdot \cos 2\psi'}{\cos (\psi - \psi')} \qquad (13)$$

and, equating the values of $\cos (p - p')$, we find the following relation between a, a', b, b', which must subsist when the vibrations are circular,

$$\left(\frac{a}{b} - \frac{a'}{b'}\right)(a^2 + b^2 - a'^2 - b'^2) = 0.$$

The vanishing of the first factor gives no circular vibration, it being introduced with the negative root of the equation $A^2 = B^2$, with which we have no concern. The other gives

$$a^2 + b^2 = a'^2 + b'^2, \text{ or } m = m',$$

which shows that the component vibrations must have equal amplitudes. Now, if for a and b we write their values $m \cdot \cos \psi$ and $m \cdot \sin \psi$, and for a' and b', respectively, $m \cdot \cos \psi'$ and $m \cdot \sin \psi'$, in either of the expressions for $\cos (p - p')$, it will reduce itself to

$$\cos (p - p') = - \cos (\psi - \psi') ; \text{ or, } p - p' = 180^\circ - (\psi - \psi').$$

Hence it appears, that the *interference of two* equal rectilinear vibrations will produce a resultant circular one, provided the difference of their phases be equal to the *supplement* of the angle their directions make with each other, so that when the molecule is just commencing its motion towards its centre, in virtue of one vibration, it shall be receding from it at an obtuse angle with this motion, in virtue of the other.

Corol. Hence, if two vibrations have equal amplitudes, but differ in their phases by a quarter of an undulation, their resultant vibration will be circular.

628. We are now in a condition to explain what becomes of the portions of the secondary waves which diverge obliquely from the molecules of the primary ones, as alluded to in Art. 595, and to explain the mode in which those which do not conspire with the primary wave mutually destroy each other. To this end, conceive the surface of any wave A B C to consist of vibratory molecules, *all in the same phase of their vibrations.* Then will the motion of any point X (fig. 130) be the same, whether it be regarded as arising from the original motion of S, or as the resultant of all the motions propagated to it from all the points of this surface. Conceive the surface A B C divided into an infinite number of elementary portions, such that the difference of distance of each consecutive pair from X shall be constant, or $= d\,f$, putting the distance of any one from that point $= f$; and let A B, B C, C D, &c., and A b, $b\,c$, $c\,d$, &c. be finite portions of the surface containing each the same number of

Fig. 130.

Mutual destruction of secondary waves.

Light. these elements, and in each of which the corresponding values of f are exactly half an undulation ($\frac{1}{2}\lambda$) greater than in the preceding, so that (for instance) $BX = AX + \frac{1}{2}\lambda$, $CX = BX + \frac{1}{2}\lambda$, &c. Then it is evident, that the vibrations which reach X simultaneously from the corresponding portions of any two consecutive ones, as of AB and BC, will be in exactly opposite phases; and, therefore, were they of equal intensity, and in precisely the same direction, would interfere with, and destroy each other. Now, first, with regard to their intensity, this depends on the magnitudes of the elements of the wave AB, from which they are derived, and on the law of lateral propagation. Of the latter, we know little, *à priori;* but all the phenomena of light indicate a very rapid diminution of intensity, as the direction in which the secondary undulations are propagated deviates from that of the primary. With respect to the former, it is evident that the elements in the immediate vicinity of the perpendicular AX, corresponding to a given increment df of the distance from X, are much larger than those remote from it; so that all the elements of the portion AB are much larger than those in BC, and these again than in those of CD, and so on. Thus the motion transmitted to X from any element in AB will be much greater than that from the corresponding one in BC, and that again greater than that from the element in CD, and so on. Thus the motion arriving at G, from the whole series of corresponding elements, will be represented by a series such as $A - B + C - D + E - F +$ &c., in which each term is successively greater than that which follows. Now it is evident that the terms approach with great rapidity to equality; for if we consider any two corresponding elements as M, N at a distance from A at all considerable, the angles XM and XN make with the surface approach exceedingly near to equality, so that the obliquity of the secondary wave to the primary, and of course its intensity, compared with that of the direct wave, is very nearly alike in both; and the elements M, N themselves, at a distance from the perpendicular, approach rapidly to equality, for the elementary triangles Mmo, Mnp are in this case very nearly similar, and have their sides mo, np equal by hypothesis. Finally, the lines MX, NX approach nearer to each other in direction so as to produce a more complete interference, as their distance from A is greater. Part III.

629. Thus we see that the terms of the series $A - B + C - D +$ &c., at a distance from its commencement, have on all accounts (viz. their smallness, near approach to equality, and disposition to interfere) an extremely small influence on its value; and as the same is true of every set of corresponding elements into which the portions AB, BC, &c. are divided, it is so of their joint effect, so that the motion of the molecule X is governed entirely by that of the portion of the wave ABC immediately contiguous to A, the secondary vibrations propagated from parts at a distance mutually interfering and destroying each others effect.

630. It is obvious, that in the case of refraction or reflexion, we may substitute for the wave AM the refracting or reflecting surface; and for the perpendicular XA the primary refracted ray, when the same things, *mutatis mutandis*, will hold good. See M. Fresnel's Paper entitled *Explication de la Réfraction dans le Système des Ondes*, published in the *Bulletin de la Société Philomatique*, October, 1821.

631. Case of a wave transmitted through a limited aperture. This is the case when the portion of the wave ABCD whose vibrations are propagated to X is unlimited, or at least so considerable, that the last term in the series $A - B + C -$ &c. is very minute compared with the first. But if this be not the case, as, if the whole of a wave except a small part about A be intercepted by an obstacle, the case will be very different. It is easy on this supposition to express by an integral the intensity of the undulatory motion of X, compared with what it would be on the supposition of no obstacle existing. For this purpose, let $d^2 s$ be the magnitude of any vibrating element of the surface, f its distance from $X = MX$, and let $\phi(\theta)$ be the function of the angle made by a laterally-divergent vibration with the direct one, which expresses its relative intensity, and which is unity when $\theta = 0$, and diminishes with great rapidity as θ increases. Then if t be the time since a given epoch, $\lambda =$ the length of an undulation, $SA = a$, the phase of a vibration arriving at X by the route SMX will be $2\pi\left(\frac{t}{T} - \frac{a+f}{\lambda}\right)$, and the velocity produced in X thereby will be represented by $a \cdot d^2 s \cdot \phi(\theta) \cdot \sin 2\pi\left(\frac{t}{T} - \frac{a+f}{\lambda}\right)$, so that the whole motion produced will be represented by

$$\iint a \cdot d^2 s \cdot \phi(\theta) \cdot \sin 2\pi \left\{\frac{t}{T} - \frac{a+f}{\lambda}\right\}$$

the integral being extended to the limits of the aperture.

632. *Corol.* 1. If but a very small portion of the wave be permitted to pass, as in the case of a ray transmitted through a very small hole, and received on a distant screen, θ and $\phi(\theta)$ are very nearly constant, so that the motion excited in X is in this case represented by

$$a \cdot \phi(\theta) \iint d^2 s \cdot \sin 2\pi \left\{\frac{t}{T} - \frac{a+f}{\lambda}\right\}.$$

We shall have occasion to revert to these expressions hereafter.

§ IV. *Of the Colours of Thin Plates.*

633. General account of the phenomena. Every one is familiar with the brilliant colours which appear on soap-bubbles; with the iridescent hues produced by heat on polished steel or copper; with those fringes of beautiful and splendid colours which appear in the cracks of broken glass, or between the laminæ of fissile minerals, as Iceland spar, mica, sulphate of lime, &c. In all these, and an infinite variety of cases of the same kind, if the fringes of colour be examined

Light. with care they will be found to consist of a regular succession of hues, disposed in the same order, and determined, obviously, not by any colour in the medium itself in which they are formed, or on whose surfaces they appear, but solely by its greater or less thickness. Thus a soap-bubble (defended from currents of air by being placed under a glass) at first appears uniformly white when exposed to the dispersed light of the sky at an open window; but, as it grows thinner and thinner by the subsidence of its particles, colours begin to appear at its top where thinnest, which grow more and more vivid, and (if kept perfectly still) arrange themselves in beautiful horizontal zones about the highest point as a centre. This point, when reduced to extreme tenuity, becomes *black*, or loses its power of reflecting light almost entirely. After which the bubble speedily bursts, its cohesion at the vertex being no longer sufficient to counteract the lateral attraction of its parts. Part III.

634. Rings formed between convex glasses.

But as it is a matter of great delicacy to make regular observations on a thing so fluctuating and unmanageable as a soap-bubble, the following method of observing and studying the phenomena is far preferable. Let a convex lens, of a very long focus and a good polish, be laid down on a plane glass, or on a concave glass lens having a curvature somewhat less than the convex surface resting on it; so that the two shall touch in but a single point, and so that the interval separating the surfaces in the surrounding parts shall be exceedingly small. If the surfaces be very carefully cleaned from dust before placing them together, and the combination be laid down before an open window in full daylight, the point of contact will be seen as a black spot in the general reflexion of the sky on the surfaces, surrounded with rings of vivid colours. A glass of 10 or 12 feet focus laid on a plane glass, will show them very well. If one of shorter focus be used, the eye may be assisted by a magnifying glass. The following phenomena are now to be attended to:

635. Order of succession of the colours.

Phenomenon 1. The colours, whatever glasses be used, provided the incident light be white, always succeed each other in the very same order; that is, beginning with the central black spot, as follows:

First ring, or first order of colours,—*black, very faint blue, brilliant white, yellow, orange, red.*

Second ring, or second order,—*dark purple or rather violet, blue, green,* (very imperfect, a yellow-green,) *vivid yellow, crimson red.*

Third ring, or third order,—*purple, blue, rich grass green, fine yellow, pink, crimson.*

Fourth ring, or fourth order,—*green,* (dull and bluish,) *pale yellowish pink, red.*

Fifth ring, or fifth order,—*pale bluish green, white, pink.*

Sixth ring, or sixth order,—*pale blue-green, pale pink.*

Seventh ring, or seventh order,—*very pale bluish green, very pale pink.* After these, the colours become so faint that they can scarcely be distinguished from white.

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On these we may remark, that the green of the third order is the only one which is a pure and full colour, that of the second being hardly perceptible, and of the fourth comparatively dull and verging to apple green; the yellow of the second and third order are both good colours, but that of the second is especially rich and splendid; that of the first being a fiery tint passing into orange. The blue of the first order is so faint as to be scarce sensible, that of the second is rich and full, but that of the third much inferior; the red of the first order hardly deserves the name, it is a dull brick colour; that of the second is rich and full, as is also that of the third; but they all verge to crimson, nor does any pure scarlet, or prismatic red, occur in the whole series.

637. Law of the breadths of the rings and thicknesses at which they appear.

Phenomenon 2. The breadths of the rings are unequal. They decrease, and the colours become more crowded, as they recede from the centre. Newton (to whom we owe the accurate description and investigation of their phenomena) found by measurement the diameters of the *darkest* (or purple) rings, just when the central black spot began to appear by pressure, and reckoning it as one of them, to be as the square roots of the even numbers 0, 2, 4, 6, &c.; and those of the brightest parts, of the several orders of colours, to be as the square roots of the odd numbers 1, 3, 5, 7, &c. Now the surfaces in contact being spherical, and their radii of curvature very great in proportion to the diameters of the rings, it follows from this that the intervals between the surfaces at the alternate points of greatest obscurity and illumination are as the natural numbers themselves 0, 1, 2, 3, 4, &c. The same measurements, when the radii of curvature of the contact surfaces are known, give the absolute magnitudes of the intervals in question. In fact, if r and r' be the curvatures of two spherical surfaces, a convex and concave, in contact, and D the diameter of any annulus surrounding their point of contact, the interval of the surfaces there will be the difference of the versed sines of the two circular arcs having a common chord D. Now (fig. 130) if A E be the diameter of the convex spherical surface A D, we have $EA : AD :: AD : DB = \frac{AD^2}{AE} = \frac{D^2}{8} r$, and in like manner $BC = \frac{D^2}{8} r'$, so that $\frac{1}{8} D^2 (r - r') = DC$, the interval of the surfaces at the point D. Thus Newton found, for the interval of the surfaces at the brightest part of the first ring, one 178000dth part of an inch; and this distance, multiplied by the even natural numbers 0, 2, 4, 6, 8, &c. gives their distance at the black centre and the darkest parts of the purple rings, and by the odd ones 1, 3, 5, &c. their intervals at the brightest parts.

638. Invariable relation between the colours and thicknesses of plates.

Phenomenon 3. If the rings be formed between spherical glasses of various curvatures, they will be found to be larger as the curvatures are smaller, and *vice versâ;* and if their diameters be measured and compared with the radii of the glasses, it will be found, that, provided the eye be similarly placed, *the same colour is invariably produced at that point, or that distance from the centre where the interval between the surfaces is the same.* Thus, the white of the first order is invariably produced at a thickness of one 178000th of an inch; the purple, which forms the limit of the first and second orders, at twice that thickness. So that there is a constant relation between the tint seen and the interval of the surfaces where it appears. Moreover, if the glasses be distorted by violent and unequal pressure, (as is easily done if thin lenses be used,) the rings lose their circular figure, and extend themselves towards the part where the irregular pressure is applied, so as to form a species of level lines each marking out a series of points where the surfaces are equidistant. Thus, too, if a

Light. cylinder be laid on a plane, the rings pass into straight lines arranged parallel to its line of contact, but following the same law of distance from that line as the rings from their dark centre, and if the glasses be of irregular curvature, as bits of window glass, the bands of colour will follow all their inequalities; yet more, if the pressure be very cautiously relieved, so as to lift one glass from the other, the central spot will shrink and disappear, and so on; each ring in succession contracting to a point, and then vanishing, so as to bring all the more distant colours successively to the centre, as the glasses recede from absolute contact. From all these phenomena it is evident, that it is the distance between the surfaces only at any point which determines the colour seen there. Part III.

639. Effect of obliquity of incidence. *Phenomenon* **4**. This supposes, however, that we observe them with the eye similarly placed, or *at the same angle of obliquity*. For if the obliquity be changed by elevating or depressing the eye, or the glasses, the diameters (but not the colours) of the rings will change. As the eye is depressed, the rings enlarge; and the same tint which before corresponded to an interval of the 178000th of an inch, now corresponds to a greater interval. This distance ($\frac{1\ in.}{178000}$) is determined by measures taken nearly at, and reduced by calculation exactly to, a perpendicular incidence. At extreme obliquities, however, the diameters of the several rings suffer only a certain finite dilatation, and Newton's measures led him to the following rule: viz. "*That the interval between the surfaces at which any proposed tint is produced, is proportional to the secant of an angle whose sine is the first of* 106 *arithmetical mean proportionals between the sines of incidence and refraction, into the glass from the air, or other medium included between the surfaces, beginning with the greater*;" or, in algebraic language, the relative index of refraction being μ, and θ the angle of incidence, and ρ that of refraction of the ray as it passes out of the rarer medium into the denser; then, if t be the interval corresponding to a given tint at the oblique incidence θ, and T at a perpendicular incidence, we shall have

$$t = \mathrm{T} \,.\, \sec u \text{ where } \sin u = \sin\theta \,\frac{1}{107} - (\sin\theta - \sin\rho)$$

but $\sin\rho = \frac{1}{\mu} \,.\, \sin\theta$, consequently we have

$$t = \mathrm{T} \,.\, \sec u;\ \sin u = \frac{106 + \frac{1}{\mu}}{107} \,.\, \sin\theta \,.= \frac{106\,\mu + 1}{107\,\mu} \,.\, \sin\theta.$$

640. Fig. 132. Rings seen through a prism. To see the rings conveniently at extreme obliquities, a prism may be used, laid on a convex lens, as in fig. 132. If the eye be placed at K, the set of rings formed about the point of contact E will be seen in the direction K H, and as the eye is depressed towards the situation I, where the ray I G intromitted from I would just begin to suffer total reflexion, the rings are seen to dilate to a certain considerable extent. When the eye reaches I, the upper half of the rings disappears, being apparently cut off by the prismatic iris of Art. 555, which is seen in that situation, but the black central spot and the lower half of the rings remains; but when the eye is still further depressed the rings disappear, and leave the central spot, like an aperture seen in the silvery whiteness of the total reflexion on the base of the prism, and dilated very sensibly beyond the size of the same spot seen in the position K H: thus proving, that the want of reflexion on that part of the base extends *beyond* the limits of *absolute contact* of the glasses, and that, therefore, the lower surface interferes with the action of the upper, and prevents its reflexion while yet a finite interval (though an excessively minute one) intervenes between them. Euler has made this an objection to the undulatory theory, but the objection rests on no solid grounds, as it is very reasonable to conclude, that the change of density or elasticity in the ether within and without a medium is not absolutely *per saltum*, but gradual. If so, and if the change take place *without* the media, the approach of two media within that limit, within which the condensation of the ether takes place, will alter the law of refraction from either into the interval separating them.

641. Fringes seen when a prism is laid on a plane glass. Fig. 133. In order, however, to see to the greatest advantage the colours reflected by a plate of air at great obliquities, the following method, first pointed out by Sir William Herschel, may be employed. On a perfectly plane glass or metallic mirror, before an open window, lay an equilateral prism, having its base next the glass or mirror very truly plane, and looking in at the side A C, fig. 133, the reflected prismatic iris, *a*, *b*, *c*, will be seen as usual in the direction E F, where a ray from E would just be totally reflected. Within this iris, and arranged parallel to it, are seen a number of beautiful coloured fringes, whose number and distances from each other vary with every change of the pressure; their breadths dilating as the pressure is increased, and *vice versâ*. They do not require for their formation, that the surfaces should be exceedingly near, being seen very well when the prism is separated from the lower surfaces by the thickness of thin tissue paper, or a fine fibre of cotton wool interposed, but in this case they are exceedingly close and numerous. If the pressure be moderate, they are nearly equidistant, and are lost, as it were, in the blue iris, without growing sensibly broader as they approach it. As the intervals of the surfaces is diminished, they dilate and descend towards the eye, appearing, as it were, to come down out of the iris. They do not require for their formation a perfect polish in the lower surface. An emeried glass, so rough as to reflect no regular image at any moderate incidence, shows them very well. The experiment is a very easy one, and the phenomena so extremely obvious and beautiful, that it is surprising it should not have been noticed and described by Newton, especially as it affords an excellent illustration of his law above stated To understand this, let E H, E K, E L be any rays from E incident at angles somewhat less than that of total reflexion on the base; they will therefore be refracted, and, emerging at the base B C, will be reflected at M N, (the obliquity of the reflexion being so great, that even rough surfaces reflect copiously and regularly enough for the purpose, Art. 558,) and will pursue the courses H D P *p*, K F Q *q*, L G R *r*, &c. entering the prism again at P, Q, R. Reciprocally, then, rays *p* P, *q* Q, &c. incident at P, Q, &c. in these directions,

Light. will enter the eye at E after traversing the interval B C N M, and being reflected at M N, and will affect the eye with the colour corresponding to that obliquity and that interval between the surfaces which is proper to each. If then we put, as above, θ for the exterior angle of incidence of the ray D H on the base of the prism, and take Part III.

$$\sin u = \frac{106\,\mu + 1}{107\,\mu} \,.\, \sin\theta = \frac{106\,\mu + 1}{107} \,.\, \sin\rho = k \,.\, \sin\rho,$$

the tint seen in the direction E H will (abstraction made of the dispersion at the surface A C) be the same with that reflected at a perpendicular incidence, by a plate of air of the thickness $T = t \,.\, \cos u = t\sqrt{1 - k^2 \,.\, \sin\rho^2}$, where $t =$ the distance between the surfaces B C, M N. There will, therefore, appear a succession of colours in the several consecutive situations of the line E H, analogous to those of the coloured rings, (except in so far as the dispersion of the side A C alters the tints by separating their component rays.)

642. But the whole series of colours will not be seen, because those which require greater obliquities than that at which total reflexion takes place, cannot be formed. In fact, the angle, reckoned from the vertical at which a tint corresponding to a thickness T in the rings would be formed, is given by the equation

$$\sin\rho = \frac{1}{k} \,.\, \sqrt{1 - \left(\frac{T}{t}\right)^2} = \frac{214}{320}\sqrt{1 - \left(\frac{T}{t}\right)^2},$$

taking $\mu = \frac{3}{2}$ for glass, which it is very nearly. Now, according to this, the central tint, or black of the first order, which is formed when $T = 0$, requires that

$$\sin\rho = \frac{1}{k} = \frac{1}{\mu - \frac{\mu - 1}{107}}$$

which being greater than $\frac{1}{\mu}$ shows that this tint lies above the situation of the iris, and cannot therefore be seen. The first visible tint will be that close to the iris, where $\sin\rho = \frac{1}{\mu}$ which gives

$$T = t\sqrt{1 - \left(\frac{k}{\mu}\right)^2} = t\sqrt{1 - \left(1 - \frac{\mu - 1}{107\,\mu}\right)^2} = t\sqrt{\frac{2\,(\mu - 1)}{107\,\mu}} = 0{\cdot}079\ t$$

nearly, or $\frac{t}{12{\cdot}25}$. Hence it appears, that these fringes would be seen, by an eye immersed in the prism, when the interval between its base and the glass it rests on is more than 12 times that at which colours are formed at a perpendicular incidence, *i. e.* at $12{\cdot}25 \times \frac{13}{178000}$, or about $\frac{1}{1100}$th of an inch, which is about the thickness of fine tissue paper. Moreover, from this value of T, we see that the first tint immediately visible below the iris ascends in the scale of the rings (*i. e.* belongs to a point nearer their centre) as the value of t diminishes, or as the prism is pressed closer to the glass; and this explains why the fringes become more numerous, and appear to come out of the iris by pressure. With regard to their angular breadth, (still to an eye immersed in the prism.) If we put $e = \frac{1^{\,inch}}{89000}$, we have, putting ρ_0, ρ_1, &c. for the values of ρ, corresponding to the several orders of visible tints,

$$\sin\rho_0 = \frac{1}{\mu}\,;\quad \sin\rho_1 = \frac{1}{k}\sqrt{1 - \left(\frac{T + e}{t}\right)^2} = \frac{1}{\mu}\sqrt{1 - \frac{\mu^2}{k^2} \,.\, 0{\cdot}079 \,.\, \frac{2\,e}{t}} = \frac{1}{\mu}\left(1 - 0{\cdot}079 \times \frac{e}{t}\right)$$

very nearly, $\sin\rho_2 = \frac{1}{\mu}\left(1 - 0{\cdot}079 \,.\, \frac{2\,e}{t}\right)$ and so on. The sines then of the incidences at which the several orders of colours are developed, beginning at the iris, increase in arithmetical progression, so that the fringes must be disposed in circular arcs parallel to the iris, and their breadths must be nearly equal, and greater the greater the pressure or the less t is, all which is conformable to observation. The refraction of the side of the prism between the eye and the base, however, disturbs altogether the succession of colours in the fringes, and in particular multiplies the number of visible alternations to a great extent, in a manner which will be evident on consideration. We have been rather more particular in explaining the origin of these fringes, and referring them to the general phenomena observed by Newton, because up to the present time we believe no strict analysis of them has been given, as well as on account of the great beauty of the phenomenon itself. If we hold the combination up to the light, and look through the base of the prism and the glass plate, so as to see the transmitted iris of Art. 556, its concavity will, in like manner, be seen fringed with bands of colours of precisely similar origin. To return now to the rings seen between convex glasses.

643. *Phenomenon* 5. If homogeneous light be used to illuminate the glasses, the rings are seen in much greater

Light.

Phenomena exhibited by homogeneous light.

number, and the more according to the degree of homogeneity of the light. When this is as perfect as possible, as, for instance, when we use the flame of a spirit lamp with a salted wick, as proposed by Mr. Talbot, they are literally innumerable, extending to so great a distance that they become too close to each other to be counted, or even distinguished by the naked eye, yet still distinct on using a magnifier, but requiring a higher and higher power as they become closer, till we can pursue them no farther, and disappearing from their closeness, and not from any confusion or running of one into the other. Moreover, they are now no longer composed of various colours, but are wholly of the colour of the light used as an illumination, being mere alternations of light and obscurity, and the intervals between them being absolutely black.

644. Contraction of the rings in the less refrangible rays.

Phenomenon 6. When the illuminating light is changed from one homogeneous ray to another, as when, for instance, the colours of the prismatic spectrum are thrown in succession on the glasses at their point of contact, at such an angle as to be reflected to the eye, then, the eye remaining at rest, the rings are seen to dilate and contract in magnitude as the illumination shifts. In red light they are largest, in violet least, and in the intermediate colours of intermediate size. Newton, by measuring their diameters, ascertained that the interval of the surfaces or thickness of the plate of air, where the violet ring of any order was seen, is to its thickness, where the corresponding red ring of the same order is formed, nearly as 9 : 14; and, determining by this method, the thickness of the plate of air where the brightest part of the first ring was formed, when illuminated in succession by the several rays proceeding from the extreme red to the extreme violet, he ascertained those thicknesses to be the halves of the numbers already set down in the second column of the Table, p. 453, expressed in parts of an inch, and which answer to the values of $\frac{\lambda}{2}$, or the lengths of a semiundulation for each ray.

Analysis of the coloured rings.

645. Synthesis of the coloured rings. Fig. 134.

This phenomenon may be regarded as an analysis of what takes place when the rings are seen in white light; for in that case they may be regarded as formed by the superposition one on the other of sets of rings of all the simple colours, each set having its own peculiar series of diameters. The manner in which this superposition takes place, or the *synthesis* of the several orders of colours, may be understood by reference to fig. 134, where the abscissæ or horizontal lines represent the thicknesses of a plate of air between two glasses, supposed to increase uniformly, and where R R′, R R″, &c. represent the several thicknesses at which the red, in the system of rings illuminated by red rings only, vanishes, or at which the darkness between two consecutive red rings is observed to happen, while R *r*, R *r*′, R *r*″, &c. represent those at which the brightness is a maximum. In like manner, let 0 0′, 0 0″, &c. be taken equal to the several thicknesses at which the orange vanishes, or at which the black intervals in the system of orange rings are seen, and so on for the yellow, green, blue, indigo, and violet rings. So that R R′, 0 0′, Y Y′, &c. are to each other in the ratio of the numbers in column 2 of the above Table, (Art. 575.) Then if we describe a set of undulating curves as in the figure, and at any point, as C in A E, draw a line parallel to A V, cutting all these curves; their several ordinates, or the portions of this line intercepted between the curves and their abscissæ, will represent the intensity of the light of each colour, sent to the eye by that thickness of the plate of air. Hence, the colour seen at that thickness will be that resulting from the union of the several simple rays in the proportions represented by their ordinates.

646. Synthesis of the several orders of colours.

The figure being laid down by a scale, we may refer to it to identify the colours of particular points. Thus, first at the thickness 0, or at A the origin of the tints, all the ordinates vanish, and this point, therefore, is black. As the thickness of the plate of air increases from 0 while yet very small, it is evident, on inspection, that the ordinates of the several curves increase with unequal rapidities, those for the more refrangible rays more rapidly than those for the less, so that the first feeble light which appears at a very small thickness A 1, will have an excess of blue rays, constituting the pure but faint blue of the first order, (Art. 635.) At a greater thickness, however, as A 2, the common ordinate passes nearly through the maxima of all the curves, being a little short of that of the red, and a little beyond that of the violet. The difference, however, is so small, that the several colours will all be present nearly in the proportions to constitute whiteness, and being all nearly at their maximum, the resulting tint will be a brilliant white. This agrees with observation; the white of the first order being, in fact, the most luminous of all; beyond this the violet falls off rapidly, the red increases, and the yellow is nearly at its maximum, so that at the thickness A 3 the white passes into yellow, and at a still greater thickness, A 4, where the violet, indigo, blue, and green, are all nearly evanescent, the yellow falling off, and the orange and red, especially the latter, in considerable abundance, the tint resulting will be a fiery orange, growing more and more ruddy. At B is the minimum of the yellow, *i. e.* of the most luminous rays. Here then will be the most sombre tint. It will consist of very little either of orange, green, blue, or even indigo; but a moderate portion of violet and a little red will produce a sombre violet purple, which, since the more refrangible rays are here all on the increase, while the less are diminishing, will pass rapidly to a vivid blue, as at the thickness denoted by A 5. At 6, where the ordinate passes through the maximum of the yellow, there is almost no red, very little orange, a good deal of green, very little blue, and hardly any indigo or violet. Here then the tint will be yellow verging to green, but the green is diminishing and the orange increasing, so that the yellow rapidly loses its green tinge, and becomes pure and lively. At 7 the predominant rays are orange and yellow, being so copious that the little red and violet with which they are mixed does not prevent the tint from being a rich, high-coloured yellow. At 8 a full orange and copious red are mixed with a good deal of indigo and a maximum of violet, thus producing a superb crimson. At C we have again a minimum of yellow; but there being at the same time a maximum of red and indigo, this point, though dark in comparison of that on either side, will still be characterised by a fine ruddy purple. This completes, and as we see faithfully represents, the second order of colours. At 9, 10 we see the origin of the vivid green of the third order, in the comparative copiousness of green, yellow, and blue rays at the former point, and of yellow, green,

and violet at the latter, while the red and orange are almost entirely absent, and thus we may pursue all the tints in the scale enumerated in Art. 635 with perfect fidelity.

647. Degradation of the tints.

As the thickness increases, however, it is clear that rays differing but little in refrangibility will differ much in intensity, as the smallest difference in the lengths of the bases of their curves being multiplied by the number of times they are repeated, will at length bring about a complete opposition, so that the maximum of one ray will fall at length on a minimum of another differing little in refrangibility, and not at all in colour. Thus, at considerable thicknesses, such as the 10th or 20th order, there will coexist both maxima and minima of every colour; since each colour, in fact, consists not of rays of one definite refrangibility, but of all gradations of refrangibility between certain limits. In consequence, the tints, as the thickness increases, will grow less and less pure, and will at length merge into undistinguishable whiteness, which, however, for this very reason, will be only half as brilliant as the white of the first order, which contains *all* the rays at their *maximum* of intensity.

648 Colours reflected by plates of different media.

Phenomenon 7. Such are the phenomena when a plate of air is included between two surfaces of glass. It is not however as *air*, but as *distance*, that it acts; for in the vacuum of an air-pump the rings are seen without any sensible alteration. If, however, a much more refracting medium, as water or oil, be interposed, the diameters of the rings are observed to contract, preserving, however, the same colours and the same laws of their breadths; and Newton found by exact measurements, that *the thicknesses of different media interposed, at which a given tint is seen, are in the inverse ratio of their refractive indices.* Thus, the white of the first order being produced in vacuo or air at the 178000th of an inch, will be produced in water at $\frac{1}{1{\cdot}336}$ part of that thickness. He found, moreover, that the law stated in Art. 639 for the dilatation of the rings by oblique incidence, holds equally good, whatever be the nature of the interposed medium. Hence it follows, that in dense media the dilatation at great obliquities is much less than in rare ones, and that in consequence a given thickness will reflect a colour much less variable by change of obliquity when the medium has a high refractive power than when low. Thus, the colours of a soap bubble vary much less by change of incidence than those of a film of air, and much more, on the other hand, than the iridescent colours on polished steel, which arise from a film of oxide formed on the heated surface.

649. Colours reflected by soap bubbles, &c.

Phenomenon 8. Surfaces of glass, or other denser medium enclosing the thin plate of a rarer, are not however necessary to the production of the colours; they are equally, and indeed more brilliantly, visible when any very thin laminæ of a denser medium is enclosed in a rarer, as in air, or in vacuo. Thus, soap bubbles, exceedingly thin films of mica, &c. exhibit the same succession of colours, arranged in fringes according to the variable thickness of the plates. The following very beautiful and satisfactory mode of exhibiting the fringes formed by plates of glass of a tangible thickness has been imagined by Mr. Talbot. If a bubble of glass be blown so thin as to burst, and the glass films which result be viewed in a dark room by the light of a spirit lamp with a salted wick, they will be seen to be completely covered with striæ, alternately bright and black, in undulating curves parallel to each other according to the varying thickness of the film. Where the thickness is tolerably uniform, the striæ are broad; where it varies rapidly, they become so crowded as to elude the unassisted sight, and require a microscope to be discerned. If the film of glass producing these fringes be supposed equal to the thousandth of an inch in thickness, they must correspond to about the 89th order of the rings, and thus serve to demonstrate the high degree of homogeneity of the light; for if the slightest difference of refrangibility existed, its effect multiplied eighty-nine times would become perceptible in a confusion and partial obliteration of the black intervals. In fact, the thickness of a plate at which alternations of light and darkness or of colour can no longer be discerned, is the best criterion of the degree of homogeneity of any proposed light, and is, in fact, a numerical measure of it. This experiment is otherwise instructive, as it shows that the property of light on which the fringes depend is not restricted to extremely minute thicknesses, but subsists while the light traverses what may be comparatively termed considerable intervals.

650. Transmitted colours.

Phenomenon 9. When the glasses between which the reflected rings are formed are held up against the light, a *set of transmitted coloured rings* is seen, much fainter, however, than the reflected ones, but consisting of tints complementary to those of the latter, *i. e.* such as united with them would produce white. Thus the centre is white, which is succeeded by a yellowish tinge, passing into obscurity, or black, which is followed by violet and blue. This completes the series of the first order. Those of the second are white, yellow, red, violet, blue: of the third, green, yellow, red, bluish green, after which succeed faint alternations of red and bluish green, the degradation of tints being much more rapid than in the reflected rings.

651. Newton's explanation of the colours of thin plates.

It was to explain these phenomena that Newton devised his doctrine of the fits of easy reflexion and transmission, mentioned in the 9th postulate of Art. 526. This doctrine we shall now proceed to develope further, and apply, as he has done, to the case in question. In addition then to the general hypothesis there assumed, it will be necessary to assume as follows:

652. Laws of the fits.

The intervals at which the fits recur, differ in different rays according to their refrangibilities, being greatest for the red and least for violet rays, and for these, and the intermediate rays, *in vacuo*, and at a perpendicular incidence, are represented in fractions of an inch by the halves of the numbers in column 2 of the Table, Art. 575.

653.

In other media, the lengths of the intervals in the course of a molecule at which its fits recur are *shorter*, in the ratio of the index of refraction of the medium to unity.

654.

At oblique incidences, or when a ray traverses a medium after being intromitted obliquely, (at an angle $= 0$ with the internal perpendicular,) the lengths of the fits are *greater* than at a perpendicular incidence, in the ratio of radius to the rectangle between the cosine of θ and the cosine of an arc u, given by the equation

$$\sin u = \frac{106\,\mu + 1}{107\,\mu} \sin \theta.$$

Light. Part III.

655. Explanation of the rings seen by homogeneous light.

Let us now consider what will happen to a luminous molecule, the length of whose fits in any medium is $\frac{1}{2}\lambda$, which, having been intromitted perpendicularly at the first surface, and traversing its thickness ($= t$), reaches the second. First, then, if we suppose t an exact multiple of $\frac{1}{2}\lambda$, it is evident that the molecule will arrive at the second surface in precisely the same phase of its fit of transmission as at the first. Of course it is placed in the very same circumstances in every respect, and having been transmitted before must necessarily be so again. Thus every ray which enters perpendicularly into such a lamina must pass through it, and cannot be reflected at its second surface. On the other hand, if the thickness of the lamina be supposed an exact odd multiple of $\frac{1}{4}\lambda$, &c. every molecule intromitted at its first surface will on its arrival at the second be in exactly the contrary phase of its fits, and, having been before in some phase of a fit of transmission, will now be in a similar phase of a fit of reflexion. It will, therefore, not necessarily be transmitted; but a reflexion, more or less copious, will take place at the second surface in this case, according to the nature of the medium and its general action on light. For it will be remembered, that every molecule in a fit of reflexion is not *necessarily* reflected. It is disposed to be so; but whether it will or no, will depend on the medium it moves in and that on which it impinges, and on the phase of its fit. Now conceive an eye placed at a distance from a lamina of unequal thickness, so as to receive rays reflected at a very nearly perpendicular incidence from it. It is evident, that in virtue of the reflexion from the first surface, which is uniform, it will receive equal quantities of light from every point. But with regard to the light reflected from the second the case is different; for in all those parts where the thickness of the lamina is an exact even multiple of $\frac{1}{4}\lambda$, *none* will be reflected, while in all those where it is an exact odd multiple of $\frac{\lambda}{4}$, a reflexion will take place; and since each molecule so reflected *retraces the path by which it arrived*, and therefore describes again the same multiple of $\frac{\lambda}{4}$; its total path described within the lamina, when it has reached the first surface again, will be an exact multiple of $\frac{\lambda}{2}$, and therefore it will penetrate that surface and reach the eye. In consequence, in virtue of the reflexion at the second surface alone, the lamina would appear black in every part where its thickness $= 0$, or $\frac{2\lambda}{4}$, or $\frac{4\lambda}{4}$, &c., and bright in those parts where its thickness $= \frac{1\lambda}{4}$, or $\frac{3\lambda}{4}$, $\frac{5\lambda}{4}$, &c. *ad infinitum*. In the intermediate thicknesses it would have a brightness intermediate between these and absolute obscurity; so that on the whole, the lamina would appear marked all over with dark and bright alternating fringes, just as we see it actually does in the experiment described, (Art. 649.) The uniform reflexion from the first surface superposed on these, will not prevent their inequality of illumination from being distinctly seen.

656. Of the rings seen by white light.

Hence it is evident, that if we take the abscissa of a curve equal to thickness of the lamina at any point, and the ordinate proportional to the intensity of the light reflected from the second surface, and returned through the first, this curve will be an undulating line, such as we have represented in fig. 134, touching the abscissa at equal distances equal to the length of a whole fit of a ray of the colour in question. Now these distances for rays of different colours being supposed such as we have assumed in Art. 652, the construction of Art. 645 holds good, and when white light falls on the lamina, its second surface will reflect a series of colours of the composition there demonstrated, and such as we actually observe, but diluted with the light uniformly reflected from every point of the first surface.

If the lamina instead of a vacuum be composed of any refracting medium, the tints will manifestly succeed each other in a similar series, but the thickness at which they are produced will be to that in a lamina of vacuum, in the ratio of the lengths of the fits in the two cases, that is, in the proportion of 1 : the index of refraction of the medium. Thus the rings seen between two object glasses including air, ought to contract when water, oil, &c. is admitted between them, as they are found to do, and, by measure, in that precise ratio.

657. Of the dilatation of the rings at oblique incidences.

At oblique incidences, θ being the angle of intromission into the lamina, $t \,.\, \sec\theta$ is the whole path of the ray between the first and second surfaces, and since $\frac{1}{2}\lambda \,.\, \sec\theta \,.\, \sec u$ is the length of the fits of the given ray at this obliquity, in order that the luminous molecule may arrive at the second surface in the same phase, and therefore be reflected with equal intensity, it must in this space have passed over the same number of these fits; hence we must have $\frac{2t \,.\, \sec\theta}{\lambda \,.\, \sec\theta \,.\, \sec u} =$ constant, or t proportional to sec u, which agrees with observation.

658. Of the transmitted rings.

All the light which is not reflected at the second surface passes through it, and forms the transmitted series of colours. These, therefore, consist of the whole incident light ($= 1$) *minus* that reflected at the first surface, (which will be a small fraction, and which we will call a,) *minus* that reflected at the second surface. Now this last will be a periodical function whose minimum is 0, and its maximum can never exceed a, because the reflexion at the second surface of a medium cannot be stronger than at the first at a perpendicular incidence. We may then represent it by $a\left(\sin\frac{2t}{\lambda}\right)^2$, and thus we have $1 - a\left\{1 + \left(\sin\frac{2t}{\lambda}\right)^2\right\}$ for the intensity of this particular coloured ray in the transmitted series, and $a\left(\sin\frac{2t}{\lambda}\right)^2$ in the reflected. Hence it is evident, that owing to the smallness of a, the difference between the brightest and darkest part of the transmitted series will be small in comparison with the whole light, and thus the alternations in homogeneous light ought to be (as they are) much less sensible than in the reflected rings, and the tints in white light much more pallid and dilute.

Light. Part III.

Thus we see that the Newtonian hypothesis of the fits affords a satisfactory-enough explanation, or rather represents with exactness all the phenomena above described. It has been even asserted, that this doctrine is really not an hypothesis, but nothing more than a pure statement of facts; for that, first, in point of mere fact, the second surface of the lamina does send light to the eye, in the bright parts of the fringes, and does not send it in the dark parts; and, secondly, that this is the same thing with saying that the light which has traversed a thickness $= (2n + 1)\frac{\lambda}{4}$ is, and that which has traversed $2n\frac{\lambda}{4}$ is not susceptible of being reflected. And, in truth, if only one ray could be regarded as being concerned, and were the light reflected at the first surface of the lamina altogether out of the question, this way of stating it would be strictly correct. But, if it can be shown, that, on any other hypothesis of the nature of light, (as the undulatory,) the second link of this argument is invalid; and that though the second surface, like the first, may reflect in every part, without regard to its thickness, its full average portion of the light that is incident on it; *yet* that afterwards, by reason of the interference of rays reflected from the first surface, such light does *not* reach the eye (being destroyed in every point of its course) from those parts where the thickness is an even multiple of $\frac{\lambda}{4}$, then it is evident, that the Newtonian doctrine is something more than a mere *aliter* statement of facts, and is open to examination as a theory. 659.

Explanation of the transmitted rings on the undulatory hypothesis.

Let us now see, therefore, what account the undulatory theory gives of these phenomena. We will begin, for a reason which will presently appear, with the *transmitted* rings. Conceive, then, a ray, the length of whose undulations in any medium is λ, to be incident perpendicularly on the first surface of a lamina of that medium whose thickness is $= t$; and (for simplicity) let its surfaces be supposed parallel, then it will be divided into two portions, the first ($= a$) reflected, and the second ($= 1 - a$) intromitted. Let θ be the phase of this portion at reaching the second surface. Here it will be again divided into two portions, the one reflected back into the medium and equal to $(1 - a) \cdot a$, or (a being small) very nearly to a, and the remainder $(1 - a) - a(1 - a)$, or nearly $1 - 2a$, transmitted. These portions, supposing no undulation, or part of an undulation, gained or lost in the act of transmission or reflexion, will both be in the phase θ. The reflected portion will again encounter the first surface in the phase $\theta + 2\pi \cdot \frac{t}{\lambda}$, will there be again partially reflected, with an intensity equal to $a \times a = a^2$, and the portion so reflected will reach the second surface in the phase $\theta + 2\pi \cdot \frac{2t}{\lambda}$, and will there be transmitted with an intensity $= (1 - a) \cdot a^2$, or nearly $= a^2$. Now, the reflexions being all perpendicular, this portion will be confounded with the portion $1 - 2a$ transmitted without any reflexion; and putting $a = \sqrt{1 - 2a} = 1 - a$ nearly, and $a' = \sqrt{a^2} = a$, a and a' will represent the amplitudes of vibration of the ethereal molecule at the posterior surface, which each of these rays tend to impress on it. Hence, its total excursion from rest will be represented by 660.

$$a \cdot \cos\theta + a' \cdot \cos\left(\theta + 2\pi \cdot \frac{2t}{\lambda}\right),$$

that is

$$(1 - a)\cos\theta + a \cdot \cos\left(\theta + 2\pi \cdot \frac{2t}{\lambda}\right).$$

$$= 1\cos\theta + a \cdot \cos\left(\theta + 2\pi \cdot \frac{2t}{\lambda}\right) - a \cdot \cos\theta.$$

The first term of this is independent of t, and represents, in fact, the incident ray in the state in which it would arrive at the second surface, had no reflexions taken place. The other two terms represent rays the former of which evidently is in complete discordance with the latter, and destroys it when t is any odd multiple of $\frac{\lambda}{4}$, (or of the half length of one of Newton's fits, a fit being, as we have seen above, equal to half an undulation,) thus leaving the ray at its emergence of the same intensity as it would have had were the lamina away; but when t is any odd multiple of half a fit, then the value of $\cos\left(\theta + 2\pi \cdot \frac{2t}{\lambda}\right) = -\cos\theta$; and the emergent ray is in this case represented by $(1 - 2a) \cdot \cos\theta$, being less than the incident ray by twice the light reflected at the first surface.

Origin of the bright and dark rings in homogeneous light. 661.

Thus if the thickness of the plate be different in different parts, the light transmitted through it to the eye will not be uniform, but will have alternate maxima and minima corresponding to the thicknesses 0 $\frac{\lambda}{4}$, $\frac{2\lambda}{4}$, $\frac{3\lambda}{4}$, &c.

If we apply to the expression above given, the general formula Art. (613) for the composition of rays in one plane, we shall find for the intensity A^2 of the ray finally emergent,

Light. Part III.

662. General expression for the transmitted ray.

$$A^2 = (1-a)^2 + 2a(1-a).\cos 2\pi.\frac{2t}{\lambda} + a^2$$

$$= 1 - 4a(1-a).\sin\left(2\pi\frac{t}{\lambda}\right)^2$$

$$= 1 - 4a.\sin\left(2\pi\frac{t}{\lambda}\right)^2$$

which shows that the several maxima are equal to the incident ray, and the minima to that ray diminished by four times the light reflected at the first surface. The difference of phase between the simple and composite emergent ray, or the value of B in the formula cited, is given by the equation,

$$\sin B = \frac{a}{A}.\sin\left(2\pi.\frac{2t}{\lambda}\right) = a.\sin\left(2\pi.\frac{2t}{\lambda}\right), \text{ neglecting } A^2,$$

so that for such media as have not a very high refractive power, this difference is always small. It is, however, periodical, and differs for different thicknesses.

663. Transmitted tints in white light expressed algebraically.

Suppose now, instead of homogeneous light, white light to fall on the lamina, and let us represent a ray of such light, as in Art. 488, by $C + C' + C'' +$ &c., or by S (C), C, C′, &c. being the intensity of the several elementary rays of all degrees of refrangibility, then will the transmitted compound beam be represented in tint and intensity by

$$C\left\{1 - 4a.\sin\left(2\pi\frac{t}{\lambda}\right)^2\right\} + C'\left\{1 - 4a.\sin\left(2\pi.\frac{t}{\lambda'}\right)^2\right\} + \&c.$$

or by

$$S.C\left\{1 - 4a.\sin\left(2\pi\frac{t}{\lambda}\right)^2\right\}.$$

Now this is the same with

$$S\left\{C(1-4a) + C\left(4a - 4a.\sin\left(2\pi.\frac{2t}{\lambda}\right)^2\right\}\right. =$$

$$= (1-4a).S(C) + 4a.S\left\{C.\cos\left(2\pi.\frac{2t}{\lambda}\right)^2\right\}.$$

The first term of this expression represents a beam of white light of the intensity $1 - 4a$. The second, a compound tint of the intensity $4a$, which, diluted with the above-mentioned white light, forms the pallid tints of the transmitted series. If we disregard this dilution, and consider only the tint in its purity as it would appear were the white light suppressed, its expression

$$4a.S\left\{C.\cos\left(2\pi.\frac{2t}{\lambda}\right)^2\right\} = 4a\left\{S(C) - S\left(C.\sin\left(2\pi.\frac{2t}{\lambda}\right)^2\right)\right\}$$

indicates that it is *complementary* to the tint represented by

$$S\left\{C.\sin\left(2\pi.\frac{2t}{\lambda}\right)^2\right\}.$$

But if we conceive a curve whose abscissa $= t$, and whose ordinate is $C.\sin\left(2\pi.\frac{2t}{\lambda}\right)^2$, it is evident that this will be precisely the undulating curve represented for each prismatic ray in fig. 134; and taking the sum of all the ordinates so drawn for each colour in the spectrum, we have the identical construction from which we derived the colours of the reflected rings in Art. 645. If, then, we take the series of tints so composed, and thence deduce their complements to white light, and dilute these complementary colours with white, in the proportion of $4a$ rays of the complementary colour to $1-4a$ of white, we shall have the series of transmitted tints which ought to result from the doctrine of interferences, and which, in fact, is observed.

664. Case of oblique transmission Fig. 135.

In the case of oblique transmission, let A C, B D, fig. 135, be the surfaces of the lamina, and A a its thickness; and let A E be the surface of a wave of which the point A has just reached the first surface of the lamina; and let S A, S C, perpendicular to it, represent rays emanating from one origin S, then will a partial reflexion take place, and its intensity will be diminished in some certain ratio $1 : 1 - a$ depending on the angle of incidence. The transmitted wave will be bent aside, taking the position A b, and advancing along A B the refracted ray; so that when it reaches the position B F, the wave without the lamina will have the corresponding position F G. Here another partial reflexion will take place depending on the interior incidence, and we may denote by $(1-a)(1-a)$ the transmitted portion, and by $(1-a).a$ the reflected portion. These portions set off together, from B, the former, with the velocity V due to the exterior medium, along the line B H parallel to S A, forming a wave which (provided S be sufficiently distant) may be regarded as a plane of indefinite extent moving uniformly with that velocity along B H. The latter portion proceeds along B C, according to the law of reflexion, with the velocity V′ due to the medium of which the lamina is composed till it reaches C, where it undergoes another partial reflexion, and proceeds back along the line C D with a diminished intensity $= (1-a)$

Light. . a^2, but with the same velocity V′ till it reaches D, having described a space $=$ B C $+$ C D $=$ 2 A B with that velocity. At D it undergoes another partial reflexion, and only a portion $=(1-a)\,(1-a)\,.\,a^2$ is transmitted, which sets off from D along the line D I (parallel to B H) with the velocity V, that is, with the same velocity as the wave along B H. This wave may also be regarded as a plane of indefinite extent perpendicular to D I, and therefore parallel to the former. But they are not coincident; for the former, having the start of the latter, will have come into a position I H K in advance of the position D L M taken by the latter, and both the waves moving forwards now with the same velocity V will preserve this distance for ever unaltered. The interval L H we may term the interval of retardation. To determine it, we have to consider that the space B H is described by the former wave with a velocity V, while the latter describes B C $+$ C D with the velocity V′, and therefore Part III.

$$\text{B H} = (\text{B C} + \text{C D})\,.\,\frac{\text{V}}{\text{V}'} = 2\,\text{A B}\,.\,\frac{\text{V}}{\text{V}'} = 2\,t\,.\,\sec\rho\,.\,\mu,$$

putting μ for the relative index of refraction of the lamina, ρ for the angle of refraction a A B, and t for the thickness A a, because

$$\text{V} : \text{V}' :: \mu : 1.$$

Again, B L $=$ B D . cos D B L $=$ D B . sin ϕ (ϕ being the angle of incidence corresponding to ρ the angle of refraction,) $=2\,a\,\text{B}\,.\,\sin\phi = 2\,t\,.\,\tan\rho\,.\,\sin\phi$. Therefore the whole interval of retardation is equal to

$$2\,t\,\{\mu\,.\,\sec\rho - \tan\rho\,.\,\sin\phi\} = \frac{2\,t\,.\,\mu}{\cos\rho}\,(1-\sin\rho^2) = 2\,\mu\,t\,.\,\cos\rho$$

because $\sin\phi = \mu\,.\,\sin\rho$.

Thus, in virtue of the two internal reflexions, each wave which before entering the medium was single, will after quitting it be double, being followed at the constant interval $2\,\mu\,t\,.\,\cos\rho$ by a feebler wave of the intensity above determined. The same being true of every wave of the system of which the ray consists, these two systems (considered as of indefinite duration) will be superposed on, and interfere with each other, according to the general principles before laid down. 665.

Let λ be the length of an undulation in the lamina, then will $\mu\,\lambda$ represent that of an undulation in the surrounding medium. This is obvious, because the velocity in the latter being to that in the former as $\mu : 1$; and the same number of undulations being propagated in the same time through a given point in both cases, they must be more crowded, and therefore occupy less space in the one than the other in the ratio of the velocities. 666. Undulations shorter in denser media.

Hence the differences of phases between the interfering systems at any point will equal 667. General expression for the transmitted ray

$$2\,\pi\,.\,\frac{\text{interval of retardation}}{\mu\,\lambda} = 2\,\pi\,.\,\frac{2\,t\,.\,\cos\rho}{\lambda} = 2\,\pi\,.\,\frac{2\,t'}{\lambda},\ \text{putting } t' = t\,.\,\cos\rho,$$

and therefore the final resulting wave will be expressed by the equation

$$\text{X} = \sqrt{(1-a)\,(1-a)}\left\{\cos\,\theta + a\,.\,\cos\left(\theta + 2\,\pi\,.\,\frac{2\,t'}{\lambda}\right)\right\},$$

which being resolved into the fundamental form A . cos (θ + B), as before, gives

$$\text{A}^2 = (1-a)\,(1-a)\,.\left\{1 + 2\,a\,.\,\cos\left(2\,\pi\,.\,\frac{2\,t'}{\lambda}\right) + a^2\right\},$$

and

$$\sin\text{B} = \frac{a\,.\,\sin\left(2\,\pi\,.\,\frac{2\,t'}{\lambda}\right)}{\sqrt{1 + 2\,a\,.\,\cos\left(2\,\pi\,.\,\frac{2\,t'}{\lambda}\right) + a^2}}.$$

Such are the general expressions for the intensity and change of origin of the compound transmitted ray. It is evident, however, that when a and a are small, which they always necessarily are in any but extreme cases, this value of A^2 reduces itself by neglecting their powers and products to 668. Case of moderate obliquities.

$$(1-a+a) - 4\,a\,.\,\sin\left(2\,\pi\,.\,\frac{t'}{\lambda}\right)^2,$$

which is exactly analogous to the expression in Art. 662, for the case of perpendicular incidence; and shows, that with the exception of a very trifling difference in the degree of dilution, the same laws of alternation in brightness, in homogeneous light, and of tint in white light, must hold good in both cases.

But there is one essential difference. The same tints will arise in the case of oblique incidence at the thickness t, which in that of perpendicular incidence is produced at the thickness $t\,.\,\cos\rho$, because $t' = t\,.\,\cos\rho$. Now this is always less than t, and therefore the tint produced at oblique incidences at the given thickness will be higher in the scale (or correspond to a less thickness) than in perpendicular; and, consequently, the rings, or fringes, so seen by transmission should dilate by inclining the lamina to the eye. The law of dilatation evidently, at moderate incidences, coincides nearly with Newton's rule; for this gives, on reduction, neglecting sin ρ^4, 669. Dilatation of the rings explained.

$$\sec u = \sec \rho \left\{ 1 - \frac{1}{2} \cdot \frac{106}{107} (\mu - 1) \cdot \tan \rho^2 \right\},$$

which does not deviate very greatly from sec ρ at moderate incidences.

670. Deviation from Newton's rule at great obliquities probably accounted for.

At great incidences the case is different, and the noncoincidence of the results of the undulatory doctrine with experiment might be drawn into an argument against it, were we sure that the law of refraction at extreme incidences, and with very thin laminæ, does not vary sensibly from that of the proportional sines. This is, indeed, highly probable, as M. Fresnel has remarked, (*Mém. sur la Diffraction, &c.*) and as we have before had occasion to observe. The inquiry into which this would lead, is, however, one of the most delicate and difficult in physical optics, and the reader must be content with this general *notice* of a *possible* explanation of one of the many difficulties which still beset the undulatory doctrine.

671. Origin of the reflected rings.

The origin of the reflected rings may be accounted for in a similar way from the partial transmission of the waves reflected from the second surface back through the first, and their interference with the waves reflected immediately from the first. The relative intensities of these waves, (in general,) are a and $(1 - a)\ (1 - a) \cdot a$; or, in the case where a and a are both small, nearly in the ratio of $a : a$, and at a perpendicular incidence, very nearly in the ratio of equality. Hence their mutual destruction in the case of complete discordance will be much more complete than in the transmitted rings, and the colours arising, much less dilute than those of the latter, agreeably to observation.

672. Loss of half an undulation.

There is, however, one consideration of importance to be attended to in the application of the undulatory doctrine to the reflected rings, which at first sight appears in the light of a powerful argument against its admissibility, *viz.* that if we apply the same reasoning to the reflected, as we have already done to the transmitted, rings, we should arrive at the conclusion, that their tints should be *precisely the same* and in the same order, beginning with a bright white in the centre; because *here*, the path traversed by the ray within the lamina vanishing, the waves reflected from the two surfaces ought to be in exact accordance, whereas it appears, by observation, that the reverse is the case, the central spot being black instead of white. It becomes necessary, then, to suppose, that in this case, half an undulation is lost or gained either by the wave reflected from the first or second surface. If this hypothesis be made, the phenomena of the reflected rings are completely represented on the undulatory system, for the compound wave reflected by the joint action of the two surfaces should be represented by the equation,

$$\mathrm{X} = \sqrt{a} \cdot \cos \theta + \sqrt{a\ (1 - a)\ (1 - a)} \cdot \cos \left\{ \theta + 2\pi \cdot \frac{2t' - \frac{1}{2}\lambda}{\lambda} \right\}$$

and if this be put equal to $\mathrm{A} \cdot \cos (\theta + \mathrm{B})$ we get

$$\mathrm{A}^2 = a + a\ (1 - a)\ (1 - a) - 2\sqrt{a\,a\ (1 - a)\ (1 - a)} \cdot \cos \left(2\pi \frac{2t'}{\lambda} \right)$$

and in the case of a and a both very small

$$\mathrm{A}^2 = (\sqrt{a} - \sqrt{a})^2 + 4 \cdot \sqrt{a\,a} \cdot \sin \left(2\pi \frac{t'}{\lambda} \right)^2$$

and at a perpendicular incidence, where $t' = t$, and where a and a may be supposed equal

$$\mathrm{A}^2 = 4a \cdot \sin \left(2\pi \frac{t}{\lambda} \right)^2$$

673. Not contrary to dynamical principles

Thus we see, that in this case the total intensity of the compound reflected wave + that of the transmitted (Art. 662) make up 1, the intensity of the incident wave; and thus, this supposition of the loss or gain of half an undulation is in no contradiction with the law of the conservation of the *vis viva*.

674. Nor to the undulatory doctrine.

In fact, however, if we consider the mode in which the undulations are propagated, at the limit between two media, we shall see nothing contrary to dynamical principles in the loss of half or any part of an undulation in the transfer—for it cannot be supposed, that the density or elasticity of the ether changes abruptly at the surfaces of media, but that there intervenes some very minute stratum in which it is variable. In this stratum, therefore, the length of an undulation is neither exactly that corresponding to the denser, nor to the rarer medium, but intermediate, and of a magnitude perpetually varying. Therefore the number of undulations *to be reckoned as added to the phase* of the ray *in traversing this stratum*, will differ from what it would be if one medium terminated, and the other commenced abruptly. Without knowing the law of density, the limits between which it undergoes its change, or the exact mode in which the partial reflexion of a wave traversing it is performed, it is impossible to subject the point to strict calculation, we must rather submit to be taught by experiment, and content ourselves with such conclusions as we can deduce from observation. In the case before us, all that observation teaches us is, that there is half an undulation more of difference in the phases of two rays that have been reflected in the manner last considered, than in those of the two whose interference forms the transmitted rays. From some curious experiments of Dr. Young, too, we may gather that it is not in all cases *strictly half* an undulation of difference to be reckoned, but rather a variable fraction depending on the nature of the contiguous media.

675. The formulæ of Art. 672 show that it is only in the case of perpendicular incidence that the tints are pure, and that in all others, and especially at great obliquities, where a and a differ considerably, there will be a

Light. dilution of white light, and this is also agreeable to experience. At a perpendicular incidence, however, the minima of each homogeneous colour ought to be absolutely evanescent; so that if we were to remove the reflection of the upper surface of an object glass laid down on a plate, (or use a prism, so as to prevent its reaching the eye,) the intervals between the rings in homogeneous light ought to appear absolutely black. In the Newtonian doctrine this should not be the case, because the light reflected from the upper surface of the lamina of included air should still remain even in the minima of the rings. This then affords a positive means of deciding between the two theories. M. Fresnel describes an experiment made for this purpose, and states the result to be unequivocally in favour of that of undulations. (*Diffraction de la Lumière*, p. 11.)

Part III. *Experimentum crucis* between the two theories

§ V. *Of the Colours of Thick Plates.*

676 Under certain circumstances rings of colours are formed by plates of transparent media of considerable thickness. The circumstances under which they appear, in one principal case, are thus described by Newton, who first observed them, and who has applied his doctrine of the fits of easy reflexion and transmission to explain them, with singular ingenuity.

Newton's experiment with a glass mirror. "Admitting a bright sunbeam through a small hole of one-third of an inch in diameter into a dark room, it was received perpendicularly on a concavo-convex glass mirror one quarter of an inch thick, having each surface ground to a sphere of six feet in radius, and the back silvered. Then holding a piece of white paper in the centre of its concavity, having a small hole in the middle of it to let the sunbeam pass, and after reflexion at the speculum to repass through it, the hole was observed to be surrounded with four or five coloured concentric rings or irises, just as the rings seen between object-glasses surround their central spot—but larger and more diluted in their colours".... "If the paper was much more distant from the mirror, or much less than six feet, the rings became more dilute and gradually vanished."... "The colours of these rings succeeded each other in the order of those which are seen between two object glasses, not by reflected but by transmitted light, *viz.* white, tawny white, black, violet, blue, greenish yellow, yellow, red, purple," &c.... "The diameters of these rings preserved the same proportion as those between the object-glasses, the squares of the diameters of the alternate bright and dark rings, reckoning the central white as a ring of the diameter 0, forming an arithmetical progression, beginning at 0. And in the case described, the diameter of the bright ring measured respectively 0, $1\frac{1}{26}$, $2\frac{3}{8}$, $2\frac{11}{12}$, $3\frac{3}{8}$."... "Lastly, in the rings so formed by reflectors of different thicknesses, their diameters were observed to be reciprocally as the square roots of the thicknesses. If the back of the mirror was silvered, the rings were only so much the more vivid."

677. These various phenomena, and a variety of similar ones, some of more, some of less complexity, according to the variation of the distance, and obliquity of the mirror, and the curvature of its surfaces, Newton has explained very happily, by considering the fits of easy reflexion and transmission of that faint portion of the light which is regularly *scattered* in all directions at the first surface of the glass, and which serves to render it visible. But for this explanation we must refer to his *Optics*, as our object here is more particularly and distinctly to show what account the undulatory doctrine gives of this phenomenon, which has hitherto been passed over rather cursorily, not without some degree of obscurity.

678. Principle of explanation on the undulatory system. There is no surface, however perfectly polished, so free from small scratches and inequalities as not to reflect and transmit, besides those principal rays which obey the regular laws of reflexion and refraction, as dependent on the general surface, other, very much feebler, portions scattered in all directions, by which the surface is rendered visible to an eye anywhere placed, but most copiously in and about the direction of the regularly reflected and transmitted rays. It is the interference of these portions, scattered at the first surface by the ray in passing and repassing through it, nearly in its own direction, that the rings in question are attributed in the undulatory doctrine.

679. Its application. Fig. 136. Let F A D, E B G be the parallel surfaces of any medium exposed perpendicularly to a homogeneous ray emanating from a luminous point C, and incident at A. The chief portion will pass straight through A, and be reflected back from B to A again. But at A a scattering takes place, and the transmitted ray A B is accompanied by a diverging cone of faint rays A *a*, A *b*, A *c*, &c., all which set out from A in the same phase of their undulations with the principal one from which they originate, so that A may be regarded as their common origin. Take Q, the focus of rays reflected at the second surface conjugate to A (if the surfaces be plane, Q and A are equidistant from B) and the cone of scattered rays, with the regularly reflected ray in its axis, will after reflexion diverge as from Q. Again, when they pass into the air again, if we take *q* the focus conjugate to Q of rays refracted at the surface F D, they will after refraction diverge from *q*, and by the nature of foci on the undulatory hypothesis, the undulations will be propagated in the air as if they had a common origin *q placed in air*; because, after refraction, the waves have the form of spheres diverging from *q*, and therefore every portion of their surfaces are equidistant from *q*; had they, therefore, really emanated from *q*, as separate rays, they must at the moment of such emanation have been all in one phase. Now, when the reflected beam reaches A a portion of it will again be scattered in a cone, having the regularly transmitted ray A G in its axis; and the rays A O, A N, A M, &c. of this cone will all have A for their origin, and will be in the same phase at their departure from A with the ray A G; but this is in the phase it would have had as emanated from *q*; hence, if we consider any point M out of the directly transmitted ray A G, it will be reached at once by a wave belonging to each diverging cone, the one along *q* M from *q* and the other along A M from A, and the difference of routes is equal to $q\,A + A\,M - q\,M$. Therefore, when M is very nearly coincident with G, this is very small and at G vanishes, or the waves are in exact accordance. As M recedes from G it increases, and when it becomes

Light. half an undulation, the waves are in complete discordance and annihilate each other, and so on alternately. Therefore, as this is true of all rays in conical surfaces round A G as an axis, equally inclined with A M, q M, if we place a white screen at G, it will appear marked with alternate dark and bright rings round a bright centre. To determine their diameters we need only put $q\,\mathrm{A} + \mathrm{A\,M} - q\,\mathrm{M} = n \cdot \frac{\lambda}{2}$, or, if we take $q\,\mathrm{A} = a$, $\mathrm{A\,G} = r$, $\mathrm{G\,M} = y$, Part III.

$$a + \sqrt{r^2 + y^2} - \sqrt{(a+r)^2 + y^2} = n\,\frac{\lambda}{2}$$

If we resolve this equation neglecting y^2, we find

$$y = \sqrt{n} \cdot \sqrt{\frac{\lambda}{a} \cdot r\,(a+r)}$$

which, on substituting 0, 1, 2, 3, &c. in succession for n, shows that the successive diameters of the alternate dark and bright rings are in the progression of the square roots of those numbers.

680. Law of the diameters of the rings. If the thickness of the plate be small compared to the distance of the screen, a will also be small, and the value of y becomes

$$y = r \cdot \sqrt{n} \cdot \sqrt{\frac{\lambda}{a}},$$

which shows that for rays of a given refrangibility the diameters of the rings are as the distance of the screen directly, and the square root of the thickness of the plate inversely.

681. Of their colours. Lastly, the diameter of a ring of the same order in different homogeneous lights, are as the square roots of the lengths of their undulations. Now, this is the very same law that governs the diameters of the rings formed between object-glasses. Consequently, if instead of homogeneous we consider white light, we ought to have a succession of coloured rings whose tints agree precisely with the transmitted series in that experiment.

682. Concentration of the rings from all points of the surface. Fig. 137. But the rays so formed, by rays scattered from a single point A, would be too feeble to be visible. If, however, we suppose the surfaces to be concentric spheres having G in their common centre, as in fig. 137, then any rays G A, G A′ falling on any points whatever of their surfaces will depict, on screens G M, G M′ respectively perpendicular to them as G, equal systems of rings having G in their common centre; and, when the arc A A′ is not very great, the screens may be regarded as coincident (for in that case $\mathrm{B\,M} - \mathrm{M\,A} = \mathrm{B\,M'} - \mathrm{M\,A'}$) and the rings from every point of the surface, exactly superposed on each other, and being thus increased in intensity in proportion to the area of the exposed surface, become visible.

683. Newton's experiment particularly considered. Now this is exactly Newton's case, for the sun being a luminary of a considerable diameter, the hole in the centre of the spheres may be regarded as a portion of the sun of that size, actually placed there. Of this, every indivisible point may be regarded as the origin of a system of waves, and as depicting on the screen its own set of rings. These, were the hole infinitely small, would be infinitely more clear and pure in their tints than the transmitted rings between object-glasses, because they are not (as in those rings) diluted with the great quantity of white light which escapes interference. But owing to the size of the hole, their centres are not exactly coincident, and therefore their tints mix and dilute each other, and that the more the larger the hole is.

684. If c be the thickness of the glass, since Q is the conjugate focus of A, on the surface B whose radius we will call $r + c$ putting $\mathrm{G\,A} = r$, we have, by Art. 249, $\mathrm{B\,Q} = \frac{r+c}{r-c} \cdot c$, $\mathrm{A\,Q} = \frac{2rc}{r-c}$; and, by Art. 248, $\mathrm{A}\,q = a = \frac{2cr}{2c - \mu(r+c)}$, taking μ for the refractive index; and when c is small compared with r, we get

$$a = \frac{2c}{\mu};\quad y = r \cdot \sqrt{n} \cdot \sqrt{\frac{\mu}{2} \cdot \frac{\lambda}{c}}$$

showing that the diameters of the rings are in this case in the subduplicate ratio of the refractive index of the glass directly, and of its thickness inversely.

685 If we reduce this value to numbers, taking $\mu = \frac{3}{2}$, $n = 4$, $r = 6$ feet $= 72$ inches, and $\lambda = \frac{2}{89000} =$ the length of an undulation for yellow rays $\frac{2}{90000}$ nearly, we find, for the diameter of the second bright ring in yellow light, (which corresponds to the brightest part of the same ring in white,)

$$2y = 72 \times \sqrt{4 \cdot \frac{3}{4} \cdot \frac{2}{90000} \cdot 4} = 2{\cdot}35,$$

which agrees almost precisely with Newton's measure $2\frac{3}{8}$, or 2·375.

686. Case of oblique incidence. When the mirror is inclined to the incident beam the phenomena become more complicated, and have been elegantly described by Newton, (*Optics*, book ii. part iv. obs. 10.) In this case, the axes of the two interfering cones of scattered rays, which are always the incident and reflected rays, are no longer coincident. But the same principles apply equally to this case in all other respects, and the reader may exercise himself in tracing their consequences.

687. Phenomena observed by the Duke of Chaulnes and The Duke de Chaulnes found similar rings to be exhibited when the surface of the mirror was covered with a thin film of milk dried on it, so as to make a delicate semitransparent coating, or even when a fine gauze or muslin was stretched before it; see the account of his experiments in the *Mém. Acad. Sci. Paris*, 1705; and

Light. Part III. Sir W. Herschel.

Sir William Herschel (*Phil. Trans.* 1807) describes a pleasing experiment, in which rings were produced by strewing hair powder in the air before a metallic mirror on which a beam of light is incident, and intercepting the reflected ray by a screen. The explanation of these phenomena seems, however, to depend on other applications of the general principle, and will be better conceived when we come to speak of the colours produced by diffraction.

688. Dr. Brewster's fringes seen in thick plates.

Dr. Brewster, in the *Transactions of the Royal Society of Edinburgh*, has described a series of coloured fringes produced by thick plates of parallel glass, which afford an excellent illustration of the laws of periodicity observed by the rays of light in their progress, whether, as in the Newtonian doctrine, we consider them as subjected to alternate fits of easy reflexion and transmission, or, as in the undulatory hypothesis, as passing through a series of phases of alternately direct and retrograde motions in the particles of ether, in whose vibrations they consist. We may here remark, once for all, that the explanations which the undulatory doctrine affords of phenomena of this description, may, for the most part, be translated into the language of the rival hypothesis; so as to afford, with more or less plausibility and occasional modifications, a result corresponding with observation. It is not, therefore, among phenomena of this class that we must look for the means of deciding between them. We shall adopt, therefore, in the remainder of this essay, the undulatory system, not as being at all satisfied of its reality as a *physical fact*, but regarding it as by far the simplest means yet devised of grouping together, and representing not only all the phenomena explicable by Newton's doctrine, but a vast variety of other classes of facts to which that doctrine can hardly be applied without great violence, and much additional hypothesis of a very gratuitous kind.

689. Described.

The fringes in question are seen when two parallel plates of glass of exactly equal thickness (portions of the same plate) are slightly inclined to each other, (at any distance,) and through them both, at nearly a perpendicular incidence, a circular luminary of 1° or 2° in diameter (a portion of the sky, for instance) is viewed. There will in this case be seen, besides the direct image, a series of lateral images reflected between the glasses, and growing fainter and fainter in succession as they are formed by 2, 4, 6, or more internal reflexions; and of which all but the first is so faint as scarcely to be visible, except in very strong lights. The direct image is colourless; but the reflected one is observed to be crossed with fifteen or sixteen beautiful bands of colour, parallel to the common section of the surfaces of the plates. Their breadth diminishes rapidly as the inclination of the plates increases. When the plates employed were 0.121 inch in thickness, and inclined at an angle of 1° 11′ to each other, the breadth of each fringe measured 26′ 50″, and at all other inclinations their breadth was inversely as the inclination. At oblique incidences its fringes are seen when the plane of incidence is at right angles to the principal section of the plates, but are at their maximum of distinctness when parallel to it.

690. Explained.

To understand their production, let us call the surfaces of the plates in order, reckoning from that on which the incident light first falls, A, *a*, B, *b*; and let us consider a ray, or system of waves emanating from a common origin at an infinite distance. Then, when this ray falls on the plates it will at every surface undergo a partial reflexion, and the remainder will be transmitted; each of the several portions will be again subdivided whenever it meets either surface. So that either image will, in fact, consist of several emergent rays, parallel in their final directions, but which have traversed the glasses by very different routes. Thus the direct or principal image will consist of

1. The chief portion of the whole incident light, refracted at A, at *a*, at B, and at *b*, and emergent parallel to the incident ray, which we will represent by A *a* B *b*.

2. A portion refracted at A, reflected at *a*, reflected again at A, refracted again at *a*, at B and at *b*, and emergent parallel to the incident beam. This we will denote thus, A *a*′ A′ *a* B *b*; the letters denoting the surfaces, the accent reflexion, and its absence refraction.

3. A portion which has undergone two similar reflexions in the interior of the second plate, and which in the same manner may be represented by A *a* B *b*′ B′ *b*.

4. Other portions which have undergone respectively four, six, &c. reflexions to infinity within either of the plates, and which may be represented by such combinations as A *a*′ A′ *a*′ A′ *a* B *b*, A *a* B *b*′ B′ *b*′ B′ *b*, or, for brevity, by A $(a'\,A')^2\,a$ B *b*, A *a* B $(b'\,B')^2\,b$, &c.; but these latter portions are too faint to have any sensible influence on the light of the direct image with which they are confounded.

691.

The first lateral reflected image will consist of four principal portions which have undergone two reflexions each, *viz.*

A *a* B′ *a*′ B *b*; A *a* B′ *a* A′ *a* B *b*; A *a* B *b*′ B *a*′ B *b*; A *a* B *b*′ *a* A′ *a* B *b*;

all which will emerge parallel. Besides these there are infinite others, formed by a greater number of reflexions, and by the portions A *a*′ A′ *a* of the incident beam reflected within the first glass; but these are all too faint materially to affect the image in question, which therefore we may regard as composed solely of the four rays just enumerated. Now if we cast our eye on the figure, (138,) we see the course pursued by each of these portions 1, 2, 3, 4; and it is evident that the first portion has traversed the thickness twice, and the interval between the glasses three times, or nearly; neglecting at present all consideration of the inclination of the plates $2\,t + 3\,i$. In like manner, the portion 2 will have traversed $4\,t + 3\,i$; the portion 3, $4\,t + 3\,i$; and the portion 4, $6\,t + 3\,i$. Hence it appears that the portions 1 and 4 differ in their routes by nearly four times the thickness of the glass, and can therefore produce no colours; but the other portions, at a perpendicular incidence, would not differ at all, and at very small inclinations of the plates, and of the incident ray, will only differ by reason of the small differences of the inclinations at which they traverse their respective thicknesses and intervals. They will, therefore, interfere so as to produce colour; and this will be dependent on the interval of retardation of one ray behind the other, arising from the varying obliquity of the ray which enters the eye. Fig. 138.

692.

Now when we look at a luminous image of sensible magnitude, the rays by which we see its several points

Light. Part III.

Isochromatic lines defined.

are incident in all planes, and at all inclinations. Hence, the image seen will appear of different colours in its different points, and the disposition of these colours will follow the law, whatever it be, which regulates the interval of retardation. The colours, therefore, will be arranged in bands, circles, or other forms, according to the form of the curves arising geometrically from the consideration of equal intervals of retardation prevailing in every point of their course. Such curves, now and hereafter, we shall term *isochromatic lines*, or lines of equal tint, measuring in all cases the *tint* numerically by the number of undulations, or parts of an undulation of mean yellow light to which the interval of retardation is equal.

693. Fig. 139. Let us, then, first consider the case when the ray is incident in a plane perpendicular to the common section. In this case, fig. 139, let K L M N be a ray formed by the union of two rays S A *a* B *b* I K L and S C E F G H K L, whose courses through the system are similar to 2 and 3, fig. 138. Draw A D perpendicular to S C, then will the interval of retardation be equal to

$$\{DC + CE + EF + FG + GH + HK\} - \{A\,a + a\,B + B\,b + b\,I + IK\}$$
$$= DC + (EF - a\,B) + (FG - IK) + 2\,(KH - B\,b),$$

the first three terms being performed in air, the last in glass. Now, without entering into a trigonometrical calculation, it is evident that this will be very small at a perpendicular incidence, and will increase rapidly as the angle of incidence varies; and that (the inclination of the plates remaining constant) it will increase by nearly equal increments, as the incidence varies by equal changes from 0 on either side of the perpendicular. Therefore, in a direction at right angles to the common section of the surfaces the tints will vary rapidly, increasing on either side of the perpendicular incidence; and at very moderate obliquities on either side, the interval of retardation will become too great for the production of colour. On the other hand, if we conceive the rays S A, S C, to be incident in a plane very nearly parallel to the principal section, then will the points K and G be situated, not, as in the figure, at different distances from P, but at very nearly the same; so that (whatever be the incidence) K I will very nearly equal G F, and for the same reason F E will very nearly equal *a* B. Moreover, in this case G K will be very nearly equal to F I, and the angles of internal incidence will be also very nearly equal, so that H G + G K will differ very little from B *b* + *b* I, and I B will be very nearly equal to G K, and therefore to I F, so that the point F will almost exactly coincide with B, and the rays S A *a* B, S C E F will coincide almost precisely, making D C = 0; and these approximate equalities and coincidences will continue for great variations in the *angle* of incidence, provided the *plane* of incidence be unaltered. The interval of retardation, then, will in this case depend very little on the angle of incidence; so that in a direction parallel to the common section of the surfaces, the tints will vary but little. Hence it appears that they will be arranged in the manner described by Dr. Brewster, *viz.* in fringes parallel to that line. Their general analytical expression is, however, rather too complex to be here set down, though very easily investigated from what has been said.

694. Fig. 140. By intercepting the principal transmitted beam in the direct image, and receiving on the eye only those portions of the rays going to form it whose curves are as in fig. 140, or the portions A *a'* A' *a* B *b*, and A *a* B *b'* B' *b*, Dr. Brewster succeeded in rendering visible a set of coloured fringes, which in general are diluted and concealed in the overpowering light of the direct beam. They originate evidently in the interference of these two rays, whose courses are each represented by $4t + i$, and would therefore be strictly equal were the plates exactly parallel. Their theory, after what has been said, will be obvious on inspection of the figure, as well as those of all the rest of the systems of fringes which he has described in that highly curious and interesting memoir.

695 Fringes between glass films. Mr. Talbot has observed, when viewing films of blown glass in homogeneous yellow light, and even in common daylight, that when two films are superposed on each other, bright and dark stripes, or coloured bands and fringes of irregular forms, are produced between them, though presented by neither separately. These are obviously referable to the same principle, the interference taking place here between rays respectively twice reflected within the upper lamina, and once reflected at the upper surface of the lower lamina, or else between rays one of which is thrice reflected in the mode represented by A *a* B' *a'* B' *a* A, and the other in that represented by A *a* B' *a* A' *a'* A, the interval between the glasses being supposed to be exactly equal to the thickness of the upper one in both cases, a condition which is sure to obtain somewhere when the laminæ are curved. A still more curious and delicate case of the production of similar fringes has been noticed by Professor Amici, to take place when two of the blue feathers of the wing of the *Papilio Idas* (a species of butterfly) are laid on each other in the field of his powerful and exquisite microscopes. These feathers he describes as small plates of perfect transparency, and uniformly and delicately striated over their whole surface. The fringes in question are formed between them, and vary in breadth, form, and situation, according to the manner in which the feathers are superposed. Their origin seems to be independent of the striæ however, and is easily understood on the principles above explained. The same may be said of the colours observed by Mr. Nicholson in combinations of parallel glasses of unequal thickness. Suppose, for instance, that instead of the plates having exactly equal thicknesses, their thicknesses t, t' differ by a very minute quantity, then the course of the rays A *a'* A' *a* B *b* and A *a* B *b'* B' *b* will (at a perpendicular incidence) be respectively $3t + i + t'$ and $t + i + 3t'$, (supposing the plates strictly parallel,) and the difference of their routes is $2t - 2t'$; so that if this be exceedingly minute, colours will arise, or, if not, may be produced by a slight inclination of the plates to each other, and so of an infinite variety of cases which may arise.

§ VI. *Of the Colours of Mixed Plates.*

696. Interference of rays not strictly coincident.

The colours hitherto described have been referred to the interference of rays rigorously coincident with each other throughout their whole course, after the point where they begin to be superimposed. Such interfering rays, or systems of waves, being united into a point on the retina, that point is agitated by the sum or difference of their actions, and the sensation produced is according. But if this coincidence be only approximate, as, if two systems of waves be propagated from origins so nearly coincident in angular situation from the eye, that their images formed on the retina shall be too close to be distinguished by the mind from the image of a single point, the impressions produced will still be confounded together; or rather, we ought to say, the mechanical action on one point will be propagated through the substance of the retina to the other, and a sensation corresponding to their mean or average effect will be produced. If, then, the rays concentered on contiguous points of the retina be in exact discordance, and of equal intensity, a mutual destruction will take place, as if they fell on one mathematical point; if in exact accordance, they will increase each others effects, and so for the intermediate states.

697. Irradiation.

To apprehend this more fully, we must consider that the impression of light appears to spread on the retina to a certain extremely minute distance all around the mathematical focus of the rays concentered by the lenses of the eye. Thus the image of a star is never seen as a *point*, but as a disc of sensible size, and that the larger as the light is stronger. Thus, too, the bright part of the new moon is seen, as it were, larger than the faintly illuminated portion of its disc projecting beyond it as an acorn cup beyond the fruit, &c. This effect is termed *irradiation*, and is manifestly the consequence of an organic action such as we have described.

698.

It follows from this, that when waves emanate from origins *undistinguishably near*, they may be regarded in their effects on the eye as emanating from origins strictly in one and the same right lines, the direction of the joint ray; and the laws of their interferences will be precisely the same, considered in their effect on vision, as if the lenses of the eye were away, and the retina were a mere screen of white paper, on a single physical point of which (*viz.* the point where the images concentered by the lenses *would* have fallen) the interfering undulations propagated simultaneously from the two origins fell, and agitated it with a vibration equal to their resultant.

699. Phenomena of mixed plates.

This premised, we are in a condition to appreciate the explanation afforded by the undulatory doctrine of the phenomena of mixed plates. They were first noticed (says Dr. Young) by him "in looking at a candle through two pieces of plate glass with a little moisture between them. He thus observed an appearance of fringes resembling the common colours of thin plates; and upon looking for the fringes by reflexion, found that the new fringes were always in the same direction as the others, but many times larger. By examining the glasses with a magnifier, he perceived, that wherever the fringes were visible, the moisture was intermixed with portions of air producing an appearance similar to dew." "It was easy to find two portions of light sufficient for the production of these fringes; for the light transmitted through the water moving in it with a velocity different from that of light passing through the interstices filled only with air, the two portions would interfere with each other and produce effects of colour according to the general law. The ratio of the velocities in water and air is that of three to four; the fringes ought therefore to appear where the thickness is six times as great as that which corresponds to the same colour in the common case of thin plates; and upon making the experiment with a plane glass and a lens slightly convex, he found the sixth dark circle actually of the same diameter as the first in the new fringes. The colours are also easily produced when butter or tallow is substituted for water, and the rings then become smaller in consequence of the greater refractive density of the oils; but when water is added so as to fill up the interstices of the oil, the rings are very much enlarged; for here the difference of velocities in water and in oil is to be considered, and this is much smaller than the difference between air and water. All these circumstances are sufficient to satisfy us of the truth of the explanation, and is still more confirmed by the effect of inclining the plates to the direction of the light; for then, instead of dilating like the colours of thin plates, these rings contract, and this is the obvious consequence of an increase of the lengths of the paths of the light which now traverses both media obliquely, and the effect is everywhere the same as that of a thicker plate. It must, however, be observed, that the colours are not produced in the whole light that is transmitted through the media; a small portion only of each pencil passing through the water contiguous to the edges of the particle is sufficiently coincident with the light transmitted through the neighbouring portions of air to produce the necessary interference; and it is easy to show that a considerable portion of the light that is beginning to pass through the water will be dissipated laterally by reflexion at its entrance, on account of the natural concavity of the surface of each portion of the fluid adhering to the two surfaces of the glass, and that much of the light passing through the air will be scattered by refraction at the second surface. For these reasons the fringes are seen when the plates are not directly interposed between the eye and the luminous object." (Young, *Phil. Trans.* 1802; *Account of some Cases of the Production of Colours.*) To see the phenomena to advantage, we may add, it is only necessary to rub up a little froth of soap and water almost dry between two plane glasses, and hold them at a distance from the eye between it and a candle, or the reflexion of the sun on any polished convex object. If two slightly convex glasses, or a plane and a convex one be used, the colours are seen arranged in rings.

§ VII. *Of the Colours of Fine Fibres and Striated Surfaces.*

700. Interference of rays reflected from points or lines very near each other. Fig. 141.

If two points supposed capable of reflecting light in all directions (as two infinitely small spheres, &c.) be so near each other as to appear to the eye as one, and if rays from a common origin reflected from them reach the eye, they will interfere; and if the light be homogeneous, its intensity will vary periodically, with an interval of retardation corresponding to the difference of their paths; if white, the colour of the mixed reflected ray will be the same as if it had been transmitted through a plate of air of a thickness equal to that difference, but deprived of its diluting white. Suppose two exceedingly fine cylindrical polished fibres to be placed at right angles to the line of sight, and parallel to each other, as in fig. 141, as A B C, *a b c*; and let S be a luminous point very distant with respect to the interval of the fibres, and E the eye, placed so as to receive the reflected rays B E, *b* E, which, by supposition, are near enough to interfere. Then the differences of phases of the rays on the retina is evidently equal to $2\pi \times \frac{(S\,b + b\,E) - (S\,B + B\,E)}{\lambda} = 2\pi \cdot \frac{b\,x + b\,y}{\lambda}$, supposing B *x* and B *y* perpendicular to S *b* and *b* E. If, then, we suppose I and *i* to be the angles of incidence of the rays S B, E B on the plane in which the axes of the two cylinders A C, *a c* lie, and put B *b* their distance equal to *a*, we have for the difference of phases

$$2\pi \cdot \frac{a}{\lambda} \cdot (\sin \mathrm{I} + \sin i).$$

Hence, if *a* remain the same, this will vary with the obliquity both of the incident and reflected ray to the plane of the axes of the fibres; and, therefore, if that plane be turned about an axis parallel to the fibres, a succession of colours analogous to the transmitted series of those of their plates, but much more vivid, will be seen, as if reflected on them.

701. Colours of scratches on striæ.

Any extremely fine scratch on a well polished surface may be regarded as having a concave, cylindrical, or, at least, a curved surface capable of reflecting the light equally in all directions; this is evident, for it is visible in all directions. Two such scratches, then, drawn parallel to each other, and then turned round an axis parallel to both in the sunshine, ought to affect the eye in succession with a series of colours analogous to those of thin plates. This is really the case. Dr. Young found, on examining the lines drawn on glass in Mr. Coventry's micrometric scales, each of them to consist of two or more finer lines exactly parallel, and at a distance of about one 10,000th of an inch. Placing the scale so as to reflect the sun's light at a constant angle, and varying the inclination of the eye, he found the brightest red to be produced at angles whose sines were in the arithmetical progression 1, 2, 3, 4.

702. Of systems of equidistant parallel lines.

In the beautiful specimens of graduation on glass and steel produced by Dr. Wollaston, Mr. Barton, and M. Fraunhofer, single lines exactly parallel to each other, and distant in some cases not more than one 10,000th of an inch, and at precisely equal intervals, are drawn with a diamond point. If the eye be applied close to a reflecting or refracting surface so striated, so as to view a distant, small, bright light reflected in it, it will be seen accompanied with splendid lateral spectra, which evidently originate in this manner. They are arranged in a straight line passing through the reflected, colourless image, and at right angles to the direction of the striæ. Their angular distances from each other, the succession of their colours, and all their other phenomena, are in perfect agreement with the above explanation. Their vividness depends on the exact equality of distance between the parallel lines, which causes the lateral images produced by each pair to coincide precisely in distance from the principal image, and thus to produce a multiplied effect. If the distance of the lines be unequal, the images from different pairs, not coinciding, blend their colours, and produce a streak, or ray of white light. This is the origin of the rays seen darting, as it were, from luminous objects reflected on irregularly polished surfaces. These colours may be transferred, by impression from the surface originally graduated, to sealing wax, or other soft body; or from steel, by violent pressure, to softer metals. It is in this way that those beautiful striated buttons and other ornaments are produced, which imitate the splendour and play of colours of the diamond.

703. Alleged analogy between colours of striated surfaces and certain musical tones considered.

Dr. Young has assimilated the colour thus produced when a beam of white light strikes on a succession of parallel equidistant lines, to the musical tone heard when any sudden sound is *echoed* in succession by a series of equidistant bars having flat surfaces situated in a direction perpendicular to the line in which they are arranged, for instance, an iron railing. It is evident that such echoes will reach the ear in succession, at precisely equal intervals of time, each being equal to the time taken by sound to traverse twice the space separating the bars; and thus producing on the ear, if the bars be sufficiently numerous, the effect of a musical sound. (*Phil. Trans.* 1801; *On the Theory of Light and Colours.*) This explanation, however, appears to us, we confess, more ingenious than satisfactory. The pitch of the musical tone produced by the echoes is independent of the sound echoed, which may be a single blow, or a *noise*, (*i. e.* a sound consisting of non-periodic vibrations,) and requires for its production a number of echoing bars sufficient to prolong the echoes a sensible time. On the other hand, the light reflected from parallel striæ depends for its colour wholly on the incident ray, being red in red light, yellow in yellow, &c.; and is produced equally well from two or from twenty, as from a million of such reflecting lines. The intensity, not the colour,—the magnitude, not the frequency of the impression made on the retina by the reflected rays, is modified by their interference. We think it necessary to point out this defect in the illustration in question, inasmuch as it has become popular for its ingenuity, and *primâ facie* plausibility; while, in reality, it is calculated to give very erroneous impressions of the analogy between sound and light.

Light. Part III.

A single scratch or furrow in a surface may, as that eminent philosopher has himself remarked, produce colours by the interference of the rays reflected from its opposite edges. A spider's thread is often seen to gleam in the sunshine with the most vivid colours. These may arise either from a similar cause, or from the thread itself as spun by the animal, consisting of several, agglutinated together, and thus presenting not a cylindrical, but a furrowed surface. 704. Colours of a spider's web, &c.

The phenomena exhibited by light reflected from and refracted through the polished surface of mother of pearl, are, no doubt, referable in great measure to the same principle, so far as they depend on the structure of the surface. Dr. Brewster has described them in a most curious and interesting Paper, (published in the *Phil. Trans.* 1814, p. 397;) and a writer in the *Edinburgh Philosophical Journal*, vol. ii. p. 117, has added some further particulars illustrative of the curious and artificial structure of this singular body. Every one knows that mother of pearl is the internal lining of the shell of a species of oyster. It is composed of extremely thin laminæ of a tough and elastic, yet at the same time hard and shelly substance, disposed parallel to the irregular concavity of the interior of the shell. When, therefore, any portion of it is ground and polished on a plane tool, the artificial surface so produced intersects the natural surfaces of the laminæ in a series of undulating curves, or level-lines, which are nearer or farther asunder, according to the varying obliquity of the artificial to the natural surfaces. As these laminæ adhere imperfectly to each other, their *feather-edges* become broken up by the action of the powders, &c. used in grinding and polishing them, so as to present a series of ridges or escarpments arranged (when any very small portion of the surface only is considered) nearly parallel to, and equidistant from each other, which are distinctly seen with a microscope, and which no polishing in the least degree obliterates or impairs. The light reflected, therefore, or dispersed on their edges, will interfere and produce coloured appearances in a direction perpendicular to that of the striæ. This is, in fact, their situation; but the phenomena are modified in a very singular manner by the peculiar form of the edges and hollows, which results, no doubt, from the crystalline structure of the pearl. That it is the configuration *only* of the surface on which they depend, is evident from the remarkable fact, that, like the colours described in Art. 702, they may be transferred, by impression, to sealing wax, gum, resin, or even metals, with little or no diminution of their brilliancy; and the impression so transferred, if examined by the microscope, is found to exhibit a faithful copy of the original striæ, though sometimes so minute as hardly to exceed one 3000th of an inch in their distance from each other. For a particular description of this very curious and beautiful class of phenomena, however, our limits oblige us to refer to the original memoirs already cited, especially as their theory is still accompanied with some obscurity. 705. Of mother of pearl,

§ VIII. *Of the Diffraction of Light.*

When an object is placed in a very small beam of light, or in the cone of rays diverging from an extremely small point, such as a sunbeam admitted through a small pin-hole into a dark chamber, or, still better, through an opening of greater size, behind which a lens of short focus is placed, so as to form an extremely minute and brilliant image of the sun from which the rays diverge in all directions, its shadow is observed to be bordered externally by a series of coloured fringes which are more distinct the smaller the angular diameter of the luminous point, as seen from the object. If this be much increased, the shadow and fringes formed by its several points, regarded each as an independent luminary, overlap and confuse each other, obliterating the colours, and producing what is called the *penumbra* of the object; but when the luminous point is extremely minute, the shadow is comparatively sharp, and the fringes extremely well defined. 706. Fringes formed exterior to the shadows of bodies in a small beam of light.

These fringes (which were first described by Father Grimaldi in a work entitled *Physico-Mathesis de Lumine*, Bologna, 1665, and afterwards more minutely by Newton in the third book of his *Optics*) surround the shadows of objects of all figures, preserving the same distance from every part, like the lines along the sea-coast in a map; only, where the object forms an acute, salient angle, the fringes curve round it; and where it makes a sharp, reentering one they cross, and are carried up to the shadow at each side, without interfering or obliterating each other. In white light three only are to be seen, whose colours, reckoning from the shadow, are black, violet, deep blue, light blue, green, yellow, red; blue, yellow, red; pale blue, pale yellow, pale red. In homogeneous light they are, however, more numerous, and of different breadths, according to the colours of the light, being narrowest in violet and broadest in red light, as in the coloured rings between glasses; and it is by the mutual superposition of the different sets of fringes for all the coloured rays that their tints are produced, and their obliteration after a few of the first orders caused. 707. Their colours, &c. described.

The fringes in question are absolutely independent of the nature of the body whose shadow they surround, and the form of its edge. Neither the density or rarity of the one, nor the sharpness or curvature of the other, having the least influence on their breadth, their colours, or their distance from the shadow; thus it is indifferent whether they are formed by the edge or back of a razor, by a mass of platina or by a bubble in a plate of glass, (which, though transparent, yet throws a shadow by dispersing away the light incident on it,) circumstances which make it clear that their origin has no connection with the ordinary refractive powers of bodies, or with any *elective* attraction or repulsions exerted by them on light; for such forces cannot be conceived as independent of the *density* of the body exerting them, however minute we might regard the sphere of their action. 708. Are independent of the body casting the shadow.

To see the fringes in question, they may be received on a smooth, white surface, and examined and measured thereon by contrivances which readily occur; this was the mode pursued by Newton. M. Fresnel, however, having (to avoid the inconvenience of intercepting the light by the interposition of the observer) received them on an emeried glass plate, was enabled, by placing himself behind it, to approach near enough to examine and measure 709. M. Fresnel's method of examining them.

them with a magnifier. In so doing, however, he observed, that when thus once brought under inspection, they continued visible, and were indeed much brighter and more distinct in the focus of the lens (as if depicted in the air) even when the emeried glass was altogether withdrawn; and this fortunate observation, by enabling him to avoid the use of a screen altogether, and to perform all his measurements of their dimensions by the aid of a micrometer, put it in his power to examine them with a degree of minuteness and precision no other way attainable, and fully adequate to the delicacy of the inquiry: for it is manifest that the fringes, being seen as they would be formed if received on a screen in the focus, may be regarded as any other optical image formed in the focus of a telescope, viewed with any magnifier, and treated in all respects as such images.

710. Their phenomena. 1st. Their distances, *inter se.*

Whatever mode of examining them we adopt, however, we shall observe the following facts:

Phenomenon 1. That, *cæteris paribus*, the distances from each other and from the border of the shadow diminishes as the screen on which they are received, or the plane in the focus of the lens in which they are formed, approaches the border of the opaque body, and ultimately coincides with it, so that they seem to have their origin close to the edge of the body.

711. They are propagated in curved lines.

Phenomenon 2. That they are not, however, *propagated in straight lines* from the edge of that body to a distance, but in hyperbolic curves, having their vertices at that edge; and therefore that it is not *one and the same* light which forms one and the same fringe at all distances from the opaque body. To explain this, conceive the distances of the fringes from each other and from the shadow measured accurately at a great variety of distances from the edge of the body; then, were they propagated in straight lines, and were each fringe really the axis of a pencil of rays emanating from a point at that edge, their intervals and distances from the shadow ought to be proportional to the distances from the edge of the body; but it is not so, in fact,—the former distances increasing as we recede from the opaque body much more rapidly at first, and less so as we recede, than according to the law of proportionality; and if the *locus* of each fringe be laid down from such measures, it will be found to be an hyperbolic curve having its convexity outwards or from the shadow. Thus in fig. 142 O is the luminous point, A the edge of the body, and G H a screen perpendicular to the straight line O A, C the border of the visible shadow, and D, E, F the places of the successive minima of the fringes in a line at right angles to the edge of the shadow. If the screen be brought nearer to the body A as at *g h*, and if *c, d, e, f* be the points corresponding to C D E F, their loci will be the hyperbolas A *c* C, A *d* D, &c.

Fig. 142.

712. The visible shadow differs from the geometrical one and is larger.

It will be noticed also that the border C of the visible shadow is not coincident with B, that of the *geometrical* one, which lies in the straight line O A, grazing the edge of the object. The deviation is difficult to perceive in the shadow of a large body, having nothing to measure from; but if we examine those of very narrow bodies, as of a hair, for instance, in such a beam of light as described, we shall find on measuring the total breadth of the shadow a full proof of this. This fact was observed by Grimaldi. The limits of the visible shadow also follows the same law of curvilinear propagation as the fringes. Thus, Newton found the shadow of a hair one 280th of an inch in diameter placed at 12 feet distance from the luminous point, to measure at 4 inches from the hair $\frac{1}{60}$ inch, or upwards of 4 diameters of the hair, at two feet, $\frac{1}{28}$ inch, or 10 diameters; while at 10 feet it measured only $\frac{1}{8}$ inch, or 35 diameters, instead of 120, which it should have done if the rays terminating the shadow had proceeded in straight lines; or rather, to speak more correctly, if the shadow were bounded by straight lines.

713. Newton's doctrine of the deflexion of light. Fig. 143.

To account for these remarkable facts, Newton supposes that the rays passing at different distances from the edges of bodies are turned aside outwards, *as if* by a repulsive force; and that those nearest are turned more aside than those more remote, as in fig. 143, where X is a section of the hair, and A D, B E, C F, &c. rays which pass at different distances beside it, and which are turned off at angles rapidly diminishing as the distance increases in directions D G, E H, F I, &c. It is manifest that the curve W Y Z, to which all these deflected rays are tangents, and within which none can enter, will be convex outwards; and its curvature will be greatest at the vertex W, and will diminish continually as it recedes from X, being, in fact, the caustic of all the deflected rays.

714. His account of the fringes. Fig. 144.

This will be the boundary of the visible shadow. To account for the fringes, he supposes (*Optics*, book iii. question 3) that each ray in its passage by the body undergoes several flexures to and fro, as in fig. 144 at *a, b, c;* and that the luminous molecules, of which that ray consists, are thrown off at one or other of the points of contrary flexure, or other determinate points of the serpentine curve described by them according to the state of their fits in which they there happen to be, or other circumstances; some outwards, as in the directions *a* A, *b* B, *c* C, *d* D, and others we may suppose inwards, as *a a*, *b β*, *c γ*, &c. With the latter we have here no concern. The former, it is evident, will give rise to as many such caustics as above described, as there are deflected rays; and each caustic, when intercepted on a screen at a distance, will depict on it the maximum of a fringe. The intervals, however, between these caustics, or minima of the fringes, will not be totally black; because the rays from the other caustics, after crossing on the confines of the shadow, or interior fringes, will pursue their course, and partially illuminate all the space beyond. Thus the fringes should be less numerous and the degradation of colour more rapid than in the coloured rings.

715. Newton's doctrine and Fresnel's objections to it considered.

This theory accounts then perfectly for the curvilinear propagation of the fringes, for their rapid degradation, for their apparently originating in the very edge of the body, (since each caustic will actually come up to that edge, as at A, fig. 142,) and for the remarkable brightness of the fringes, especially the first, which really contains in itself all the light which would have passed into the region B C between the visible and geometrical shadows. It should appear, therefore, that M. Fresnel, in the objections he has taken against these points of the Newtonian doctrine of inflexion in his excellent work *Sur la Diffraction de la Lumière*, (§ 1, p. 15, 17, 19,) must have formed a very inadequate conception of the doctrine he opposes, which, if viewed in the light he has there placed it in, would indeed deserve no other epithet than puerile, and must be looked upon as quite unworthy of its illustrious author; and were these the only difficulties to be explained, we should certainly not be justified

in passing a hasty sentence on it. Other objections advanced by the same eminent philosopher, however, are more serious, and refer to a phenomenon of which the doctrine of deflective forces seems incapable of giving any account; but of which, in justice to Newton we ought to add, it does not appear that he was aware, or its importance could not fail to have struck him.

716. Dilatation of the fringes by the approach of the radiant point.

Phenomenon 3. All other things remaining the same, let the opaque body A be brought nearer the luminous point O, (fig. 142.) The fringes then, formed at the same distance as before behind A, are observed to dilate considerably in breadth,—preserving, however, the same relative distances from each other, and from the border of the shadow. This fact is evidently incompatible with the idea of their being caused by any deflecting force emanating from the opaque body, since it is inconceivable that such a force should depend on the distance the light has travelled from another point no way related to the body.

717. Dr. Young's explanation of the fringes on the undulatory doctrine. Objections against it.

To explain the diffracted fringes on the undulatory doctrine, Dr. Young conceived the rays passing near the edge of the opaque body to interfere with those reflected very obliquely *on* its edge, and which in the act of reflexion had lost half an undulation, as in the case of the reflected rings. This supposition would, in fact, lead us to conclude the existence of a series of fringes propagated hyperbolically, and perfectly resembling those really existing. M. Fresnel, however, has shown that a minute though decided difference exists between their places, as given by this theory and by direct measurement; and has, moreover, remarked, that were this the true explanation, they could hardly be supposed absolutely independent of the figure of the edge of the opaque body, which experience shows they are; and that in cases where this edge is extremely sharp, the small quantity of light which *could* be reflected from it would be insufficient to interfere with that passing by it, so as to form fringes so bright as we see them. These objections appear conclusive, especially as the supposition of a reflexion on the edge of the body is unnecessary, since a more strict application of the undulatory doctrine, assisted by the principle of interferences, will be found to afford a full and precise explanation of all the facts, regarding the opaque body as merely an obstacle *bounding* the waves propagated from the luminous point on one side.

718. Fresnel's explanation Fig. 145.

To show this, let us consider a wave A M F propagated from O, and of which all that part to the right of A (fig. 145) is intercepted by the opaque body A G; and let us consider a point P in a screen at the distance A B behind A, as illuminated by the undulations emanating simultaneously from every point of the portion A M F, according to the theory laid down in Art. 628, *et seq.* For simplicity, let us consider only the propagation of undulations in one plane. Put A O $= a$, A B $= b$, and suppose $\lambda =$ the length of an undulation; and drawing P N any line from P to a point near M, put P F $= f$, N M $= s$, P B $= x$; then, supposing P very near to B, and with centre P radius P M describing the circle Q M, we shall have $f =$ P Q + Q N $= \sqrt{(a+b)^2 + x^2} - a$ + Q N $= b + \frac{x^2}{2(a+b)}$ + Q N. Now, Q N is the sum of the versed sines of the arc s to radii O M and P M, and is therefore equal to $\frac{s^2}{2\,\mathrm{O\,M}} + \frac{s^2}{2\,\mathrm{P\,M}} = \frac{s^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{a+b}{2\,a\,b} \cdot s^2$; so that, finally,

$$f = b + \frac{x^2}{2(a+b)} + \frac{(a+b)\,s^2}{2\,a\,b}.$$

Now, if we recur to the general expression demonstrated in Art. 632, for the motion propagated to P from any limited portion of a wave, we shall have in this case $a \cdot \phi(\theta) = 1$, because we may regard the obliquity of all the undulations from the whole of the efficacious part of the surface A M N as very trifling, when P is very distant from A in comparison with the length of an undulation. And as we are now only considering undulations propagated in one plane, that expression becomes merely $\mathrm{V} = \int ds \cdot \sin 2\pi\left(\frac{t}{\mathrm{T}} - \frac{f}{\lambda}\right)$, and the corresponding expression for the excursions of a vibrating molecule at P will be

$$\mathrm{X} = \int ds \cdot \cos 2\pi\left(\frac{t}{\mathrm{T}} - \frac{f}{\lambda}\right).$$

If then we put for f its value, and take

$$2\pi\left(\frac{t}{\mathrm{T}} - \frac{b}{\lambda} - \frac{x^2}{2\lambda(a+b)}\right) = \theta; \qquad s \cdot \sqrt{\frac{2(a+b)}{a\,b\,\lambda}} = \nu,$$

and consider that in those expressions t and x remain constant, while s only varies, the latter will take the form

$$\mathrm{X} = \sqrt{\frac{a\,b\,\lambda}{2(a+b)}} \cdot \left\{\cos\theta \cdot \int d\nu \cdot \cos\left(\frac{\pi}{2}\nu^2\right) + \sin\theta \cdot \int d\nu \cdot \sin\left(\frac{\pi}{2}\nu^2\right)\right\},$$

which shows that the total wave on arriving at P may be regarded as the resultant of two waves $\mathrm{X}' \cdot \cos\theta$ and $\mathrm{X}'' \cdot \sin\theta$, differing in their origin by a quarter-undulation, and whose amplitudes X' and X'' are given by the expression

$$\mathrm{X}' = \sqrt{\frac{a\,b\,\lambda}{2(a+b)}} \cdot \int d\nu \cos\frac{\pi}{2}\nu^2; \qquad \mathrm{X}'' = \sqrt{\frac{a\,b\,\lambda}{2(a+b)}} \cdot \int d\nu \cdot \sin\frac{\pi}{2}\nu^2,$$

the integrals being taken between limits of ν corresponding to $s = -$ A M, and $s = +\infty$. Consequently,

Light. since $s = AM = PB \times \frac{a}{a+b} = \frac{a\,x}{a+b}$, and $v = s\sqrt{\frac{2\,(a+b)}{a\,b\,\lambda}}$, Part III.

the limits of v must be

$$v = -x\sqrt{\frac{2\,a}{(a+b)\,b\,\lambda}} \text{ and } v = +\infty.$$

719. Rule for determining the illumination of any point in the screen. Hence, to determine the intensity of the light at any point P on the screen, we must first of all calculate the values of these integrals; and having thus determined X′ and X″, the square root of the sum of their squares $\sqrt{X'^2 + X''^2}$ will represent the amplitude of a single vibration, the resultant of both, (Art. 615;) and the sum of their squares simply $(X'^2 + X''^2)$, the intensity of the light, or the sensation produced in the eye.

720. Maxima and minima numerically estimated. M. Fresnel, in the work already cited, has given a table of the values of these integrals for limits successively increasing from 0 up to ∞, (at which latter limit each is equal to $\frac{1}{2}$, as may readily be proved;) and, calculating on this, he finds that the intensity of the light, without the limit of the geometrical shadow, passes through a series of maxima and minima according to the following table:

Table of the Maxima and Minima for the Exterior Fringes, and of the Corresponding Intensities of the Light illuminating them.

	Values of v.	Intensities of the light.		Values of v.	Intensities of the light.
First maximum	1.2172	2.7413	Fourth minimum ..	3.9372	1.7783
First minimum	1.8726	1.5570	Fifth maximum	4.1832	2.2206
Second maximum ..	2.3449	2.3990	Fifth minimum	4.4160	1.8014
Second minimum ..	2.7392	1.6867	Sixth maximum	4.6069	2.1985
Third maximum ..	3.0820	2.3022	Sixth minimum	4.8479	1.8185
Third minimum	3.3913	1.7440	Seventh maximum..	5.0500	2.1818
Fourth maximum ..	3.6742	2.2523	Seventh minimum ..	5.2442	1.8317

In this it is to be remarked, that no minimum is zero, and that the difference between the successive maxima and minima diminishes very rapidly as the values of v increase, which explains the rapid degradation of their tints.

721. Illumination of the border of the geometrical shadow. If the point P be situated on the very edge of the geometrical shadow, its illumination should on this theory be $(\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$. To compare this with the illumination of the same point, were the opaque body removed, we have only to consider, that at a great distance from the shadow the light must be the same, whether the body be there or not. Now the limit to which the maxima and minima approximate is 2, which therefore represents the uniform illumination beyond the fringes; so that the light on the border of the geometrical shadow is equal to $\frac{1}{4}$ of the full illumination from the radiant point.

722. Illumination within the shadow. Within the shadow we have only to make s or v negative. This does not alter the values of the integrals, but it does their limits, which must in that case be taken not from $v = -x\sqrt{\frac{2\,a}{(a+b)\,b\,\lambda}}$ to $+\infty$, but from $v = +x\sqrt{\frac{2\,a}{(a+b)\,b\,\lambda}}$ to $+\infty$. The computations have been executed by M. Fresnel, who finds that no periodical increase or decrease here takes place, but that the light degrades rapidly and constantly within the geometrical shadow to total darkness.

723. Visible shadow larger than the geometrical. The actual visible shadow then is marked by no sudden defalcation of light, and it will depend on the judgment of the eye where to establish its termination. If we regard all that part as shadow which is less illuminated than the general light of the screen beyond the fringes, then the visible shadow will extend considerably beyond the geometrical one, and this explains why the shadows of small bodies are so much dilated, as we have seen they are.

724. If we would know the breadths of the several fringes, we have only to find the values of x in the equation

$$x = v\,.\,\sqrt{\frac{(a+b)\,b}{a}}\,\frac{\lambda}{2},$$

where v has in succession the several values set down in the foregoing table. If we consider the variation of x for successive values of a and b, we shall see the origin both of the curvilinear propagation of the fringes, and of their dilatation on the approach of the luminous point. In fact if we regard, first, the relation between b and x, or the locus of any fringe regarded as a curve, having the line A B for an abscissa and B P as an ordinate, we have $x^2 = v^2\,\frac{\lambda}{2}\left(b + \frac{b^2}{a}\right)$, which is the equation of an hyperbola having its convexity outwards and passing through A. Secondly, on the other hand, if we regard a as the variable quantity and b as constant, we see that for one and the same distance from the screen, the breadths of the fringes increase as a diminishes; the increments of

their squares, as the incident rays from being parallel become more divergent, being directly as their divergence. Thirdly, for equal values of λ, a, and b, x is proportional to ν; so that the breadths of the several fringes are always in the same ratio to each other, and form a progression the same with those of the values of ν in the foregoing table. Lastly, the breadths of the fringes for different coloured rays are as the square roots of the lengths of their undulations.

725. The accordance of this theory with experiment, so far as it regards the distances of the fringes from the shadow and from each other, has been put to a severe test by M. Fresnel, and found perfect. It were to be wished, however, that he had stated somewhat more precisely the instrumental means by which he determined the place of the border of the *geometrical shadow*, from which his measures are all stated to be taken; and which, being marked by no phenomenon of maximum or minimum, might be liable to uncertainty if judged of by the eye alone. This, however, in no way invalidates the accuracy of the final conclusions, as the intervals between the fringes are distinctly marked, and susceptible of exact measurement. The dilatation of the fringes on the approach of the luminous point is, perhaps, the strongest fact in favour of the undulatory doctrine, and in opposition to that of inflection, which has yet been adduced. It seems hardly reconcilable to any received ideas of the action of corpuscular forces, to suppose the force of deflection exerted by the edge of a body on a passing ray, to depend on the distance which the ray has passed over before arriving at that edge from an arbitrarily assumed origin. M. Fresnel has placed this argument in a strong light, in his work already cited.

726. Fringes observed by Grimaldi within narrow shadows.

Besides the exterior fringes above described, there are others formed in certain circumstances within the shadows of bodies which afford peculiarly apt illustrations of the principle of interferences. The first class of phenomena of this kind was noticed by Grimaldi, who found that when a long, narrow body is held in a small diverging beam of light, the shadow received on a screen at a distance will be marked in the direction of its length with alternate streaks or fringes brighter and darker than the rest. These are more or less numerous, according as the distance of the shadow from the body is smaller or greater in proportion to the breadth of the latter. To study the phenomena more minutely, Dr. Young passed a sunbeam through a hole made with a fine needle in thick paper, and brought into the diverging beam a slip of card one-thirtieth of an inch in breadth, and observed its shadow on a white screen at different distances. The shadow was divided by parallel bands, as above described, but *the central line was always white.* That these bands originated in the interference of the light passing on both sides of the card, Dr. Young demonstrated beyond all controversy, by simply intercepting the light on one side by a screen interposed between the card and the shadow, leaving the rays on the other side to pass freely, in the manner represented in fig. 146, where O is the hole, A B the card, E F its shadow, and C D the intercepting body receiving on its margin the margin of the shadow of the edge B of the body. In this arrangement all the fringes which had before existed in the shadow E F immediately disappeared, although the light inflected on the edge A was allowed to retain its course, and must have necessarily undergone any modification it was capable of receiving from the proximity of the other edge B. The same result took place when the intercepting screen was placed as at *c d before* the edge B of the body, so as to throw its own shadow on the margin B of the card.

Dr. Young's fundamental interference.

Fig. 146.

727. Explanation.

Without entering minutely into the rationale of this phenomenon, which, however, the formulæ of the preceding articles enable us fully to do, by considering the illumination of any point X between E and F as arising from the whole wave *a* A B *b*, *minus* the portion A B, and which M. Fresnel has done at full length, and with great success, in his Memoir already so often cited; we shall content ourselves with showing how fringes or alternations of colour must originate in such circumstances; in fact, if we join A X, B X, the difference of routes of the waves arriving at X by the paths O A X, O B X is equal to B X − A X. It is therefore nothing in the middle of the shadow E F, which ought therefore to be illuminated by double the light deflected into the shadow at that distance by either edge, Art. 722, which will be less as the object is larger, and the shadow broader. But on either side of the middle B X − A X increases; and when it attains a value equal to half an undulation, the waves are in complete discordance, and therefore the middle bright portion will be succeeded by a dark band on either side, and these again by bright ones, and so on.

728. Grimaldi's crested fringes.

An elegant variation of this experiment of Dr. Young is afforded by a phenomenon described by Grimaldi. When a shadow is formed by an object having a rectangular termination; besides the usual external fringes there are two or three alternations of colours, beginning from the line which bisects the angle, disposed, within the shadow on each side of it, in curves which are convex towards the bisecting line, and which converge towards it as they become remote from the angular point. These fringes are the joint effect of the light spreading into the shadow from each outline of the object, and interfering as above; and that they are so, is proved by placing a screen within a few inches of the object, so as to receive only one edge of the shadow, when the whole of the fringes disappear. If, on the other hand, the rectangular point of the screen be opposed to the point of the shadow, so as barely to receive the angle of the shadow on its extremity, the fringes will remain undisturbed. (Young, Experiments and Calculations relating to Physical Optics, *Phil. Trans.*, 1803.)

729. Case of diffraction through a small circular aperture.

Such are some of the more remarkable appearances produced within and beyond the shadows of narrow bodies. Let us next consider the effect of transmitting a beam through a very narrow aperture. And the first case is when the aperture is circular. Suppose, for instance, we place a sheet of lead, having a small pin-hole pierced through it, in the diverging cone of rays from the image of the sun, formed by a lens of short focus, and in the line joining the centres of the hole and focus prolonged place a convex lens or eye-glass, behind which the eye is applied. The image of the hole will be seen through the lens as a brilliant spot, encircled by rings of colours of great vividness, which contract and dilate, and undergo a singular and beautiful alternation of tints, as the distance of the hole from the luminous point on the one hand, or on the eye-glass on the other, is changed. When the latter distance is considerable, the central spot is white, and the rings follow nearly the order of the colours of thin plates. Thus, when the diameter of the hole was about $\frac{1}{30}$th of an inch, its distance

Light. (*a*) from the luminous point about 6 feet 6 inches, and its distance (*b*) from the eye-lens 24 inches, the series of colours was observed to be, Part III.

1st order. White; pale yellow; yellow; orange; dull red.
2d order. Violet; blue (broad and pure;) whitish; greenish yellow; fine yellow; orange red, very full and brilliant.
3rd order. Purple; indigo blue; greenish blue; pure, brilliant green; yellow green; red.
4th order. Good green, but rather sombre and bluish; bluish white; red.
5th order. Dull green; faint bluish white; faint red.
6th order. Very faint green; very faint red.
7th order. A trace of green and red.

730. Table of colours of central spot and surrounding rings. When the eye-lens and hole are brought nearer together, the central white spot contracts into a point and vanishes, and the rings gradually close in upon it in succession, so that the centre assumes in succession the most surprisingly vivid and intense hues. Meanwhile the rings surrounding it undergo great and abrupt changes in their tints. The following were the tints observed in an experiment made some years ago, (July 12, 1819,) the distance between the eye-glass and luminous point ($a + b$) remaining constant, and the hole being gradually brought nearer to the former.

$b =$	Central Spot.	Surrounded by
24.00	White	Rings as in the foregoing Article.
18.00	White	The two first rings confused, the red of the 3rd and green of the 4th orders splendid.
13.50	Yellow	Interior rings much diluted, the 4th and 5th greens and 3rd, 4th and 5th reds the purest colours.
10.00	Very intense orange	All the rings are now much diluted.
9.25	Deep orange red	The rings all very dilute.
9.10	Brilliant blood red	The rings all very dilute.
8.75	Deep crimson red	The rings all very dilute.
8.36	Deep purple	The rings all very dilute.
8.00	Very sombre violet	A broad yellow ring.
7.75	Intense indigo blue	A pale yellow ring.
7.00	Pure deep blue	A rich yellow.
6.63	Sky blue	A ring of orange, from which it is separated by a narrow, sombre space.
6.00	Bluish white	Orange red, then a broad space of pale yellow, after which the other rings are scarcely visible.
5.85	Very pale blue	A crimson red ring.
5.50	Greenish white	Purple, beyond which yellow verging to orange.
5.00	Yellow	Blue, orange.
4.75	Orange yellow	Bright blue, orange red, pale yellow, white.
4.50	Scarlet	Pale yellow, violet, pale yellow, white.
4.00	Red	White, indigo, dull orange, white.
3.85	Blue	White, yellow, blue, dull red.
3.50	Dark blue	Orange, light blue, violet, dull orange.

731. Fresnel's analysis of this case. The series of tints exhibited by the central spot is, evidently, so far as it goes, that of the reflected rings in the colours of thin plates. The surrounding colours are very capricious, and appear subject to no law. They depend, indeed, on very complicated and unmanageable analytical expressions, with which we shall not trouble the reader, but content ourselves with presenting the explanation given by M. Fresnel of the changes of tint of the central spot in white light, and its alternations of light and total darkness observed by him in an homogeneous illumination. Let then a and b be the distances of a small hole whose radius is r from the luminous point, and a screen placed behind the hole perpendicularly to the ray passing directly through its centre. Then if we consider any infinitely narrow annulus of the hole whose radius is z, and breadth dz, this annulus will send to the central point of the screen a system of waves whose intensity is proportional to the area of the annulus, or $2\pi z\,dz$, but whose phase of undulation differs from that of the central ray, by reason of the difference of the paths described by them. Now, calling f the distance of each point in the annulus from the centre of the screen, we have $f^2 = b^2 + z^2$, and, in like manner, if f' be the distance of the luminous point from the same annulus, $f'^2 = a^2 + x^2$, so that $(f + f') - (a + b)$ the difference of paths, or interval of retardation, is equal to $\frac{z^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{z^2\,(a+b)}{2\,a\,b}$. Hence, the general expression in Art. 632 for the amplitude of the total wave, incident on the centre of the screen in this particular case, is equivalent to

$$X = \int 2\pi z\,dz \,.\, \sin 2\pi \left\{\frac{t}{T} - \frac{z^2\,(a+b)}{2\,a\,b\,\lambda}\right\},$$

Light. Part III.

or, integrating, which from the peculiar form of the differential is in this case easy,

$$X = \frac{a\,b\,\lambda}{a+b}\left\{\text{const} + \cos 2\pi . \left(\frac{t}{T} - \frac{z^2\,(a+b)}{2\,a\,b\,\lambda}\right)\right\}$$

which, extended from $z = 0$ to $z = r$, gives

$$X = \frac{a\,b\,\lambda}{a+b}\left\{\cos\lambda\left(\frac{t}{T} - \frac{(a+b\,.\,r^2}{2\,a\,b\,\lambda}\right) - \cos 2\pi\,\frac{t}{T}\right\}$$

$$= \frac{a\,b\,\lambda}{a+b}\left\{\sin\frac{\pi\,(a+b)\,r^2}{a\,b\,\lambda}\,.\,\sin 2\pi\,\frac{t}{T} + \left(\cos\frac{\pi\,(a+b)\,r^2}{a\,b\,\lambda} - 1\right).\cos 2\pi\,\frac{t}{T}\right\}.$$

This expresses, as we have before remarked in a similar case, (Art. 718,) two partial waves differing by a quarter-undulation, and expressing it, as in that case, by $X = X'\,.\cos\theta + X''\,.\sin\theta$, where $\theta = \frac{t}{T}$, we find for the intensity A^2 of their resultant

$$A^2 = X'^2 + X''^2 = 4\left(\frac{a\,b\,\lambda}{a+b}\right)^2\left(\sin\frac{\pi\,(a+b)\,r^2}{2\,a\,b\,\lambda}\right)^2.$$

732. Illumination of the central spot compared with the total illumination. Fresnel's theorem.

To make use of this, however, we must compare it with what would be the direct illumination of the centre of the screen, if the aperture were infinite, *i. e.* if the direct light from the luminous point shone full upon it. To this case, however, neither our formula nor our reasoning are applicable; for if we make r infinite in this expression, it becomes illusory, and presents no satisfactory sense, and in our reasoning we have neglected to consider the law of diminution of the intensity of the oblique waves, or regarded $\phi(\theta)$ in Art. 631 as invariable, which in this extreme case is far from the truth. We must, therefore, have recourse to another method. Now, M. Fresnel has demonstrated (and our limits oblige us to take his demonstration for granted) that this total illumination is equal to one-fourth of that which the centre of the screen would receive from an opening of such a radius, that the difference of routes of a ray passing through the centre, and one diffracted at the circumference, shall be an exact semi-undulation, *i. e.* in which $\frac{r^2\,(a+b)}{2\,a\,b} = \frac{\lambda}{2}$, or $r = \sqrt{\frac{a\,b\,\lambda}{a+b}}$. If then we substitute this for r in the above formula, and put C for the whole illumination, we get, *on the same scale*,

$$C = \left(\frac{a\,b\,\lambda}{a+b}\right)^2.\left(\sin\frac{\pi}{2}\right)^2 = .\left(\frac{a\,b\,\lambda}{a+b}\right)^2$$

and, consequently,

$$A^2 = 4\,C\left(\sin\frac{\pi\,(a+b)\,r^2}{2\,a\,b\,\lambda}\right)^2.$$

733. The colours those of the reflected rings.

In this expression r, a, b are independent of λ, and therefore the value of A^2 is of the form $4\,C\left(\sin 2\pi\,.\,\frac{\beta}{\lambda}\right)^2$ where $B = \frac{(a+b)\,r^2}{4\,a\,b}$. Hence, if we suppose light of all colours to emanate from the luminous point, the compound tint produced in the central point of the screen will be represented by $S\left\{4\,C\,.\left(\sin 2\pi\,\frac{\beta}{\lambda}\right)^2\right\}$ and will therefore, by Art. 673, be the same with that reflected by a plate of air whose thickness is B, or $\frac{(a+b)\,r^2}{4\,a\,b}$ which increases as b diminishes when $a+b$ remains constant. Thus we see the origin of the succession of colours of the central spot in the Table above recorded, which is the more satisfactory, as that experiment was made without reference to, and indeed in ignorance of, this elegant application of M. Fresnel's general principles, the merit of which is due (as he himself states) to M. Poisson.*

734. Poisson's theorem for the illumination in the centre of a small circular shadow.

Another very curious result of M. Poisson's researches is this, that the centre of the shadow of a very small circular opaque disc, exposed to light diverging from a single point, is precisely as much illuminated by the diffracted waves as it would be by the direct light, if the disc were altogether removed. We cannot spare room for the demonstration of this singular theorem. It has been put to the test of experiment by M. Arago, with a small metallic disc cemented on a very clear and homogeneous plate of glass, and with full success.

735. Case of diffraction through two apertures very near each other. Fig. 147 and 148.

When the light is transmitted through two equal apertures, placed very near each other, the rings are formed about each as in the case of one; but besides these arise a set of narrower, straight, parallel fringes bisecting the interval between their centres, and at right angles to the line joining them. If the apertures be unequal, these fringes assume the form of hyperbolas, having the aperture in their common focus. Besides these also two other sets of parallel rectilinear fringes (in the case of equal apertures) go off in the form of a St. Andrew's cross from the centre at equal angles with the first set. See figures 147, 148. When the apertures are more numerous or varied in shape, the variety and beauty of the phenomena are extraordinary; but of this more presently.

M. Fresnel has shown, that when the light from a single luminous point is received on two plane mirrors

* The coincidence in the higher orders of colours was, however, in our experiments less complete, and especially the green of the third order, which was wanting altogether in some cases.

very slightly inclined to each other, so as to form two almost contiguous images, if these be viewed with a lens, there will be seen between them a set of fringes perpendicular to the line joining them. These are evidently analogous to those produced by the two holes in the experiments last described. The experiment is delicate; for if the surfaces of the reflectors at the point where they meet be ever so little, the one raised above or depressed below the other, so as to render the difference of routes of the rays greater than a very few undulations, no fringes will be seen. But it is valuable, as demonstrating distinctly that the borders of the apertures in the preceding experiment have nothing to do with the production of the fringes, the rays being in this case abandoned entirely to their mutual action after quitting the luminous point. An exactly similar set of fringes is formed if, instead of two reflectors, we use a glass, plane on one side, and on the other composed of two planes, forming a very obtuse angle, as in fig. 149. This being interposed between the eye-lens E and the luminous point S, forms two images S and S′ of it; and the interference of the rays S E and S′ E from these images, forms the fringes in question.

736. Fresnel's experiment with two mirrors inclined to each other.

Fig. 149.

737. Effect of interposing a denser medium in one of two interfering rays.

Since the production of the fringes and their places with respect to the images of the luminous point, depends on the difference of routes of the interfering rays, it is evident, that if, without altering their paths, we alter the *velocity* of one of them with respect to the other, during the whole or a part of its course, we shall produce the same effect. Now, the velocity of a ray may be changed by changing the medium in which it moves. In the undulatory system, the velocity of a ray in a rarer medium is greater than in a denser. Hence, if in the path of one of two interfering rays we interpose a parallel plate of a transparent medium denser than air, (at right angles to the ray's course,) we shall increase its interval of retardation, or produce the same effect as if its course had been prolonged. If then a *thick* plate of a dense medium, such as glass, be interposed in one of the rays which form visible fringes, they will disappear; because the interval of retardation will be thus rendered suddenly equal to a great number of undulations, whereas the production of the fringes requires that the difference of routes shall be very small. If, however, only a very thin lamina be interposed, they will remain visible, but shift their places. Thus, in fig. 150, let S A, S B be rays transmitted through the small apertures A, B from the luminous point S, and received on the screen D C E, these forming a set of fringes of which C, the middle one, will be *white*. Let D, E be the dark fringes immediately adjacent on either side; and things being thus disposed, let a thin film of glass or mica G be interposed in *one* of the rays S A, its thickness being such that the ray in traversing it shall just be retarded half an undulation. Then will the rays A E, B E, which before were in complete discordance, be now in exact accordance, and there will be formed at E a bright fringe instead of a dark one. On the other hand, the ray A C will now be half an undulation behind B C, instead of in complete accordance with it, so that at C there will be formed a dark fringe, and so on. In other words, the whole system of fringes will be formed as before, but will have shifted its place, so as to have its middle in E instead of in C, *i. e.* will have moved *from* the side on which the plate of the dense medium is interposed. It is evident, that if the plate G be thicker, the same effect will take place in a greater degree.

Fig. 150.

Displacement of the fringes by it explained.

738. Mode of putting this to the test of experiment.

To make the experiment, however, it must be considered that the refractive power of glass, or indeed of any but gaseous media, is so great, that any plate of manageable thickness would suffice to displace the fringes so far as to throw them wholly out of sight. But we shall succeed, if, instead of a single plate G placed over *one* aperture A, we place two plates G, *g* of very nearly equal thicknesses, (such as will arise from two nearly contiguous fragments of one and the same polished plate,) one over each aperture; or we may vary the thickness of the plate traversed by either ray by inclining it, so as to bring it within the requisite limits. This done, the effect observed is precisely that described; the fringes shift their places *from* the thicker plate, without sustaining any alteration in other respects. This elegant experiment affords a strong indirect argument in favour of the undulatory system, and in opposition to that of emission, since it proves that the rays of light are *retarded* in their passage through denser media, agreeably to what the undulatory system requires, and contrary to the conclusions of the corpuscular doctrine.

Argument against the corpuscular system.

739. Arago and Fresnel's method of determining refractions of gases.

MM. Arago and Fresnel have taken advantage of this property, to measure the relative refractive powers of different gases, or of the same in different states of temperature, pressure, humidity, &c. It is manifest, that if any considerable portion of the path of *one* of the interfering rays be made to pass through a tube closed at both ends with glass plates, and the other through equal glass plates *only*, the fringes will be formed as usual. But if the tube be *exhausted*, or warmed, or cooled, or filled with a gas of different refractive density, a displacement of the fringes will take place, which (if they be received in the focus of a micrometer) may be measured with the greatest delicacy. Knowing the amount of their displacement, as compared with the breadth of the fringes, we know the number of undulations gained or lost by one ray on the other; and hence, knowing the internal length of the tube, we have the ratio of the refracting power of the medium it contains to that of air. What renders this method remarkable is, that *there is actually no conceivable limit to the precision of which it is susceptible*, since tubes of any length may be employed, and micrometers of any delicacy.

740. Fraunhofer's experiments on diffraction and interference. His apparatus.

The phenomena of diffraction, and those arising from the mutual interference of several very minute pencils of rays emanating from a common origin, have been investigated by M. Fraunhofer with great care and extraordinary precision, by the aid of a very delicate apparatus devised and executed by himself.

This apparatus consisted of a repeating, 12-inch theodolite, reading to every 4″, carrying, attached to its horizontal circle, a plane circular disc of six inches in diameter, having its axis precisely coincident with that of the theodolite, and having its own particular divisions independent of those of the theodolite. In the centre of this disc was placed vertically a metallic screen, having in it one or more narrow, vertical, rectangular slits, or other apertures, and so fixed as to have the middle of its aperture, or system of apertures, exactly coincident with the axis of the instrument. Attached to the great circle of the theodolite, horizontally, was a telescope, having its object-glass three inches and a half from the centre, and its axis directed exactly to it, and precisely parallel to the plane of the limb, and provided with a delicate micrometer, whose parallel threads were exactly vertical.

The instrument being insulated on a support of stone, a beam of solar light was directed by a heliostat, through a very narrow slit, also exactly vertical, having a breadth of one hundredth of an inch, and distant 463½ inches from the centre of the theodolite, so as to fall on the screen, and, being transmitted through its apertures, to be received into the telescope. It is manifest that the eye-glass of the telescope will here view the fringes, &c. as they are formed in its focus. The magnifying power of the telescope used by Fraunhofer varied from 30 to 50 times.

741. Fringes produced by a single narrow aperture.

M. Fraunhofer first examined the effect produced by the diffraction of the light through a single slit,—the breadth of which he first determined with the greatest precision by means of a micrometer-microscope, with which he assures us that he found it practicable to appreciate so minute a quantity as 1-50,000th of an inch. The slit being then placed on the apparatus, and accurately adjusted before the object-glass of the telescope, which was directed exactly to the aperture in the heliostat, the image of the latter was formed in its focus, accompanied by lateral fringes, which by the effect of the magnifying power were dilated into broad and brilliant prismatic spectra. The distances of the red ends of these spectra from the middle point, or white central image, were then measured accurately by means of the micrometer. The result of a great number of experiments with apertures of all breadths from one-tenth to one-thousandth of an inch, agreed to astonishing precision with each other, and with the following laws, *viz.* that (under the circumstances of the experiment,)

Their laws and dimensions.

1. *The angles of deviation of the diffracted rays, forming similar points of the systems of fringes produced by different apertures, are inversely as the breadths of the apertures.*

2. *That the distances of similar rays (the extreme red, for instance,) from the middle in the several spectra, constituting the successive fringes, form in each case an arithmetical progression whose difference is equal to its first term.*

3. That calling γ the breadth of the aperture, in fractions of a Paris inch, the angular distances L', L'', L''', &c. in parts of a circular arc to radius unity, of the extreme red rays in each fringe from the middle line, are respectively represented by $L' = \frac{L}{\gamma}$, $L'' = 2 . \frac{L}{\gamma}$, $L''' = 3 . \frac{L}{\gamma}$, &c. where $L = 0.0000211$, and a similar law holds for all the other coloured rays, different values being assigned to L for each.

742. Newton's experiment with two knife edges.

This conclusion agrees perfectly with the result of an experiment related by Newton in the IIId Book of his *Optics*. He ground two knife edges truly straight, and placed them opposite to each other, so as to be in contact at one end, and at the other to be at a small distance, such that the angle included between them was about 1° 54′, thus forming a slit whose breadth at their intersection was evanescent, and at 4 inches from that point ⅛th of an inch, and in the intermediate points, of course, of every intermediate magnitude. Exposing this in a sunbeam emanating from a very small hole at 15 feet distance, he received their shadows on a white screen behind them, and observed that when they were received very near to the knife edges, (as at half an inch,) the fringes exterior to the shadow of each edge ran parallel to its border without sensible dilatation, till they met and joined without crossing, at angles equal to that contained between the knife edges. But when the shadows were received at a great distance from the knives, the fringes had the form of hyperbolas, having for one asymptote the shadow of the knife to which they respectively belonged, and for the other a line perpendicular to that bisecting the angle of the two shadows, each fringe becoming broader and more distinct from the shadow which it bordered, as it approached the angle. These hyperbolas crossed without interfering, as represented in fig. 151. Their points of crossing, Newton found, however, not to be at a constant distance from the angle included between the projections of the knife edges, but to vary in position with the distance from the knives, at which the shadow is received on the screen; and hence, he says, "I gather that the light which makes the fringes upon the paper, is not the same light at all distances of the paper from the knives; but when the paper is held very near the knives, the fringes are made by light which passes by their edges at a less distance, and is more bent than when the paper is held at a greater distance from the knives." Newton, however, left these curious researches, which could hardly have failed to have led in his hands to a complete knowledge of the principles of diffraction—unfinished; being, as he says, interrupted in, and unwilling to resume them: doubtless, owing to the chagrin and opposition his optical discoveries produced to him. An unmeet reward, it must be allowed, for so noble a work, but one of which, unhappily, the history of Science affords but too many parallels.

Fig. 151.

743. Case when the edges of the aperture were at different distances from the origin of the light.

The above were the results obtained by M. Fraunhofer when the two edges of the aperture were both in a plane perpendicular to the incident rays; but when the same effective breadth was procured, by inclining a larger aperture obliquely, so as to reduce its actual breadth in the ratio of the cosine of its incidence to radius, or by limiting the incident ray by two opaque edges at different distances from the object-glass of the telescope, the phenomena were very different. To accomplish this, two metallic plates were fixed upright on the horizontal plate of the theodolite, having their edges exactly vertical, and precisely at opposite extremities of a diameter. Then, by turning the plate round on its axis, the passage allowed to the light between them could be increased or diminished at pleasure. The phenomena, then, were as follows. When the opening allowed to the light was considerable, as 0.02 or 0.04 inch (Paris,) the fringes were exactly similar to those observed when the edges were equidistant from the object-glass; but as the opening diminished, they ceased to be symmetrical on both sides of the middle line, those on the side of that edge of the aperture nearest to the telescope becoming broader than those on the other, which, on their part, undergo no sensible alteration. As the aperture contracts, this inequality increases, till at length the dilated fringes begin to disappear in succession, the outermost first, which they do by suddenly acquiring an extraordinary magnitude, so as to fill the whole field of the telescope, and thus, as it were, losing themselves. While these are thus vanishing, those on the other side remain quite unaltered till the last is gone, when they all disappear at once, which happens at the moment that the opening is reduced to nothing by the two edges covering each other.

744. Case of a small, circular aperture. When the aperture placed before the object-glass, instead of being a straight line, was a small, circular hole, and the aperture of the heliostat, in like manner, a minute circle, the phenomena of the rings were observed, and their diameters could be accurately measured by the micrometer. The results of these measurements led M. Fraunhofer to the following laws: 1st, that for apertures of different diameters, the diameters of the rings are inversely as those of the apertures forming them; 2dly, that the distances from the centre of the maxima of extreme red rays (or of rays of any given refrangibility) in the several rings of one and the same system, form an arithmetical progression, whose difference is somewhat less than its first term. Thus, calling γ the diameter of the aperture, and putting $L = \frac{0.0000214}{\gamma}$ and $l = \frac{0.0000257}{\gamma}$, he found $L' = l$, $L'' = l + L$, $L''' = l + 2L$, &c., where L' L'', &c. represent the angular semidiameters of the several rings expressed in arc of a circle to radius unity. The near coincidence of the value of L in this case, with that in the case of a linear aperture, and the small, but decided difference of the values of the first term of the progression in the two cases, are very remarkable.

745. Case of a very narrow, annular aperture. When the aperture was a very narrow, circular annulus, such as might be traced with a steel point on a gilt disc of glass, of whatever diameter, the image was a circular spot, surrounded in like manner by coloured rings, the diameters of which depended nowise on the *diameter*, but only on the *breadth* of the annulus, being in fact (as might be expected) the very same as the intervals between similar opposite fringes, on both sides of the central line in the image produced by a linear aperture of equal breadth.

746. Interference of many rays passing through gratings. But the most curious parts of M. Fraunhofer's investigations are those which relate to the interference of rays transmitted through a great many narrow apertures at once. When these apertures are exactly equal, and placed at exactly equal distances from one another, phenomena of a totally different kind from those originating in a single aperture are seen. In his first experiments of this kind he formed a *grating* of wire, by stretching a very fine wire across a frame, in the form of a narrow, rectangular parallelogram, whose shorter sides were screws tapped in the same die, and therefore precisely similar; across these screws in the consecutive intervals between their threads the wires were stretched, and of course could not be otherwise than parallel and equidistant. The diameter of the wire was 0.002021 Paris inch, the intervals between them each 0.003862, and the grating consisted of 260 such wires. When this apparatus was placed precisely vertical before the object-glass of his telescope, and illuminated by a narrow line of light 0.01 inch in breadth, also exactly vertical, forming the aperture of the heliostat, the image of this was seen in the telescope, colourless, well defined, and in all respects precisely as it would have been seen without the interposition of any grate or aperture at all, occupying the centre of the field, only less bright. On either side of this was a space perfectly dark, after which succeeded a series of prismatic spectra, which he calls spectra of the second class, not consisting of tints melting into each other, according to the law of the coloured rings, or any similar succession of hues depending on a regular degradation of light, but of perfectly *homogeneous* colours; so much so, as to exhibit the same dark lines crossing them as exist in the purest and best defined prismatic spectrum. In the disposition of things already described, the first, or nearer spectrum is completely insulated, the space between it and the central image, as well as between it and the second spectrum, being quite dark. The violet ends of the spectra are inwards, and the red outwards; but the violet end of the third spectrum is superposed on the red end of the second, so as in place of a dark interval to produce a purple space; and as we proceed farther from the middle, the spectra become more and more confounded, but not less than thirteen may easily be counted on each side by the aid of a prism refracting them transversely, so as to separate their overlapping portions.

Spectra of the second class.

747. Ratio of the coloured spaces in them. The measurement of the distances of similar points in the several spectra are rendered susceptible of the utmost precision by means of the dark lines which cross them. A very remarkable peculiarity of these spectra must, however, be here noticed, *viz.* that although the dark lines hold exactly the same places in the order of colours, or, in other words, correspond to precisely the same *degrees of refrangibility*, as in the prismatic spectra formed by refraction, yet the ratio of the intervals between them, or the breadths of the several coloured spaces, differ entirely in the two cases. Thus, in the diffracted spectra, the interval between the lines C and D (fig. 94) is very nearly double of that between G and H, while in a spectrum formed by a flint-glass prism of an angle of 27°, the proportion is reversed, and in a water prism of the same angle C D : G H :: 2 : 3.

748. Their laws. In the diffracted fringes formed by a single aperture, their distances (as we have seen) from the axis depends only on the breadth of the aperture, being inversely as that breadth. In the spectra formed by a great number, their distances from the central image depends neither on the breadths of the apertures nor on the intervals between them, but on the sum of these quantities, that is, on the distances between the middle points of the consecutive apertures, (or, in the case before us, on the distances between the axes of the wires.) By a series of measures performed with the utmost care and precision on wire gratings of a great variety of dimensions, M. Fraunhofer ascertained the following laws and numerical values.

749. 1. For different gratings, if we call γ the breadth of each of the interstices through which the light passes, and δ that of each of the opaque intervals between them, the magnitudes of spectra of the same order, and the distances of similar points in them from the axis, is inversely as the sum $\gamma + \delta$.

750. 2. The distances of similar points, (*i. e.* of similar colours or similar fixed lines,) in the several consecutive spectra formed by one and the same grating from the axis, constitute an arithmetical progression whose difference is equal to its first term.

751. 3. For the several refrangibilities corresponding to the fixed lines B, C, D, E, &c. the first term of this progression is numerically represented by the respective fractions which follow, being the lengths of the arcs, or their sines to radius unity.

$$B = \frac{0.00002541}{\gamma + \delta}; \quad E = \frac{0.00001945}{\gamma + \delta}; \quad H = \frac{0.00001464}{\gamma + \delta}.$$

$$C = \frac{0.00002422}{\gamma + \delta}; \quad F = \frac{0.00001794}{\gamma + \delta}; \text{ &c.}$$

$$D = \frac{0.00002175}{\gamma + \delta}; \quad G = \frac{0.00001587}{\gamma + \delta};$$

752. Case of extremely close gratings. Methods of constructing them.

These results were all, however, deduced from gratings so coarse as to allow of our regarding the angles of diffraction as proportional to their sines; but when extremely fine gratings are employed, the spectra are formed at great distances from the axis, and the analogy of other similar cases, as well as theory, would lead us to substitute sin B, sin C, sin D, &c. in the place of B, C, D, &c. This, M. Fraunhofer found by experiment to be really the case. The construction of gratings proper for these delicate purposes, however, was no easy matter. Those employed by him were nothing more than a system of parallel and equidistant lines ruled on plates of glass covered with gold-leaf, or with the thinnest possible film of *grease;* by the former of these methods he found, that the proximity of the lines might be carried to the extent of placing about a thousand in the inch, but when he would draw them still closer, the whole of the gold-leaf was scraped off. When the surface was covered with a film of grease so thin as to be almost imperceptible to the sight, (although the intervals were in this case transparent,) no change was produced in the optical phenomena, so far as the spectra were concerned, only the brightness of the central image being increased. By this means he was enabled to obtain a system of parallel lines at not more than half the distance from each other that could be produced on gold-leaf: but beyond this degree of proximity, he found it impossible to carry the ruling of equidistant lines on any film of grease or varnish. But this being still far short of his wishes, he had recourse to actual engraving with a diamond point on the surface of the glass itself, and by this means was enabled to rule lines so fine as to be absolutely invisible under the most powerful compound microscope, and so close that 30,000 of them lie in a single Paris inch. When so excessively near, however, no accuracy of machinery will ensure that perfect equidistance which is essential to the production of the spectra now under consideration, and he found it impossible to succeed in placing them nearer than 0.0001223, (or about 8200 to the inch,) with such a degree of precision as to enable him to distinguish the fixed lines in the spectra; and, if it be considered, that a deviation to the extent of the hundredth part of the just interval frequently occurring, is sufficient to obliterate these, and that to produce the spectra in sufficient brightness to affect the eye, some hundreds or even thousands must be ruled, we shall be enabled to form some conception of the difficulties to be encountered in researches of this kind. For a detail of some of these, and of the methods employed by him to count their number and measure their distances, we must refer to his original Memoir, (read to the Royal Bavarian Academy of Sciences, June 14, 1823.)

753. The spectra modified by the form of the striæ in the gratings.

In the course of these researches, M. Fraunhofer met with a very singular and instructive peculiarity in one of the engraved glass-gratings used by him; which, although it produced spectra equidistant on either side of the axis, yet gave always those on one side a much greater degree of brightness than those on the other. Attributing this to the *form* of the furrows being sharper terminated on one side than on the other, owing either to the figure of the diamond point or the manner of its application, he endeavoured to produce a similar structure of the striæ in a film of grease spread on glass, by purposely applying the engraving tool obliquely, and the attempt proved successful.

754. Case of inclined gratings. Unsymmetrical spectra of the second class.

When the incident rays from the opening in the heliostat fell obliquely on the grating, it might be supposed that the phenomena would be the same as those exhibited by a closer grating, having intervals less in proportion of the cosine of the angle of incidence to radius. But the analogy of the unsymmetrical fringes produced by a single aperture, whose sides lie in a plane oblique to the incident ray, may lead us to expect a different result, and experiment confirms the surmise; thus, M. Fraunhofer found, that on inclining a grating, whose intervals $(\gamma + \delta)$ were each equal to 0.0001223 inch, so as to make the angle of incidence 55° with the perpendicular, the distance of the first fixed line D from the axis on the one side of the axis was 15° 6′, and on the other no less than 30° 33′, or more than double.

755. Theoretical considerations.

The facts deduced by M. Fraunhofer in the above detailed researches are certainly extremely curious. The most interesting and remarkable point about them is the perfect homogeneity of colour in the spectra, indicating a saltus, or breach of continuity, in the law of intensity of each particular coloured ray in the diffracted beam. For it is obvious, that taking any one refrangibility (that corresponding to the fixed line C, for example,) the expression of its intensity in functions of its distance from the axis must be (analytically speaking) of such a nature as to vanish completely for every value of that distance, excepting for a certain series in arithmetical progression, or, as it is called, a *discontinuous* function; so that the curve representing such value, having the distance from the axis for its abscissa, must be a series of points arranged above the axis at equal intervals; or, at least, a curve of the figure represented in fig. 151, in which certain extremely narrow portions, equidistantly arranged, start up to considerable distances from the axis, while all the intermediate portions lie so close to that line as to be confounded with it. The manner in which such a function can be supposed to originate from the summation of a series of the values of $\int d\nu \cdot \sin \frac{\pi}{2} \nu^2$ and $\int d\nu \cdot \cos \frac{\pi}{2} \nu^2$, (Art. 718,) taken successively between limits corresponding to the boundaries of the several interstices, involves too many complicated considerations to enter into in this place. M. Fraunhofer, meanwhile, states the following general expression, as the result of his own investigations founded on the principle of interferences. Let n indicate the order of any

Fraunhofer's formula.

spectrum, reckoned from the axis; ϵ the distance from the middle of one interstice to that of the adjacent one $= \gamma + \delta$; λ the length of an undulation of an homogeneous ray; σ the angle of incidence of the ray from the luminous point on the grating; and y the length of a perpendicular let fall from the micrometer thread of the telescope, (or from the point in the focus of its object-glass, where that particular homogeneous ray in that spectrum is found,) on the plane of the grating. Then, if the angular elongation of that ray from the axis be called $\theta^{(n)}$, we shall have, in general,

$$\text{cotan}\,\theta^{(n)} = \frac{\sqrt{\{\epsilon^2 - (\epsilon \,.\, \sin\sigma + n\lambda)^2\} \,.\, \{4y^2 + \epsilon^2 - (\epsilon \,.\, \sin\sigma + n\lambda)^2\}}}{2y\,(\epsilon \,.\, \sin\sigma + n\lambda)}.$$

In this equation, n is to be regarded as $+$ for the spectra which lie on the side of the axis on which the incident ray makes an obtuse angle with the plane of the grating, and negative for the spectra on the other side. This formula he states to be rigorous, and independent of any approximation. When y is very great (as it, in fact, always is,) compared with ϵ and λ, this reduces itself simply to

$$\text{cotan}\,\theta^{(n)} = \frac{\sqrt{\epsilon^2 - (\epsilon \,.\, \sin\sigma + n\lambda)^2}}{\epsilon \,.\, \sin\sigma + n\lambda}, \text{ or } \sin\theta^{(n)} = \frac{\epsilon \,.\, \sin\sigma + n\lambda}{\epsilon}.$$

756. Lengths of undulations of the rays B, C, D, &c. assigned by Fraunhofer.

This formula, applied to M. Fraunhofer's measures of the distances of the same fixed lines in successive spectra on either side of the axis, in the case of inclined gratings, represents them with perfect exactness. When the gratings are perpendicular to the ray $\sigma = 0$, and the equation becomes $\sin t^{(n)} = \frac{n\lambda}{\epsilon}$, which is the law before noticed for symmetrical spectra. And hence, too, it appears that the values of λ, or the lengths of the undulations for the several rays designated by C, D, E, &c., are no other than the numerators of the fractions in Art. 751, expressed in parts of a Paris inch, which thus become data of the utmost value in the theory of light, from the great care and precision with which they have been fixed, and for the possibility of identifying them at all times.

757. Diffracted spectra produced by reflexion.

If the unruled surface of the glass grating be covered with black varnish, and the light reflected from the ruled surface be received in the telescope, the very same phenomena are seen as if the light had been transmitted through the glass, and the same analytical expression, according to M. Fraunhofer, applies to both cases.

758. Alleged limit to the powers of microscopes.

A curious consequence of this expression is, that if ϵ, the distance between the lines, be less than λ, and the light fall perpendicularly on the grating, so that $\sin\sigma = 0$, we shall have $\sin\theta^{(n)} > 1$, and therefore $\theta^{(n)}$ imaginary. It appears, therefore, that lines drawn on a surface distant from each other by a less quantity than one undulation of a ray of light, produce no coloured spectra. Hence, such scratches, or inequalities, on polished surfaces, have no effect in disturbing the regularity of reflexion or refraction, and produce no dimness or mistiness in the image; if less distant from each other than this limit. M. Fraunhofer seems inclined to conclude further, that an object of less linear magnitude than λ can in consequence never be discerned by microscopes, as consisting of parts: a conclusion which would put a natural limit to the magnifying power of microscopes, but which we cannot regard as following from the premises.

759. Spectra produced by composite gratings.

When the intervals of the parallel interstices are unequal, and disposed with no regularity, the light of the diffracted spectra of different combinations is confounded together, and a white misty streak at right angles to the direction of the lines arises; but when they are *regularly unequal*, so that the same intervals recur in regular periods, if we call E ($= \epsilon' + \epsilon'' + \epsilon''' + $ &c.) the interval between any two distant by a whole period, we shall have, for the law of the lateral spectra, the equation $\sin\theta^{(n)} = \frac{n\lambda}{E}$. And the spectra so formed, are still observed to consist of homogeneous light, exhibiting the fixed lines with great distinctness. A very curious, and, as far as concerns the practical measurement of the phenomena, useful observation has been made by M. Fraunhofer on the spectra so formed by these *composite* gratings, *viz.* that although they follow the same law in respect of their distances from the axis, yet the successive spectra differ greatly in intensity, some being so faint as to be scarce perceptible, while the immediately adjacent ones will often be very intense. Owing to this cause, spectra of the higher orders, which in a simple grating the interval of whose interstices is represented by E, are confused and obliterated by the encroachment of those adjacent, are often very distinct when formed by a *composite* grating, the period of recurrence of whose similar interstices is $E = \epsilon' + \epsilon'' + \epsilon''' + $ &c. Thus, M. Fraunhofer was never able through a simple grating to see the fixed lines C and F in the spectrum of the 12th order, reckoning from the axis, while in a composite grating, consisting of three systems of lines continually repeated, whose intervals ϵ', ϵ'', ϵ''' were to each other as 25 : 33 : 42, these fixed lines as well as the lines D and E, were distinctly seen in the 12th spectrum, owing to the almost total disappearance of the 10th and 11th. Nay, even the fixed line E in the 24th spectrum could be seen, and its distance from the axis measured with this grating.

Singular phenomenon noticed by Fraunhofer respecting the intensity of the spectra.

760. Various stages of the phenomena. Spectra of the first class

Such are the extreme cases of the phenomena as produced by a single aperture, and by an infinite, or, at least, very great number; but the intermediate steps and gradations by which one set of phenomena pass into the other, remain to be traced. When a single interstice is left open in a grating, the spectra are formed as described in Art. 741. These, M. Fraunhofer calls *spectra of the first class*, and their colours are not homogeneous, but graduate into one another.

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When two contiguous interstices are left open, the spectra of the first class appear as before; but between the axis and the first spectrum on either side appear other spectra, which M. Fraunhofer terms imperfect spectra of the second class, their colours being similar to those of the first class, and no fixed lines being visible in them.

When three adjacent interstices are left open, a third set of spectra, or *spectra of the third class*, are formed between the axis and the nearest of the imperfect spectra of the second class. Besides these, no new *classes* of spectra arise by a further increase of the number of interstices; but these undergo a series of modifications as the interstices grow more numerous. These are chiefly as follows:

Spectra of the third class.

762. Modifications of these spectra by increasing the number of interfering rays. Formula for spectra of third class.

The spectra of the third class grow narrower, and approach the axis, till at last they run together and form by their union the colourless, well-defined image of the opening of the heliostat in the axis of the whole phenomenon. By a series of exact measurements, M. Fraunhofer found their breadths to be inversely as the number of interstices by which they are produced in the same grating, and inversely as the intervals of the interstices for different ones; and in general, that $\gamma + \delta = \epsilon$ representing this interval, m the number of interstices used, and n the *order* of the spectrum, $\theta^{(n)}$ the distance of extremity of the red rays in that spectrum is given by the equation

$$\theta^{(n)} = \frac{n}{m} \times \frac{0.0000208}{\epsilon}.$$

763. Transition from imperfect to perfect spectra of second class.

As the spectra of the third class contract into the axis, they leave a dark space between it and the first spectrum of the second class. This and the other spectra of that class meanwhile grow continually more vivid and homogeneous in respect of colour; till at length, when the number of interfering rays is very much increased, the fixed lines begin to appear in them, and they acquire the character of *perfect* spectra of the second class.

M. Fraunhofer next examined the phenomena produced by immersing in media of different refractive powers the gratings used, when he found all the phenomena precisely similar; but the distances at which the several spectra were formed from the axis, to be less than when in air, in the inverse ratio of the refractive indices.

764. Phenomena of gratings immersed in fluids.

A very beautiful and splendid class of optical phenomena has been investigated and described by M. Fraunhofer, which arise by substituting for the gratings used in the above experiments very small apertures of regular figures, such as circles and squares, either singly or arranged in regular forms, in great numbers; as, for instance, when two equal wire gratings are crossed at right angles. Fig. 151 is a representation of the phenomenon produced when the light is received on the object-glass of the telescope through two circular holes of the diameter 0.02227 inch, placed at a distance of 0.03831 inch centre from centre. Each compartment is a separate spectrum. In the bands *a a*, *b b* we see here plainly the origin and minute structure of the vertical and crossed fringes described in Art. 735. The appearances vary as the number of apertures is increased, the spectra growing purer and more vivid. That which arises when two equal wire gratings are crossed, is figured in M. Fraunhofer's work, and is one of the most magnificent phenomena in Optics.

765. Substitution of very minute apertures for gratings.

766. Rings seen about the stars in telescopes.

When we look at a bright star through a very good telescope with a low magnifying power, its appearance is that of a condensed, brilliant mass of light, of which it is impossible to discern the shape for the brightness; and which, let the goodness of the telescope be what it will, is seldom free from some small ragged appendages or rays. But when we apply a magnifying power from 200 to 300 or 400, the star is then seen (in favourable circumstances of tranquil atmosphere, uniform temperature, &c.) as a perfectly round, well-defined planetary disc, surrounded by two, three, or more alternately dark and bright rings, which, if examined attentively, are seen to be slightly coloured at their borders. They succeed each other nearly at equal intervals round the central disc, and are usually much better seen and more regularly and perfectly formed in refracting than in reflecting telescopes. The central disc, too, is much larger in the former than in the latter description of telescope.

767. Spurious discs of the stars.

These discs were first noticed by Sir William Herschel, who first applied sufficiently high magnifying powers to telescopes to render them visible. They are not the real bodies of the stars, which are infinitely too remote to be ever visible with any magnifiers we can apply; but *spurious*, or unreal images, resulting from optical causes, which are still to a certain degree obscure. It is evident, indeed, to any one who has entered into what we have said of the law of interferences, and from the explanation given in Art. 590 and 591 of the formation of *foci* on the undulatory system, that (supposing the mirror or object-glass rigorously aplanatic) the focal point in the axis will be agitated with the united undulations, in complete accordance, from every part of the surface, and must, of course, appear intensely luminous; but that as we recede from the focus in any direction in a plane at right angles to the axis, this accordance will no longer take place, but the rays from one side of the object-glass will begin to interfere with and destroy those from the other, so that at a certain distance the opposition will be total, and a dark ring will arise, which, for the same reason, will be succeeded by a bright one, and so on. Thus the origin both of the central disc and the rings is obvious, though to calculate their magnitude from the data may be difficult. But this gives no account of one of the most remarkable peculiarities in this phenomenon, *viz.* that the apparent size of the disc is different for different stars, being uniformly larger the brighter the star. This cannot be a mere illusion of judgment; because when two unequally bright stars are seen at once, as in the case of a close double star, so as to be directly compared, the inequality of their spurious diameters is striking; nor can it be owing to any real difference in the stars, as the intervention of a cloud, which reduces their brightness, reduces also their apparent discs till they become mere points. Nor can it be attributed to irradiation, or propagation of the impression from the point on the retina to a distance, as in that case the light of the central disc would encroach on the rings, and obliterate them; unless, indeed, we suppose the vibrations of the retina to be performed according to the same laws as those of the ether, and to be capable of interfering with them; in which case, the disc and rings seen on the retina will be a resultant system, originating from the interference of both species of undulations.

Explanation of the rings on the principle of interferences.

768. Phenomena produced by apertures of various figures.

Not to enter further, however, on this very delicate question, we shall content ourselves with stating some of the phenonena we have observed, as produced by diaphragms, or apertures of various shapes variously applied to mirrors and object-glasses, and which form no inapt supplement to the curious observations of Fraunhofer on the effect of *very minute* apertures, of which they are in some sort the converse.

769. Circular apertures. When the whole aperture of a telescope is limited by a *circular* diaphragm, whether applied near to, or at a distance from, the mirror or object-glass, the disc and rings enlarge in the inverse proportion of the diameter of the aperture. When the aperture was much reduced (as to one inch, for a telescope of 7 feet focal length) the spurious disc was enlarged to a planetary appearance, being well defined, and surrounded by one ring only, strong enough to be clearly perceived, and faintly tinged with colour in the following order, reckoning from the centre of the disc. White, very faint red, black, very faint blue, white, extremely faint red, black. When the aperture was reduced still farther (as to half an inch) the rings were too faint to be seen, and the disc was enlarged to a great size, the graduation of light from its centre to the circumference being now very visible, giving it a hazy and cometic appearance, as in fig. 152.

Fig. 152.

770. Annular apertures. When annular apertures were used the phenomena were extremely striking, and of great regularity. The exterior diameter of the annulus being three inches, and the interior $1\frac{1}{4}$, the appearance of Capella was as in fig. 153, and of the double star Castor, as in 154. As the breadth of the annulus is diminished, the size of the disc and breadth of the rings diminish also, (contrary to what took place in Fraunhofer's experiments with extremely narrow annuli, and obviously referring the present phenomena to different principles,) at the same time the number of visible rings increases. Fig. 155, 156, and 157 exhibit the appearance of Capella with annular apertures of 5.5 inch $-$ 5 inch (*i. e.* whose exterior diameter $=$ 5.5 and interior $=$ 5) of $0.7-0.5$, of $2.2-2.0$. In the last case the disc was reduced to a hardly perceptible round point, and the rings were so close and numerous as scarcely to admit being counted, giving, on an inattentive view, the impression of a mere circular blot of light. When the breadth of the annulus was reduced to half this quantity, the intervals between the rings could no longer be discerned. The dimensions of the rings and disc, generally, seem to be proportional to $\frac{r'-r}{r}$.

Fig. 153 to 157.

771. Another set of rings. Fig. 158 and 159. Besides the rings immediately close to the central disc, however, others of much greater diameter and fainter light, like halos, are seen with annular apertures, which belong (in Fraunhofer's language) to spectra of a different class. With a single annulus they are too faint to be distinctly examined, but with an aperture composed of two annuli, as in fig. 158, they are very distinct and striking, presenting the phenomenon in fig. 159, (in which it is to be understood that light is represented in the engraving by darkness, and darkness by light.)

772. Image produced by a triangular aperture. When the aperture was in the form of an equilateral triangle, the phenomenon was extremely beautiful; it consisted of a perfectly regular, brilliant, six-rayed star, surrounding a well-defined circular disc of great brightness. The rays do not unite to the disc, but are separated from it by a black ring. They are very narrow, and perfectly straight; and appear particularly distinct in consequence of *the total destruction of all the diffused* light which fills the field when no diaphragm is used; a remarkable effect, and much more than in the mere proportion of the light stopped. Fig. 160 is a representation of this elegant appearance. The same arises when, in place of an equilateral triangle, the aperture is the difference of two concentric, equilateral triangles similarly situated.

Fig. 160.

773. When out of focus. As a triangle has but three sides and three angles, it seems singular that a *six*-rayed star should be produced. Supposing three to arise from the angles, and three from the sides, it might be expected that some sensible difference should exist in the alternate rays, marking their different origin. When the telescope is in perfect focus, however, all the rays are precisely alike; but if thrown out of focus, their difference of origin becomes apparent. Fig. 161 represents the phenomenon then seen, in which the alternate branches are seen to consist of a series of fringes parallel to their length, and the others of small arcs of similar fringes immediately adjacent to the vertices of the hyperbolas to which they belong, and which consequently cross the rays in a direction perpendicular to their length. As the telescope is brought better in focus, the hyperbolas approach their asymptotes, and are confounded together in undistinguishable proximity; and thus three rays arise composed of continuous lines of light, and three intermediate ones composed of an infinite number of discontinuous points placed infinitely near each other. To represent analytically the intensity of the light in one of these *discontinuous* rays would call for the use of functions of a very singular nature and delicate management.

Fig. 161.

774. Application to the construction of a position-micrometer. The phenomenon just described affords in certain cases a very perfect position-micrometer for astronomical uses. If the diaphragm be turned round, the rays turn with it; and if a brilliant star (as α Aquilæ) have near it a very small one, the diaphragm may be so placed as to make one of the rays pass through the small star, which thus remains like a bead threaded on a string, and may be examined at leisure. If then the position of the diaphragm be read off on a graduation properly contrived, the relative situations of the two stars become known. We have satisfied ourselves by trial of the practicability of this; and by proper contrivances the principle may be made available in cases which at first sight appear to present considerable difficulties.

775. Three circular apertures. Fig. 162. Fig. 163. Fig. 164. When three circular apertures, having their centres at the angles of an equilateral triangle, were used, the image consisted of a bright central disc. Six fainter ones in contact with it, and a system of very faint halo-like rings surrounding the whole as in fig. 162. When, however, three equal and similar *annular* apertures were thus disposed, the appearance when *in focus* was as in fig. 153, being exactly the same as if two of them were closed. But when thrown a little out of focus, the difference was perceived. Fig. 163 represents the appearance in this case, each of the apertures then produces its own central disc and system of rings, whose intersections give rise to the system of intersectional fringes there depicted. As the telescope is brought better in focus these disappear, and the phenomenon is as in fig. 164; the centres gradually approaching, and the rings blending till the point of complete coincidence is attained.

776. An aperture in the form of the difference between two concentric squares produced not an eight, but a four-rayed star. The rays, however, were not, as in the case of the triangular aperture, uninterrupted fine lines, gradually tapering away from the centre to their extremities, but composed of distinct alternating obscure and

Light. bright portions, as represented in fig. 165. The portions nearest the central disc (which is circular) were composed of bands transverse to the direction of the rays, and tinged with prismatic colour. Similar bands, no doubt, existed in the more distant portions, which extended to a great length.

Part III. Square apertures. Fig. 165.

An aperture consisting of fifty squares, each of about half an inch in the side, regularly disposed at intervals so as to leave spaces between them in both directions equal in breadth to the side of each, produced an image totally different from that described by Fraunhofer as resulting from the crossing of two equal very close gratings, though the distribution and shape of the apertures were the same in both cases. It was as represented in fig. 166, consisting of a white, round, central disc, surrounded by eight vivid spectra, disposed in the circumference of a square, beyond which were arranged in the shape of a cross, triple lines of very faint spectra extending to a great distance.

777. Effect of very numerous square apertures Fig. 166.

When the aperture consisted of numerous equilateral triangles regularly disposed, as in fig. 167, the image presented the very beautiful phenomenon represented in fig. 168, consisting of a series of circular discs arranged in six diverging rays from the central one, and each surrounded with a ring. The central disc was colourless and bright; the rest more and more strongly coloured and elongated into spectra, according to their degree of remoteness from the centre. These are only a few of the curious and beautiful phenomena depending on the figures of the apertures of telescopes, which afford a wide field of further inquiry, and one at least as interesting to the artist as to the philosopher.

778. Fig. 167.

PART IV.

OF THE AFFECTIONS OF POLARIZED LIGHT.

§ I. *Of Double Refraction.*

Exceptions to the law of ordinary refraction numerous.

779. When a ray of light is incident on the surface of a transparent medium, a portion of it is reflected, at an angle equal to that of incidence, another small portion (so small, however, that we shall neglect its consideration) is dispersed in all directions, serving to render the surface visible, and the rest enters the medium and is refracted. The *law* of refraction, or the rule which regulates the path of this portion within the medium, has been explained in the preceding parts; and no exceptions to it, as a general law, have hitherto been noticed. It is, however, very far from general; and, in fact, obtains only where the refracting medium belongs to one or other of the following classes, *viz.*

Classes of bodies in which it holds.

Class 1. Gases and vapours.

2. Fluids.

3. Bodies solidified from the fluid state too suddenly to allow of the regular crystalline arrangement of their particles, such as glass, jellies, &c., gums, resins, &c., being chiefly such as in the act of cooling pass through the viscous state.

4. Crystallized bodies, having the cube, the regular octohedron, or the rhomboidal dodecahedron for their primitive form, or which belong to the *tessular* system of Mohs. A very few exceptions (probably only apparent ones, arising from our imperfect knowledge of crystallography) exist to the generality of this class.

The solid bodies belonging to these classes, moreover, cease to belong to them when forcibly compressed or dilated, either by mechanical violence, or by the unequal action of heat or cold, which brings their particles into a state of strain, such as in extreme cases to produce their disruption, as is familiarly seen in the cracking of a piece of glass by heat too suddenly and partially applied. The class of fluids too admits some exceptions, at least when very minutely considered; but the deviation from the ordinary law of refraction in these cases is of so microscopic a kind, that we shall at present neglect to regard it.

Double refraction.

780. All other bodies, comprehending all crystallized media, such as salts, gems, and crystallized minerals, not belonging to the system above mentioned; all animal and vegetable bodies in which there is any disposition to a regular arrangement of molecules, such as horn, mother of pearl, quill, &c.; and, in general, all solids when in a state of unequal compression or dilatation, act on the intromitted light according to very different laws, dividing the refracted portion *into two distinct pencils*, each of which pursues a rectilinear course so long as it continues within the medium, according to its own peculiar laws, but without further subdivision. This phenomenon is termed double refraction. It is best and most familiarly seen in the mineral termed Iceland spar, which is, in fact, carbonate of lime in a regular crystalline form. This is generally obtained in oblique parallelepipeds, easily reduced by cleavage to regular, obtuse rhomboids, and is not uncommonly met with in a state of limpid transparency, on which account, as well as by reason of its remarkable optical properties, it easily attracted attention. Bartholinus, in 1669, appears to have been the first to give any account of its double refraction, which was afterwards more minutely examined by Huygens, the first proposer of the undulatory theory of light, whose researches on this phenomenon form an epoch in the history of Physical Optics little if at all less important than the great discovery of the different refrangibility of the coloured rays by Newton. To Huygens we owe the discovery of the law of double refraction in this species of medium. Newton, misled by some inaccurate measurements, (a thing most unusual with him,) proposed a different one; but the conclusions of Huygens, long and unaccountably lost sight of, were at length established by unequivocal experiments by Dr. Wollaston, since which time a new impulse has been given to this department of Optics; and the successive labours of Laplace. Malus, Brewster, Biot, Arago, and Fresnel present a picture of emulous and successful research, than which nothing prouder has adorned the annals of physical science since the developement of the true system of the universe. To enter, however, into the history of these discoveries, or to assign the share of honour which each illustrious labourer has reaped in this ample field forms no part of our plan. Of the splendid constellation of great names just enumerated, we admire the living and revere the dead far too warmly and too deeply to suffer us to sit in judgment on their respective claims to priority in this or that particular discovery; to balance the mathematical skill of one against the experimental dexterity of another, or the philosophical acumen of a third. So long as "one star differs from another in glory,"—so long as there shall exist varieties, or even incompatibilities of excellence,—so long will the admiration of mankind be found sufficient for all who truly merit it. Waving, then, all reference to the history of the subject, except in the way of incidental remark, or where the necessity of the case renders it unavoidable, we shall present the reader with as

systematic an account as we are able, of the present state of knowledge with respect to the laws and theory of Double Refraction. The Huygenian law having been demonstrated to apply rigorously to the case for which he himself devised it, as well as to a very large class of other bodies, we shall begin with that class, and proceed afterwards to consider more complicated cases.

781. Axes of double refraction.

In all crystallized bodies, then, which possess double refraction, it is found that that portion of a ray of ordinary light incident on any natural or artificially polished surface which enters the body is separated into *two equal pencils* which pursue rectilinear paths, making with each other an angle not of constant magnitude, but varying according to the position which the incident ray holds with respect to the surface, and to certain fixed lines, or axes within the crystal, and which lines are related in an invariable manner to the planes of cleavage, or other fixed planes or lines in the primitive form of the crystal. Now, it is found that in every crystal there is at least one such fixed line, along which if one of these two pencils be transmitted the other is so also, so that in this case the two pencils coincide, the angle between them vanishing. Moreover, no crystal has yet been discovered in which more than two such lines exist. These lines are called the optic axes. All double refracting crystals, then, at present, may be divided into such as have one, and such as have two, optic axes.

782. Rays moving along the axes suffer ordinary refraction only.

When a ray penetrates the surface of a crystal so as to be transmitted undivided along the optic axis; or when, moving within the crystal along that line, it meets the surface and passes out, whatever be the inclination of the surface, its refraction is always performed according to the ordinary law of the proportional sines. Thus, in this particular case, the crystal acts precisely as an uncrystallized medium, (some rare instances excepted, of which more hereafter.)

But in all other cases the law is essentially different, and (for one portion of the divided pencil, at least) of a very singular and complicated nature. This we shall first proceed to explain in the simpler case of crystals with one optic axis. But, first, we must explain somewhat more distinctly, what we mean by axes and fixed lines within a crystal. Suppose a mass of brickwork, or masonry, of great magnitude, built of bricks, all laid parallel to each other. Its exterior form may be what we please; a cube, a pyramid, or any other figure. We may cut it (when hardened into a compact mass) into any shape, a sphere, a cone, or cylinder, &c.; but the edges of the bricks within it lie still parallel to each other; and their directions, as well as those of the diagonals of their surfaces, or of their solid figures, may all be regarded as so many axes, *i. e.* lines having (so long as the mass remains at rest) a determinate position, or rather *direction* in space, no way related to the exterior surfaces, or linear boundaries of the mass, which may cut across the edges of the bricks in any angles we please. Whenever, then, we speak of fixed lines, or axes *of*, or *within*, a crystal, we always mean directions in space parallel to each of a system of lines drawn in the several elementary molecules of the crystal, according to given geometrical laws, and related in a given manner to the sides and angles of the molecules themselves. We must conceive the axis, then, of a crystallized mass not as a single line having a given *place*, but as any line whatever having a given *direction* in space, *i. e.* parallel to the *axis of each molecule*, which is a line having a determinate place and position within it.

783. What is meant by axes and fixed lines within a crystal.

784.

In the remainder of this section, when we speak of the *axis* or *axes* of a crystallized mass or surface generally, we mean the direction of the optic axis or axes of its molecules, or of a crystal similar and similarly situated to any one of them.

Of the Law of Double Refraction in Crystals with One Optic Axis.

785. Enumeration of crystals having a single axis in classes.

This class of crystals comprises all such as belong to Mohs's rhombohedral system, or which have the acute or obtuse rhomboid, or regular six-sided prism, for their primitive form, as well as all which belong to his pyramidal system, or whose primitive form is either the octohedron with a square base, the right prism with a square base, or the bi-pyramidal dodecahedron. All such crystals Dr. Brewster has shown to have but one axis, which is that to which the primitive form is symmetrical, *viz.* in the rhomboid, the axis of the figure, or line joining the two angles formed by three equal plane angles; in the hexagonal prism, the geometrical axis of the prism; in the octohedron, or square based prism, a line drawn through the centre of the base at right angles to it. The cases in accordance with the rule are so numerous, and the exceptions, once believed to be so, have so often disappeared on the attainment of a more perfect knowledge of the crystalline forms of the excepted minerals, that when any case of disagreement seems to occur, we are justified in attributing it rather to our own incorrect determination of this datum, than to want of generality in the rule itself.

786. Refraction of the ordinary ray in this class of crystals.

In all crystals of this class, one of the two equal pencils into which the refracted ray is divided follows the ordinary law of Snellius and Descartes, having a constant index of refraction (μ), or invariable ratio of the sine of incidence to that of refraction, whatever be the inclination of the surface by which it enters; so that its velocity within the medium, when once entered, is the same in whatever direction it traverses the molecules; and with respect to this ray the crystal comports itself as an uncrystallized medium. This, then, is called the *ordinary* pencil.

787. Huygens's law for the velocity of the extraordinary ray.

To understand the law obeyed by the other, or *extraordinary* portion of the divided pencil, let us consider it as fairly immersed in the medium, and pursuing its course among the molecules. Then its velocity will not, as in the case of the ordinary ray, be the same in whatever direction it traverses them, but will depend on the angle it makes with the axis; being a minimum when its path within the crystal is parallel to the axis, and a maximum when at right angles to it, or *vice versâ;* and in all intermediate inclinations of an intermediate magnitude according to the following law. Let an ellipsoid of revolution, either oblate or prolate, as the case

may be, be conceived, having its axis of revolution coincident in direction with the axis of the crystal, and its polar to its equatorial radius in the ratio of the minimum and maximum velocities above mentioned, *i. e.* as the velocity of a ray moving parallel to that of one perpendicular to the axis. Then in all intermediate positions, the radius of this spheroid parallel to the ray will represent its velocity on the same scale that its polar and equatorial radii represent the velocities in their respective directions.

788. Its connection with the law of extraordinary refraction.

This is the Huygenian law of velocities, in its most simple and general form. It does not at first sight appear what this has to do with the law of extraordinary refraction; but the reader who has considered with the requisite attention what has been said in Art. 539, 540, with prospective reference to this very case, will easily perceive that, the law of velocity of the ray within the medium once established, it becomes a mere matter of pure Geometry to deduce from it the law of extraordinary refraction, whether we adopt the Corpuscular theory, and employ Laplace's principle of least action, as in that Article; or whether, preferring the Undulatory hypothesis, we substitute for this principle the equivalent one of swiftest propagation, as explained in Art. 587, 588. We should observe, however, that the Huygenian law, as just stated, is worded in conformity with the undulatory doctrine, in which the velocity in a denser medium is supposed *slower* than in a rarer. But when we use the principle of least action, we must invert the use of the word, or, which comes to the same thing, suppose the the velocity in the medium to be *inversely* proportional to the radius of the ellipsoid. The results being necessarily the same in both cases, we shall use at present the language of the Corpuscular system.

789. Investigation of the latter from the former law.

Retaining, then, the notation of Art. 540, the law of refraction will be derived from the equation $V \cdot S + V' \cdot S =$ a minimum, where V is the velocity *without*, and V′ that *within* the medium, and where S and S′ are the spaces described without and within it, in the passage of a ray from point to point. Let a and b be the polar and equatorial semiaxes of the ellipsoid above spoken of, (which we shall call the ellipsoid of double refraction,) and let α, β, γ be the coordinates of the point (A) without the crystal, and α', β', γ' those of one (B) within it, through which the ray is supposed to pass, and x, y, z the coordinates of a point in the surface of the crystal, on which it must be incident, so as to be capable of passing from A to B in the manner required by the law of extraordinary refraction; and let ϕ be the angle which the interior portion S′ makes with the axis of the crystal.

Expression for the radius of the spheroid of refraction.

Then will the radius of the spheroid parallel to this portion (by conic sections) be expressed by

$$r = \frac{a\,b}{\sqrt{b^2 \cdot \sin\phi^2 + a^2 \cos\phi^2}} = \frac{a\,b}{W}; \quad (1)$$

where a is the equatorial, and b the polar radius of the spheroid. Now, if we take μ to represent the index of ordinary refraction, since we have, generally, $V' = \frac{\text{const}}{r}$; and since, when $r = b$ the extraordinary and ordinary rays coincide, and therefore $V' = \mu V$, consequently we must have $\mu V = \frac{\text{const}}{b}$, and const $= b\,\mu V$, so that we shall get

$$V' = \mu V \cdot \frac{b}{r}, \text{ and } \frac{V'}{V} = \mu \cdot \frac{b}{r}.$$

790. Introduction of the principle of least action or swiftest propagation. Fig. 169.

In general, as we have already seen, the condition of least action affords the equation

$$d\{V S + V' S'\} = 0, \text{ or } V \cdot d S + V' \cdot d S' + S' \cdot d V' = 0; \quad (2)$$

But to make use of this, we must express V′, S, and S′, in terms of variable quantities relating to a point any how taken in the surface of the crystal. Whether this point be expressed by rectangular or polar coordinates is no matter: it will be more convenient, however, to use polar. Let, then, C (fig. 169) be the point of incidence of the ray A C on the surface H a O b, and about C as a centre describe a sphere. Let Z C z be the perpendicular to the surface at C, and let P C p be the position of the axis of the crystal. The plane Z P H z p O Z perpendicular to the surface, and passing through the axis, is called the principal section of the surface. Let Z A a, z B b be vertical planes, containing the incident and refracted rays, and join B p by the arc of a great circle. Then it is evident, that this arc will be equal to ϕ.

791. Suppose, now, the axis of the x to be parallel to H C the projection of the axis of the crystal, and since we may choose the plane of the x, y, as we please, let it coincide with the refracting surface, so that $z = 0$. Then dropping the perpendiculars A M, M m, B N, N n, and putting $\lambda =$ Z P $= z\,p =$ angle between the axis and perpendiculars.

$\varpi =$ O $a =$ inclination of the plane of incidence to the principal section.
$\varpi' =$ O $b =$ inclination of plane of refraction to ditto.
$\theta =$ angle Z C A $=$ Z A $=$ angle of incidence
$\theta' = z$ C B $= z$ B $=$ angle of refraction.

We shall have as follows:

$$A C = S; \; A M = \gamma; \; C m^2 = (a - x)^2; \; M m^2 = (\beta - y)^2;$$

consequently,

$$\left.\begin{aligned} a - x &= \gamma \cdot \tan\theta \cdot \cos\varpi; \; \beta - y = \gamma \cdot \tan\theta \cdot \sin\varpi; \; S = \frac{\gamma}{\cos\theta} \\ \text{and, similarly,} \\ a' - x &= \gamma' \cdot \tan\theta' \cdot \cos\varpi'; \; \beta' - y = \gamma' \cdot \tan\theta' \cdot \sin\varpi'; \; S' = \frac{\gamma'}{\cos\theta'} \end{aligned}\right\} \quad (3)$$

Light. Now, differentiating these equations, and considering that $d(\alpha - x) = d(\alpha' - x)$ and $d(\beta - y) = d(\beta' - y)$ Part IV. we get

$$d(\tan\theta \,.\, \cos\varpi) = \frac{\gamma'}{\gamma} \,.\, d(\tan\theta' \,.\, \cos\varpi');$$

$$d(\tan\theta \,.\, \sin\varpi) = \frac{\gamma'}{\gamma} \,.\, d(\tan\theta' \,.\, \sin\varpi');$$

which equations, developed and reduced, afford the following,

$$\left.\begin{aligned} \frac{d\theta}{d\theta'} &= \frac{\gamma'}{\gamma} \cdot \left(\frac{\cos\theta}{\cos\theta'}\right)^2 \cdot \cos(\varpi - \varpi'); & \frac{d\theta}{d\varpi'} &= \frac{\gamma'}{\gamma} \,.\, \cos\theta^2 \,.\, \tan\theta' \,.\, \sin(\varpi - \varpi'); \\ \frac{d\varpi}{d\theta'} &= \frac{\gamma'}{\gamma} \cdot \frac{\sin(\varpi' - \varpi)}{\tan\theta \,.\, \cos\theta'^2}; & \frac{d\varpi}{d\varpi'} &= \frac{\gamma'}{\gamma} \cdot \frac{\tan\theta'}{\tan\theta} \cos(\varpi' - \varpi); \end{aligned}\right\} \quad (4)$$

which are necessary conditions, in order that the point C may remain on the surface.

But since S, S′, V′ may be regarded as functions of θ' and ϖ', which are the polar coordinates we propose to use as independent variables, we shall have 792.

$$dV' = \frac{dV'}{d\theta'} d\theta' + \frac{dV'}{d\varpi'} d\varpi';$$

and, moreover,

$$dS = \frac{\gamma \,.\, \sin\theta}{\cos\theta^2}\left(\frac{d\theta}{d\theta'} d\theta' + \frac{d\theta}{d\varpi'} d\varpi'\right); dS' = \frac{\gamma' \,.\, \sin\theta'}{\cos\theta'^2} d\theta',$$

so that, substituting their values in the equation (2,) we get

$$\begin{aligned} 0 = &\left\{V \,.\, \frac{\gamma \,.\, \sin\theta}{\cos\theta^2} \cdot \frac{d\theta}{d\theta'} + V' \,.\, \frac{\gamma' \,.\, \sin\theta'}{\cos\theta'^2} + \frac{\gamma'}{\cos\theta'} \cdot \frac{dV'}{d\theta'}\right\} d\theta' \\ &+ \left\{V \,.\, \frac{\gamma \,.\, \sin\theta}{\cos\theta^2} \cdot \frac{d\theta}{d\varpi'} + \frac{\gamma'}{\cos\theta'} \cdot \frac{dV'}{d\varpi'}\right\} d\varpi'; \end{aligned} \quad (5)$$

in which the coefficients of each of the two independent differentials being separately made to vanish, we get

$$\left.\begin{aligned} \frac{dV'}{d\theta'} &= -V \,.\, \frac{\gamma}{\gamma'} \cdot \frac{\sin\theta \,.\, \cos\theta'}{\cos\theta^2} \cdot \frac{d\theta}{d\theta'} - V' \,.\, \tan\theta' \\ \frac{dV'}{d\varpi'} &= -V \,.\, \frac{\gamma}{\gamma'} \cdot \frac{\sin\theta \,.\, \cos\theta'}{\cos\theta^2} \cdot \frac{d\theta}{d\varpi'} \end{aligned}\right\}. \quad (6)$$

In these, substituting the values of $\frac{d\theta}{d\theta'}$ and $\frac{d\theta}{d\varpi'}$ found in equation (4,) we obtain the following

$$\left.\begin{aligned} \frac{dV'}{d\theta'} &= -V \,.\, \frac{\sin\theta}{\cos\theta'} \,.\, \cos(\varpi - \varpi') - V' \,.\, \tan\theta' \\ \frac{dV'}{d\varpi'} &= -V \,.\, \sin\theta \,.\, \sin\theta' \,.\, \sin(\varpi - \varpi') \end{aligned}\right\}; \quad (7)$$

These are the very same equations with those deduced by Laplace and Malus, by a more abstruse and complicated calculus, from the primary dynamical relations of the problem; and from them it is easy to express, in general, the law of refraction corresponding to any given law of velocities, for we have only to put them under the form

$$V \,.\, \sin\theta \,.\, \cos\varpi \,.\, \cos\varpi' + V \,.\, \sin\theta \,.\, \sin\varpi \,.\, \sin\varpi' = -V' \,.\, \sin\theta' - \cos\theta' \,.\, \frac{dV'}{d\theta'}$$

$$V \,.\, \sin\theta \,.\, \cos\varpi \,.\, \sin\varpi' - V \,.\, \sin\theta \,.\, \sin\varpi \,.\, \cos\varpi' = \frac{1}{\sin\theta'} \frac{dV'}{d\varpi'};$$

and multiplying the first by $\cos\varpi'$, and the second by $\sin\varpi'$, and adding, we get

$$V \,.\, \sin\theta \,.\, \cos\varpi = \frac{\sin\varpi'}{\sin\theta'} \,.\, \frac{dV'}{d\varpi'} - \cos\theta' \,.\, \cos\varpi' \,.\, \frac{dV'}{d\theta'} - \sin\theta' \,.\, \cos\varpi' \,.\, V'; \quad (8)$$

and, again, multiplying the first by $\sin\varpi'$, and the second by $-\cos\varpi'$, and adding, we find

$$V \,.\, \sin\theta \,.\, \sin\varpi = -\frac{\cos\varpi'}{\sin\theta'} \frac{dV'}{d\varpi'} - \cos\theta' \,.\, \sin\varpi' \,.\, \frac{dV'}{d\theta'} - \sin\theta' \,.\, \sin\varpi' \,.\, V'; \quad (9)$$

Light. Now, the second members of these equations, (when V′ the velocity of the extraordinary ray is any function of ϕ the angle it makes with the axis, or of its position within the crystal,) is always explicitly given in terms of θ' and ϖ', so that, calling P and Q their values so expressed, we have at once Part IV.

$$\tan \varpi = \frac{Q}{P}\,;\ \cos \varpi = \frac{P}{\sqrt{P^2 + Q^2}}\,;\ \sin \theta = \sqrt{P^2 + Q^2}\,;$$

so that ϖ and θ are directly expressed in terms of ϖ' and θ'; and, therefore, the direction in which a ray, moving anyhow within the crystal will emerge, is known, and *vice versâ*.

793. It only remains to execute these processes in the case before us. To this end (for simplicity) we shall put V = 1, and suppose (since a and b, the semiaxes of the spheroid, are arbitrary) $b = \frac{1}{\mu}$, or $\mu = \frac{1}{b}$, and put W for the radical $\sqrt{a^2 . \cos \phi^2 + b^2 . \sin \phi^2}$, when we shall have

$$V' = \frac{W}{a\,b}\,;\ dV' = \frac{a^2 - b^2}{a\,b} \cdot \frac{\cos \phi}{W} . d\,(\cos \phi).$$

Now in the spherical triangle Z B p we have, the side Z $p = \lambda$; Z B $= \theta'$, angle p Z B $= \varpi'$, and side p B $= \phi$, therefore, by spherical trigonometry,

$$\cos \phi = \cos \lambda . \cos \theta' + \sin \lambda . \sin \theta' . \cos \varpi', \qquad (10)$$

and differentiating separately with respect to θ' and ϖ',

$$\frac{d . \cos \phi}{d\,\theta'} = -\cos \lambda . \sin \theta' + \sin \lambda . \cos \theta' . \cos \varpi'$$

$$\frac{d . \cos \phi}{d\,\varpi'} = -\sin \lambda . \sin \theta' . \sin \varpi'.$$

If, then, we write these values in the partial differences of V′ in the equations (8) and (9,) they will become

$$\sin \theta . \cos \varpi = -\frac{1}{a\,b\,W}\left\{W^2 . \sin \theta' . \cos \varpi' + (a^2 - b^2) \cos \phi\,[\sin \lambda\,(1 - \cos \varpi'^2 . \sin \theta'^2) - \cos \lambda . \sin \theta' . \cos \theta' . \cos \varpi']\right\}$$

$$\sin \theta . \sin \varpi = -\frac{1}{a\,b\,W}\left\{W^2 . \sin \theta' . \sin \varpi' - (a^2 - b^2) \cos \phi\,[\sin \lambda . \sin \varpi' . \cos \varpi' . \sin \theta'^2 + \cos \lambda . \sin \theta' . \cos \theta' . \sin \varpi']\right\}.$$

In these, let $b^2 + (a^2 - b^2) \cos \phi^2$ be put for W^2, and, bearing in mind that the value of $\cos \phi$ is as given in the equation (10,) we shall see that they will reduce themselves respectively to

$$\sin \theta . \cos \varpi = -\frac{1}{a\,b\,W}\left\{b^2 . \sin \theta' . \cos \varpi' + (a^2 - b^2) . \sin \lambda . \cos \phi\right\}$$

that is, by reason of (10,)

$$\left.\begin{aligned} -\sin \theta . \cos \varpi &= \frac{(a^2 - b^2) . \cos \lambda . \sin \lambda . \cos \theta' + (a^2 . \sin \lambda^2 + b^2 . \cos \lambda^2) . \cos \varpi' . \sin \theta'}{a\,b\,W} \\ \text{and} \\ -\sin \theta . \sin \varpi &= \frac{b^2 . \sin \theta' . \sin \varpi'}{a\,b\,W} \end{aligned}\right\}. \qquad (11)$$

794. These equations, conjointly with the equations expressing the value of W in terms of $\cos \phi$, and of $\cos \phi$ in terms of θ' and ϖ', afford a complete solution of the problem in the case when a ray passes out of a crystal into air, and suffice to determine both the inclination of the refracted ray to the surface, and the inclination of the plane in which it lies to the principal section.

For brevity, let us put

$$a^2 . \sin \lambda^2 + b^2 . \cos \lambda^2 = A\,;\quad a^2 . \cos \lambda^2 + b^2 . \sin \lambda^2 = B\,;\quad (a^2 - b^2) . \sin \lambda . \cos \lambda = C\,; \qquad (12)$$

and, dividing the second of the equations (11) by the first, we find

$$\tan \varpi = \frac{b^2 . \tan \theta' . \sin \varpi'}{A . \tan \theta' . \cos \varpi' + C} \qquad (13)$$

which gives immediately the inclination of the plane of emergence to the principal section, or, as it is sometimes termed, the *azimuth* of the emergent ray.

Light. Part IV.

Reciprocally, if having given the angle of incidence and azimuth of a ray incident externally on the crystal, we would find the angle of refraction and azimuth of the intromitted ray, we must find θ' and ϖ' from the above equations in terms of θ and ϖ. This may be thus accomplished:

795. Given the path of the ray without, required that within the crystal.

Take $$x = \tan\theta' . \cos\varpi', \text{ and } y = \tan\theta' . \sin\varpi',$$

then $$x^2 + y^2 = \tan\theta'^2, \text{ and } \cos\theta'^2 = \frac{1}{1 + x^2 + y^2};$$

and, moreover, $$\tan\varpi = \frac{b^2 y}{A x + C};$$

now, since $W^2 = b^2 + (a^2 - b^2) . \cos\phi^2$,

$$= \cos\theta'^2 \left\{ \frac{b^2}{\cos\theta'^2} + (a^2 - b^2) (\cos\lambda + \sin\lambda . \tan\theta' . \cos\varpi')^2 \right\}$$

the second of the equations (11) becomes, by squaring,

$$(\sin\theta . \sin\varpi)^2 \left\{ \frac{b^2}{\cos\theta'^2} + (a^2 - b^2) . (\cos\lambda + \sin\lambda . \tan\theta' . \cos\varpi')^2 \right\} = \frac{b^2}{a^2} (\tan\theta' . \sin\varpi')^2,$$

that is,

$$a^2 (\sin\theta . \sin\varpi)^2 \{ b^2 (1 + x^2 + y^2) + (a^2 - b^2) (\cos\lambda + x . \sin\lambda)^2 \} = b^2 y^2,$$

that is, developing

$$a^2 . (\sin\theta . \sin\varpi)^2 \{ A x^2 + 2 C x + B + b^2 y^2 \} = b^2 y^2.$$

Now we have $$A x + C = \frac{b^2 y}{\tan\varpi}, \text{ and } x = \frac{b^2 y}{A . \tan\varpi} - \frac{C}{A}.$$

And, on substitution, this equation will be found to take the form $p y^2 + q = 0$, and being resolved to give

$$y = \tan\theta' . \sin\varpi' = \frac{a^2 . \sin\theta . \sin\varpi}{\sqrt{A - a^2 . \sin\theta^2 (A . \sin\varpi^2 + b^2 . \cos\varpi^2)}}, \qquad (14)$$

and substituting this in the value of x, we find

$$x = \tan\theta' . \cos\varpi' = \frac{a^2 b^2}{A} . \frac{\sin\theta . \cos\varpi}{\sqrt{A - a^2 . \sin\theta^2 \{ A . \sin\varpi^2 + b^2 . \cos\varpi^2 \}}} - \frac{C}{A}. \qquad (15)$$

These equations are identical with those demonstrated by Malus in his *Théorie de la Double Réfraction*, with some slight differences of notation only, arising from our having reckoned ϖ and ϖ' from the opposite point of the circle.

796. Particular applications.

The values of A, B, and C depend only on a, b, and λ, that is, on the peculiar nature of the crystal, which determines the ratio of the axes of the spheroid of double refraction, and on the inclination of the axis to the surface on which the ray is incident. The former are constant for one and the same crystal, however the surface be placed; the latter is constant for any given surface. Hence it appears, that the general law of extraordinary refraction, when we confine ourselves to the consideration of a surface given in position with respect to the axis, resolves itself into an infinite variety of particular laws, some of which we shall now consider.

797. 1st. When the axis is perpendicular to the surface.

Case 1. $\lambda = 0$, the surface perpendicular to the axis; $A = b^2$; $B = a^2$; $C = 0$, and the equations (14) and (15) become

$$\tan\theta' . \sin\varpi' = \frac{a^2}{b} . \frac{\sin\theta . \sin\varpi}{\sqrt{1 - a^2 . \sin\theta^2}}; \quad \tan\theta' . \cos\varpi' = \frac{a^2}{b} . \frac{\sin\theta . \cos\varpi}{\sqrt{1 - a^2 . \sin\theta^2}};$$

these equations (as well as Equation 13) give $\varpi' = \varpi$, so that in this case the plane of refraction is the same with that of incidence, and the extraordinary ray is not deviated out of the vertical plane. Hence, we get simply

$$\tan\theta' = \frac{a^2}{b} . \frac{\sin\theta}{\sqrt{1 - a^2 . \sin\theta^2}}; \qquad (16)$$

which expresses the law of extraordinary refraction in this case. If $\theta = 0$, $\theta' = 0$, or the ray incident perpendicularly passes unrefracted along the axis. If $\theta = 90°$, $\tan\theta' = \frac{a^2}{b\sqrt{1 - a^2}}$. Now if we put $b = \frac{1}{\mu}$ and $a = \frac{1}{\mu'}$, this becomes

$$\tan\theta' = \frac{\mu}{\mu'\sqrt{\mu'^2 - 1}}; \qquad (17)$$

which, μ and μ' being each greater than unity, is always real, so that the ray can enter the crystal however oblique its incidence.

Light. Part IV.

Case 2. When the axis lies in the surface, or $\lambda = 90°$; $A = a^2$; $B = b^2$; $C = 0$, and the equations become

798. 2d. When the axis lies in the surface.

$$\tan\theta' \,.\, \sin\varpi' = \frac{a \,.\, \sin\theta \,.\, \sin\varpi}{\sqrt{1 - \sin\theta^2 \left\{ a^2 \,.\, \sin\varpi^2 + b^2 \,.\, \cos\varpi^2 \right\}}}; \qquad (18)$$

$$\tan\theta' \,.\, \cos\varpi' = \frac{b^2}{a} \,.\, \frac{\sin\theta \,.\, \cos\varpi}{\sqrt{1 - \sin\theta^2 \left\{ a^2 \,.\, \sin\varpi^2 + b^2 \,.\, \cos\varpi^2 \right\}}}; \qquad (19)$$

$$\tan\varpi' = \frac{a^2}{b^2} \,.\, \tan\varpi = \left(\frac{\mu}{\mu'}\right)^2 \,.\, \tan\varpi \,. \qquad (20)$$

The latter of these equations shows that the extraordinary ray deviates from the plane of incidence. The amount of this deviation is nothing when the plane of incidence coincides with the principal section, but increases on either side of it till it attains a certain magnitude, the deviation being *from* the axis, or the plane of refraction making a greater angle with the axis than that of incidence. The two planes then approach each other, and when $\varpi = 90°$, $\tan\varpi = \infty$, $\tan\varpi' = \infty$, and, consequently, $\varpi' = 90°$, or the plane of refraction coincides with that of incidence.

799. Case of refraction in the principal section.

The equations (18) and (19) show that in the present case, the refracted ray does not describe a conical surface about the perpendicular when the incident one does so, and therefore that the law of refraction varies in every different azimuth. Two cases deserve express notice, *viz.* those in which the plane of incidence is coincident with the principal section, and when perpendicular to it. In the former, $\varpi = 0$ and $\varpi' = 0$, so that we have

$$\tan\theta' = \frac{b^2}{a} \,.\, \frac{\sin\theta}{\sqrt{1 - b^2 \,.\, \sin\theta^2}}. \qquad (21)$$

A remarkable relation holds good in this case between the angles of refraction of the ordinary and extraordinary ray, their tangents being to each other in a given ratio. In fact, if we find (θ') = the angle of refraction for the ordinary ray, we have $\sin(\theta') = \frac{1}{\mu} \,.\, \sin\theta = b \,.\, \sin\theta$, and, consequently,

$$\tan\theta' = \frac{b}{a} \,.\, \frac{\sin(\theta')}{\sqrt{1 - \sin(\theta')^2}} = \frac{b}{a} \,.\, \tan(\theta). \qquad (22)$$

In the latter case, when the plane of refraction is at right angles to the axis, $\varpi = \varpi' = 90°$, and we get

$$\tan\theta' = \frac{a \,.\, \sin\theta}{\sqrt{1 - a^2 \,.\, \sin\theta^2}}; \quad \sin\theta' = a \,.\, \sin\theta. \qquad (23)$$

800. Case of refraction at right angles to the principal section.

In this case, therefore, the sine of incidence is in a given ratio to that of refraction, and the extraordinary refraction is performed according to the same law as the ordinary, only with a different index, *viz.* μ', or $\frac{1}{a}$, instead of μ, or $\frac{1}{b}$. Hence, if we consider only this particular case, the medium will appear to have two indices of refraction, an ordinary and an extraordinary one.

801. Experimental method of determining the spheroid of double refraction.

It was by a careful examination of these cases, that Dr. Wollaston was enabled to verify the Huygenian law. The circumstance last mentioned puts it in our power to determine in the case of any particular crystal the axes of its spheroid of double refraction. We have only to cut a prism of it, having its refracting angle parallel to the axis, and ascertain its indices of refraction according to the principles laid down in the former part of this Essay, and calling them μ and μ', the semiaxes of the spheroid will be respectively $\frac{1}{\mu}$ and $\frac{1}{\mu'}$. Thus, in the instance of carbonate of lime, which Malus examined with the utmost care, he found the two values of a and b to be respectively equal to the numbers 0.67417 and 0.60449, having determined $\mu' = 1.4833$, and $\mu = 1.6543$. (*Théorie de la Double Réfraction*, p. 199.)

802. In this arrangement, however, it is not possible to decide simply from the phenomena of refraction, which is the ordinary, and which the extraordinary ray. There are, however, infallible and easy criteria, as we shall speedily show. Meanwhile, we may for the present content ourselves with observing, that as a moderate deviation from the exact azimuth $\varpi = 90°$ imparts to the extraordinary ray a deviation from the plane of incidence which does not happen to the ordinary one, this may serve for a criterion to distinguish them in certain cases.

803. Law of the increment of velocity.

The square of the velocity of the ordinary ray within the medium is $\mu^2 V^2$, or μ^2, that is, $\frac{1}{b^2}$, and is constant. That of the extraordinary is V'^2, or $\frac{W^2}{a^2 b^2}$, that is to say, $\frac{\cos\phi^2}{b^2} + \frac{\sin\phi^2}{a^2}$,

or,
$$V'^2 = \frac{1}{b^2} - \left(\frac{1}{b^2} - \frac{1}{a^2}\right) \,.\, \sin\phi^2.$$

Light. Part IV.

The square of the velocity of the extraordinary ray is therefore (in the corpuscular doctrine) diminished by a quantity proportional to the square of the sine of the inclination of the ray within the crystal, to the axis. We say diminished, in the algebraical sense of the word, supposing $a > b$, this agrees with common parlance; but if $a < b$, then it will be increased. This gives rise to the subdivision of the crystallized bodies now treated of into two classes, which have by some been termed attractive and repulsive: by others, positive and negative, which seems preferable, as the former phrases involve theoretical considerations. *Positive* crystals are, then, such as have a less than b, or in which the spheroid of double refraction is *prolate*. In these the coefficient $-\left(\frac{1}{b^2} - \frac{1}{a^2}\right)$ which we call k is positive, and the square of the velocity, or $v^2 + k \,.\, \sin\theta^2$, (where $v = \frac{1}{b}$ = velocity of the ordinary ray within the medium,) is increased by the action of the medium, and is a minimum in the axis. In the negative class the coefficient k is negative, $a > b$, or the spheroid of double refraction is *oblate*, and the velocity of the extraordinary ray is a maximum along the axis. In positive crystals, therefore, the index of ordinary refraction (μ) is *less* than that of extraordinary; in negative, greater. To the former class belong quartz, ice, zircon, apophyllite, (when uniaxal;) and to the latter, Iceland spar, tourmaline, beryl, emerald, apatite, &c. The negative class, as far as our present knowledge extends, far out-numbers the positive among natural and artificial crystals. They were first distinguished by M. Biot.

Division of crystals into positive and negative.

804. Undulations propagated in spheroidal surfaces.

In the undulatory doctrine the velocity is the reciprocal of what it is in the corpuscular doctrine, and is therefore *directly* as the radius of the spheroid of double refraction. Hence a wave propagated within the crystal from any point will run over in the same time in different directions, distances proportional to the radii of the spheroid parallel to those directions; and therefore at any instant the surface of the whole wave will be itself a spheroid similar to the spheroid of double refraction. This is Huygens's conception of the subject. It requires us to regard the crystal, or the ether within the crystal through which the undulation is propagated, as having different elasticities in different directions. As far as regards the molecules of a solid body there is no apparent impossibility or improbability in such an idea, but the contrary; but if we regard the propagation of the light within the medium to take place by the elasticity of the ether only, we must then suppose *its* molecules in crystallized bodies to be in a very different physical state from what they are in free space, and either to be in some manner connected with the solid particles, (forming atmospheres, for instance, about them,) or as subjected to laws of mutual action which approximate to those governing the molecules of solid bodies; and partaking, themselves, of a regular crystalline arrangement and mutual dependency.

805. Malus's further researches.

To pursue the particular applications of the general formulæ (13,) 14,) and (15) farther, would be far beyond our limits. The reader who is curious on this very interesting part of Physical Optics, and who wishes to be delighted and instructed by a combination of consummate mathematical skill with sound experimental research, which may deservedly be cited as a model of the kind, will find every thing which relates to the subject in its best form in the work, already so often cited, of Malus, *Théorie de la Double Réfraction*, which gained the mathematical prize of the French Institute in 1810. To the theory of the internal reflexion of the extraordinary ray which offers many remarkable particularities, as there delivered, we must especially refer him, as well as to his investigation of the foci of lenses formed of doubly refracting crystals, of which we shall here only extract the results, in the single case of a double convex lens having the axis of double refraction in the direction of the axis of the lens.

Foci of a doubly refracting lens.

Let r, r' be the radii of the anterior and posterior surfaces of the lens, both supposed convex.

d = distance of the radiant point in the axis.

a, b = the equatorial and polar radii of the spheroid of double refraction, as above.

D = distance of the conjugate focus behind the lens for extraordinary rays.

Δ = extraordinary focal length for parallel rays.

F = ordinary focal length for parallel rays.

Then shall we have for the general expression of D,

$$\mathrm{D} = -\frac{a^2\, b\, d\, r\, r'}{d\,(r+r')\,(2\,b^2 - a^2 - a^2\, b) - a^2\, b\, r\, r'}; \qquad \mathrm{F} = \frac{-\,b\, r\, r'}{(r+r')\,(1-b)}.$$

If the lens be equi-convex, or $r = r'$,

$$\mathrm{D} = -\frac{a^2\, b\, r\, d}{2\,(2\,b^2 - a^2 - a^2\, b)\, d - a^2\, b\, r}; \qquad \Delta = -\frac{a^2\, b\, r}{2\,(2\,b^2 - a^2 - a^2\, b)};$$

$$\mathrm{F} = -\frac{b\, r}{2\,(1-b)}; \qquad \Delta - \mathrm{F} = -2\,\mathrm{F} \,.\, \frac{a^2 - b^2}{2\,b^2 - a^2 - a^2\, b}.$$

In the case of Iceland spar, these last equations become

$$\mathrm{D} = -r \,.\, 88{,}2286; \qquad \mathrm{F} = -r \,.\, 0{,}7642; \qquad \mathrm{D} - \mathrm{F} = -\mathrm{F} \,.\, 114{,}4546;$$

and in the case of rock crystal (quartz)

$$\mathrm{D} = -r \,.\, 0.9628; \qquad \mathrm{F} = -r \,.\, 0{,}8958; \qquad \mathrm{D} - \mathrm{F} = -\mathrm{F}.0{,}0748.$$

806. To represent, in general, the course of any extraordinarily refracted ray, Huygens has giving the following construction, (fig. 170.) Let H E D be the elliptic section of the spheroid of double refraction by the surface, and R C the incident ray falling on C its centre, and B C K the orthographic projection of the ray R C on the

surface. Let H M E be the portion of the spheroid within the crystal, whose axis passes through C, and may be anyhow inclined to the surface. Then will the surface of this spheroid be the boundary of the wave propagated from C as a centre, after the lapse of a given time. Draw C O in the plane R C K at right angles to R C, and make O K (perpendicular to O K, or parallel to R C) equal to the space described by light in the medium exterior to the crystal in the same given time. This will determine the point K in the line B C K. Through K draw K T perpendicular to B K, and about K T as an axis let a plane revolve passing through K T, till it touches the surface of the spheroid in I. Join C I, and C I is the extraordinary refracted ray.

Huygens's construction for extraordinary refraction. Fig. 170.

807. The demonstration of this construction (granting the principle of spheroidal undulations) is evident, if we consider the manner in which the general wave, a perpendicular to whose surface forms what we term a ray of light, (at least in singly refracting media,) arises from the reunion of all the elementary waves propagated from every part of the surface, (Art. 586.) In this construction, if we conceive a plane wave from an infinitely distant luminary perpendicular to R C to move along R C, every point in the line C K will become *in succession*, and every point in the line C D perpendicular to C K, or parallel to K T *simultaneously*, a centre of vibration. The general wave, therefore, will be a surface touching all ellipsoids described about each point of the surface, having their axes parallel, their generating ellipses similar, and their linear dimensions proportional to the distance of their centre from the line K T. Of course it can be no other than the tangent plane I K T drawn as above.

Demonstration from the principle of spheroidal undulations.

808. This then will be the form and position of the general wave within the crystal. Now if we consider only that very minute portion of it which emanates from C, it is evident that I is the corresponding point in it; and therefore C I is necessarily the direction of the ray, because I is the point on which that portion of the general wave transmitted through a very small aperture at C would fall.

809. Thus we see, that in the case of the extraordinary ray, we are no longer to regard the *ray* as a perpendicular to the surface of the wave. It is propagated obliquely to that surface. So soon, however, as the wave emerges into the ambient medium, the usual law of perpendicular propagation is restored.

Oblique propagation of the extraordinary ray

810. To show the identity of the law of extraordinary refraction resulting from this construction with that expressed by the general equations (13,) (14,) and (15,) we have only to translate it into analytical language. This has been done by Malus, in his work above referred to; and the reader may also consult Biot's *Traité Général de Physique*, for a more elementary exposition of the process, which is one of considerable complexity, for which reason we shall not embarrass ourselves with it here.

811. Some very remarkable and important consequences follow from this mode of viewing the subject. It appears that when a plane wave is incident on a doubly refracting surface, the transmitted *extraordinary wave* is also plane, and advances with a uniform velocity in a direction oblique to itself. Consequently the velocity is also uniform in a direction perpendicular to itself. Moreover, its common section with the surface is always parallel to K T, or to the common section of the incident wave with the same surface. Hence, it is evident, that it moves in the same way as an ordinarily transmitted wave would do, and at any instant has the same position that such a wave would have, provided the index of refraction in the latter case were properly assumed. The only difference is, that the motions of the vibrating molecules, of which they respectively consist, are executed in different planes. Now, when this wave emerges from the medium, it obeys the same laws as on its entry, only reversed; so that it still continues a plane wave, and its common section with the surface of *emergence* remains unaltered.

Form and position of the extraordinary ray.

812. Hence it follows, that if we cut a prism of any doubly refracting crystal with one axis, and transmit through it a ray incident in a plane at right angles to the edge of the prism, the ordinary and extraordinary ray will both emerge in that plane, and their separation will take place in a plane containing the incident and ordinarily-refracted ray, and will therefore be, apparently, such as would arise from attributing two ordinary refractive powers to the medium. It is only when the edge of the prism is oblique to the plane of incidence, that the extraordinary ray can deviate from the plane containing the incident and ordinarily refracted rays.

Consequences in double refraction through prisms.

813. We see, then, that in the theory of extraordinary refraction, it is necessary to consider, as distinct, two things, which, in that of ordinary, are one and the same, *viz.* the velocity of the *luminous waves*, and the *velocity of the rays of light*. This distinction will require to be very carefully kept in view hereafter, when we come to treat of the law of refraction in crystals with two axes of double refraction. For this, however, we are not yet prepared, as the knowledge of this law presupposes an acquaintance with a multitude of facts relative to the polarization of light, of which we have yet said nothing. It will suffice here to mention, that the whole doctrine of double refraction has recently undergone a great revolution; one, indeed, which may be said to have changed the face of Physical Optics, in consequence of the researches of M. Fresnel. It had all along been taken for granted, that in crystals possessed of double refraction, one of the pencils followed the ordinary law of proportional sines. It had, moreover, been ascertained, by experiments hereafter to be related, that the difference of the squares of the velocities between the two pencils is in all cases proportional to the product of the sines of the angles contained between the extraordinary ray (as it was termed) and the two axes, or directions in which the refraction is single. It was hence concluded, that the velocity of the *extraordinary pencil* was in all cases represented by $\sqrt{v^2 + k \cdot \sin\phi \cdot \sin\phi'}$, v being that of the ordinary one, and k a constant depending on the nature of the crystal, and ϕ, ϕ' the angles in question. This granted, there would be no difficulty in determining the form of the surface of double curvature, which should be substituted for the Huygenian spheroid; so as to render the same construction with that described in Art. 806, or the general formulæ in Art. 792, applicable to this case. In fact, if we call a the semi-angle between the two axes, and conceive three coordinates x, y, z, of which x bisects that angle, the plane of the x, y containing both axes, it is easy to see, by spherical trigonometry, that we must have

Velocity of *luminous waves*, and of *rays of light* distinguished.

Theory of double refraction in bi-axal crystals deferred, and why.

Light. Part IV.

$$\cos\phi = \frac{x \,.\, \cos a + y \,.\, \sin a}{\sqrt{x^2 + y^2 + z^2}}; \qquad \cos\phi' = \frac{x \,.\, \cos a - y \,.\, \sin a}{\sqrt{x^2 + y^2 + z^2}}.$$

Hence, since r ($\sqrt{x^2 + y^2 + z^2}$) the radius of the surface of the wave, is always equal to

$$\frac{1}{V'}, \text{ or } \frac{1}{\sqrt{v^2 + k^2 \,.\, \sin\phi \,.\, \sin\phi'}},$$

a simple substitution would give at once the equation of its surface as referred to the three coordinates x, y, z; namely,

$$0 = (k^2 - v^4)\,(x^2 + y^2 + z^2)^2 + 2\,(x^2 + y^2 + z^2)\,(v^2 - k^2 x^2 \,.\, \cos a^2 - k^2 y^2 \,.\, \sin a^2) + k^2\,(x^2 \,.\, \cos a^2 + y^2 \,.\, \sin a^2)^2 - 1,$$

which it would be easy then to transform into functions of r, ϖ, and θ, as required for the application of the general analytical formulæ by the usual substitutions

$$z = r \,.\, \sin\theta; \qquad y = r \,.\, \sin\theta \,.\, \sin\varpi; \qquad x = r \,.\, \sin\theta \,.\, \cos\varpi.$$

The researches of M. Fresnel, however, as before remarked, have destroyed the basis on which this theory rested, by demonstrating the non-existence of an ordinarily refracted ray in the case of crystals with two axes. The theory which he has substituted in its place, however, and which it is impossible to regard otherwise than as one of the finest generalizations of modern science, we must reserve for a more advanced place in this essay. We shall now proceed to treat

Of the Polarization of Light.

814. The phenomena which belong to this division of our subject are so singular and various, that to one who has only studied the subject of Physical Optics under the relations presented in the foregoing pages, it is like entering into a new world,—so splendid as to render it one of the most delightful branches of experimental inquiry; and so fertile in the views it lays open of the constitution of natural bodies, and the minuter mechanism of the universe, as to place it in the very first rank of the physico-mathematical sciences, which it maintains, by the rigorous application of geometrical reasoning its nature admits and requires. The intricacy as well as variety of its phenomena, and the unexampled rapidity with which discoveries have succeeded each other in it, have hitherto prevented the possibility of embodying it satisfactorily in a systematic form; but, after the rejection of numberless imperfect generalizations, it seems at length to have acquired that degree of consistency as to enable us—not, indeed, to deduce every phenomenon, by distinct steps, from one general cause—but to present them, at least, in something like a regular succession; to show a mutual dependence between their several classes, which is a main step to a complete generalization; and to dispense with the bewildering detail of an immense multitude of individual facts, which, having served their purpose in the inductive process, must in future be considered as having their interest merged in that of the laws from which they flow.

§ II. *General Ideas of the Distinction between Polarized and Unpolarized Light.*

815. In all the properties and affections of light which we have hitherto considered, we have regarded it as presenting the same phenomena of reflexion and transmission, both as respects the direction and intensity of the reflected or transmitted beam, however it may be presented to the reflecting or refracting surface, provided the angle of incidence, and the plane in which it lies, be not varied. And this is true of light in the state in which it is emitted immediately from the sun, or from other self-luminous sources. A ray of such light, incident at a given angle on a given surface, may be conceived to revolve round an axis coincident with its own direction; or, which comes to the same thing, the reflecting or refracting surface may be actually made to revolve round the ray as an axis, preserving the same relative situation to it in all other respects, and no change in the phenomena will be perceived. For instance, if in a long cylindrical tube we fix a plate of glass, or any other medium at any angle of inclination to the axis; and then, directing the tube to the sun, turn the whole apparatus round on its axis, the intensity of the reflected or refracted ray will suffer no variation, and its direction (if deviated) will revolve uniformly round with the apparatus, so that if received on a screen connected invariably with the tube, it will continue to fall on the very same point in all parts of its rotation. Or we may receive the light from a piece of white hot iron at any angle on any medium, and its phenomena will be precisely the same, whether the iron be at rest, or be made to revolve round an axis coincident with the direction of the ray.

816. Polarized rays have acquired fixed relations to external space.

But, if instead of employing a ray immediately emitted from a self-luminous source, we subject to the same examination a ray that has undergone some reflexions, refractions, or been in any one of a great variety of ways subjected to the action of material bodies, we find this perfect uniformity of result no longer to hold good. It is no longer indifferent in what plane, with respect to the ray itself, the reflecting or refracting surface is presented to it. It seems to have acquired *sides;* a right and left, a front and back; and the *intensity*, though not the direction of the reflected or transmitted portion, depends materially on the position with respect to these

Light. Illustration.

sides, in which the plane of incidence lies, though every thing else remains precisely the same. In this state it is said to be polarized. The difference between a polarized and an ordinary ray of light can hardly be more readily conceived than by assimilating the latter to a cylindrical, and the former to a four-sided prismatic rod, such as a lath or a ruler, or other long, flat, straight stick. It is evident that the cylinder, if inclined to any surface at a given angle in a given plane, may be turned round its own axis without altering its relations to the plane, while those of the prism will vary essentially according to the position of its sides. Let us suppose, for instance, (it is but a simile, which we do not wish the reader to dwell on for a moment, or to imagine that any analogy is hereafter intended to be established,) that we had occasion to thrust such a rod into a surface composed of detached fibres, all lying in one direction, or of scales or laminæ arranged parallel to one another, we should find a much greater facility of penetration on presenting its broad side in the direction of the laminæ or fibres, than transverse to them. A thin sheet may be slipped between the bars of a grating, which would present an insuperable obstacle to it if presented cross-wise.

817. Property of the tourmaline and other crystals.

But, to be more particular, and to give a more clear conception of the marked distinction which exists between a polarized and an unpolarized ray. There are many crystallized minerals, which when cut into parallel plates are sufficiently transparent, and let pass abundance of light with perfect regularity, but which, nevertheless, at its emergence is found to have acquired that peculiar modification here in question. One of the most remarkable of these is the tourmaline. This mineral crystallizes in long prisms, whose primitive form is the obtuse rhomboid, having its axis parallel to the axis of the prism. The lateral faces of these prisms are frequently so numerous as to give them an approach to a cylindrical or cylindroidal form. Now if we take one of these crystals, and slit it (by the aid of a lapidary's wheel) into plates parallel to the axis of the prism of moderate and uniform thickness, (about $\frac{1}{20}$ of an inch,) which must be well-polished, luminous objects may be seen through them, as through plates of coloured glass. Let one of these plates be interposed perpendicularly between the eye and a candle, the latter will be seen with equal distinctness in every position of the axis of the plate with respect to the horizon, (by the axis of the plate is meant any line in it parallel to the axes of its molecules, or to the axis of the prism from which it was cut.) And if the plate be turned round on its own plane, no change will be perceived in the image of the candle. Now, holding this first plate in a fixed position, (with its axis vertical, for instance,) let a second be interposed between it and the eye, and turned round slowly in its own plane, and a very remarkable phenomenon will be seen. The candle will appear and disappear alternately at every quarter revolution of the plate, passing through all gradations of brightness, from a maximum down to a total, or almost total, evanescence, and then increasing again by the same degrees as it diminished before. If now we attend to the position of the second plate with respect to the first, we shall find that the *maxima* of illumination take place when the axis of the second plate is parallel to that of the first, so that the two plates have either the same positions with respect to each other that they had in the original crystal, or positions differing by 180°, while the minima, or evanescences of the image, take place exactly 90° from this parallelism, or when the axes of the two plates are exactly crossed. In tourmalines of a good colour, the stoppage of the light in this situation is total, and the combined plate (though composed of elements separately very transparent and of the same colour) is perfectly opake. In others it is only partial; but however the specimens be chosen, a very marked defalcation of light in the crossed position takes place. We shall at present suppose that the specimens employed possess the property in question in its greatest perfection. Now it is evident that the light which has passed through the first plate has acquired in so doing a property totally distinct from those of the original light of the candle. The latter would have penetrated the second plate equally well in all its positions; the former is incapable altogether of penetrating it in some positions, while in others it passes through readily, and these positions correspond to certain *sides* which the ray has acquired, and which are parallel and perpendicular respectively to the axis of the first plate. Moreover, these *sides* once acquired, are retained by the ray in all its future course, (provided it be not again otherwise modified by contact with other bodies,) for it matters not how great the distance between the two plates, whether they be in contact or many inches, yards, or miles asunder, not the least variation is perceived in the phenomenon in question. If the position of the first plate be shifted, the sides of the transmitted ray shift with it, through an equal angle, and the second will no longer extinguish it in the position it at first did, but must be brought into a position removed therefrom, by an angle equal to that through which the first plate has been made to revolve.

818. Selection of proper tourmalines.

A great many other crystallized bodies besides the tourmaline possess this curious property, and several in great perfection. The tourmaline, however, is one easily procured, and being exceedingly useful in optical experiments, we would recommend the reader who has any desire to familiarize himself with the practical manipulations of this branch of optical science, to provide himself with a good pair of corresponding plates of this mineral, cut and polished as above directed. The colour is a point of great moment. Those of a blue or green colour possess the property in question very imperfectly; the yellow varieties, unless when verging to greenish brown, are equally improper, the best colour is a hair-brown, or purplish brown, and they may be slit and polished by any lapidary.

819. Various modes of polarizing light.

But it is not only by such means that the *polarization* of a pencil of light may be operated, nor is this the only character which distinguishes polarized from ordinary light. We shall, therefore, describe in order, the principal means by which the polarization of light may be performed, and the assemblage of characters which are invariably found to coexist in a ray when polarized.

The chief modes by which the polarization of light may be effected, are

1st. By reflexion at a proper angle from the surfaces of transparent media.

2d. By transmission through a regularly crystallized medium possessed of the property of double refraction.

3d. By transmission through transparent, uncrystallized plates in sufficient number, and at proper angles.

Light. 4th. By transmission through a variety of bodies, such as agate, mother-of-pearl, &c. which have an approach to a laminated structure, and an imperfect state of crystallization. Part IV.

The characters which are invariably found to coexist in a polarized ray, being the chief of those by which it may be most easily recognised as polarized, are— 820. Characters of a polarized ray of light.

1. Incapability of being transmitted by a plate of tourmaline, as above described, when incident perpendicularly on it, in certain positions of the plate; and ready transmission in others, at right angles to the former.

2. Incapability of being reflected by polished transparent media at certain angles of incidence, and in certain positions of the plane of incidence.

3. Incapabiltiy of undergoing division into two equal pencils by double refraction, in positions of the doubly refracting bodies, in which a ray of ordinary light would be so divided.

Besides which, there might be enumerated a vast variety of other characters, which, however, it will be better to regard as *properties* at once of polarized light, and of the various media which affect it. It cannot fail to be remarked, that all these characters are of the *negative kind*, and consist in denying to polarized light properties which ordinary light possesses, and that they are such as affect the intensity of the ray, not its direction. Thus, the direction which a polarized ray will take under any circumstances of the action of media, is never *different* from what an unpolarized ray might take, and from what a portion of it at least actually does. For instance, when an unpolarized ray is separated by double refraction into two equal pencils, a polarized ray will be divided into two unequal ones, one of which may even be altogether evanescent, but their directions are precisely the same as those of the pencils into which the unpolarized ray is divided. Hence we may lay it down as a general principle, that the *direction* taken by a polarized ray, or by the parts into which it may be divided by any reflexions, refractions, or other modifying causes, may always be determined by the same rules as apply to unpolarized light; but that the relative *intensities* of these portions differ from those of similar portions of unpolarized light, according to certain laws which it is the business of the optical inquirer to ascertain. Affect the intensity and not the direction of the ray.

§ III. *Of the Polarization of Light by Reflexion.*

When a ray of direct solar light is received on a plate of polished glass or other medium, a portion more or less considerable is always reflected. The intensity of this portion depends only on the nature of the medium and on the angle of incidence, being greater as the refractive power of the former is greater, and as the ray falls more obliquely on the surface. But it is, moreover, found, that at a certain angle of incidence, (which is therefore called the *polarizing angle*,) the reflected ray possesses all the characters above enumerated, and is therefore *polarized*. 821. Light polarized by reflexion.

This remarkable fact was discovered by Malus in 1808, when accidently viewing, through a doubly refracting prism, the light of the setting sun reflected from the glass windows of the Luxembourg Palace in Paris. On turning round the prism, he was surprised to observe a remarkable difference in the intensity of the two images; the most refracted alternately surpassing and falling short of the least in brightness, at each quadrant of the revolution. This phenomenon connecting itself in his mind with similar phenomena produced by rays which had undergone double refraction, and with which, from the researches he was then engaged in, he was familiar, led him to investigate the circumstances of the case with all possible attention, and the result was the creation of a new department of Physical Optics. So true it is, that a thousand indications pass daily before our eyes which might lead to the most important conclusions. The seeds of great discoveries are everywhere present and floating around us, but they fall in vain on the unprepared mind, and germinate only where previous inquiry has elaborated the soil for their reception, and awakened the attention to a perception of their value. 822. Discovery by Malus.

To make this new property acquired by the reflected ray evident by experiment, let any one lay down a large plate of glass on a black cloth, on a table before an open window, and placing himself conveniently so as to look obliquely at it, let him view the reflected light of the sky, (or, which is better, of the clouds if not too dark,) from the whole surface, which will thus appear pretty uniformly bright. Then let him close one eye, and apply before the other a plate of tourmaline, cut as above directed, so as to have *its axis in a vertical plane*. He will then observe the surface of the glass, instead of being as before equally illuminated, to have on it, as it were, an obscure cloud, or a large blot, the middle of which is totally dark. If this be not seen at first, it will come into view on elevating or depressing the eye. If the inclination of a line drawn from the centre of the dark spot to the eye be measured, it will be found to make an angle of about 33° with the surface of the glass. If now, keeping the eye fixed on the spot, the tourmaline plate (which it is convenient to have set in a small circular frame for such experiments) be turned slowly round in its own plane, the spot will grow less and less obscure, and *when the axis of the tourmaline is parallel to the reflecting surface*, (or horizontal,) will have disappeared completely, so as to leave the surface equally illuminated, and, on continuing the rotation of the tourmaline, will appear and vanish alternately. 823. Experiment

It appears from this experiment, that the ray which has been reflected from the surface of the glass at an inclination of 33°, or an incidence of 57°, has thereby been deprived of its power to penetrate a tourmaline plate whose axis lies in the plane of incidence. It has therefore acquired the same character, or (so far as this goes, at least) undergone the same modification as if, instead of being reflected on glass, it had been transmitted through a tourmaline plate, whose axis was perpendicular to the plane of reflexion. 824.

It has, moreover, acquired all the other enumerated characters of a polarized ray. And, first, it has become

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incapable of reflexion at the surface of glass, or other transparent media at certain definite angles, and in certain positions of the plane of incidence. To show this experimentally, let a piece of polished glass have one of its surfaces roughened, and blackened with melted pitch or black varnish, so as to destroy its internal reflexion, and let this be fixed on a stand, so as to be capable of varying at will the inclination of its polished surface to the horizon, and of turning it round a vertical axis in any azimuth. A very convenient stand of this kind is figured in fig. 171, consisting of a cylindrical support A sliding in a vertical tube B, attached to a round base F like a candlestick, and carrying an arm C, which can be set to any angle of inclination to the horizon by means of a stiff shoulder joint D. To this arm the blackened glass E is fixed, having its plane parallel to the axis of the joint D. Let this apparatus be set on a table, so that the rays reflected from a pretty large plate of glass G, at an angle of about 57° (of incidence) shall be received on the glass E, which ought to be inclined with its polished surface looking downwards, and making an angle of about 73° with the horizon, see Art. 842. Then let the observer apply his eye near the glass E, so as to see the glass G reflected in it, and slowly turn the stand F round in a horizontal plane, keeping always the reflected image of G in view. He will then perceive, that at a certain point of the rotation of the stand, the illumination of this image, which in other situations is very bright, will undergo a rapid diminution, and at last wholly disappear, and (if the glass G be large enough) the same appearance of a cloud or large dark spot will then be visible upon it. If the inclination of the arm C D be correct, it will be easy to find such a position by turning the stand a little backwards and forwards, as shall make the centre of this spot totally black; if not, bring it to as great a degree of obscurity as possible by the horizontal motion, then, holding fast the stand, vary a little one way or another the inclination of the reflector E, and a very complete obscurity will readily be attained.

825. Experiment. The polarized ray incapable of a second reflexion, &c. Fig. 171.

826. Another mode of making the experiment. Fig. 172.

Another, and, for some experimental purposes, a better way of exhibiting the same phenomenon, is to take two metallic or pasteboard tubes, open at both ends, and fitting into each other so as to turn stiffly. Into each of these, at the end remote from their junction, fix with wax, or in a frame, a plate of glass, blackened at the back as above described, so as to make an angle of 33° with the axis of the tube, as represented in fig. 172. Then having placed the tube containing one of the plates (A) so that the light from any luminary, reflected at the plate shall traverse the axis of the tube, fix it there, and the reflected ray will be again reflected at B, and on its emergence may be received on a screen or on the eye. Now make the tube containing the reflector B revolve within the other, so that that reflector shall revolve round the ray A B as an axis, preserving the same inclination. Then will the twice reflected ray revolve with equal angular motion, and describe a conical surface. But in so doing, it will be observed to vary in intensity, and at two points of the revolution of the tube B will *disappear altogether*. Now if we attend to the position of the reflectors at this moment, it will be found that *the planes of the first and second reflexion make a right angle.*

827. By repeating these experiments with all sorts of reflecting media, and determining by exact measurement the angles at which the original ray must be incident that polarization shall take place, and those at which a polarized ray ceases to be reflected, the following laws have been ascertained to hold good, previous to announcing which a definition will be necessary.

828. Plane of polarization defined.

Definition. The plane of polarization of a polarized ray is the plane in which it must have undergone reflexion, to have acquired its character of polarization; or that plane passing through the course of the ray perpendicular to which it cannot be reflected at the polarizing angle from a transparent medium; or, again, that plane in which, if the axis of a tourmaline plate exposed perpendicularly to the ray be situated, no portion of the ray will be transmitted. Also, a polarized ray is said *to be polarized in* its plane of polarization, as just defined.

829. Sides of a polarized ray.

The plane of polarization of any polarized ray is to be considered as one of the *sides* of the ray which thus, in all its future progress, carries with it certain relations to surrounding fixed space, which must be regarded, while they continue unchanged, as inherent in the ray itself, and as having no further any relation to the particular mode in which they originated.

830. Laws of polarization by reflexion. Law 1.

The laws of polarization by reflexion are these:

Law 1. All reflecting surfaces are capable of polarizing light if incident at proper angles; only, metallic bodies, or bodies of very high refractive powers, appear to do so but imperfectly, the reflected ray not *entirely* disappearing in circumstances when a perfectly polarized ray would be completely extinguished. Of this more hereafter.

831. Law 2. Brewster's law of the tangents.

Law 2. Different media differ in the angles of incidence at which they polarize light; and it is found, that these angles may always be determined from the following simple and elegant relation, discovered by Dr. Brewster after a laborious examination of an infinite variety of substances.

The tangent of the polarizing angle for any medium is the index of refraction belonging to that medium.

Thus, the indices of refraction of water, crown-glass, and diamond, being respectively 1.336, 1.535, and 2.487, their respective polarizing angles will be 53° 11′, 56° 55′, and 68° 6′. For diamond, however, or bodies of very high refractive powers, we must understand by the polarizing angle, that angle of incidence at which the reflected ray approximates most nearly to the character of a ray completely polarized.

832. All the colours not polarized at one incidence.

It follows from this law, that one and the same medium does not polarize all the coloured rays at the same angle, and that therefore the disappearance of the reflected pencil can never be total, except where the incident ray is homogeneous. This will account *in some degree* for the want of complete polarization of a white ray, reflected at any angle from highly refractive media, which are generally also highly dispersive. Of the reality of the fact, it is easy to satisfy oneself by a very simple experiment, which we have often made. Receive a sunbeam on a plane glass, with the back roughened and blackened, at an incidence (θ) nearly equal to the polarizing angle (a,) and let the reflected ray pass into a darkened room, and fall on another similar glass, which may be held in the hand, so as to reflect the ray in a plane at right angles to that of the first reflection, and

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also at an angle (θ') nearly equal to the polarizing angle (a') of the second plate. It will be easy to find a position where the reflected ray (which must be received on a white screen) very nearly vanishes; but no adjustment of the angles of incidence θ and θ' will produce a *total* disappearance. When the disappearance is most nearly total, the reflected light is coloured of a neutral purple; the yellow, or most luminous rays, being now totally extinguished. In this position, if θ remain constant, and θ' the incidence on the second plate be varied a little on one side or the other of the polarizing angle a', the reflected ray assumes on the one hand a pretty intense blue-green, and on the other a ruddy plum colour or amethyst red. The several changes of tint, arising from variations of incidence on both plates, were observed to be as follows: Proved by experiment.

1st.	$\theta < a$;	$\theta' < a'$;	Reflected ray, Strong green.
		Intermediate,	——— White.
		$\theta' > a'$;	——— Pale red or amethyst.
2d.	$\theta = a$;	$\theta' < a'$;	——— Strong blue green.
		$\theta' = a'$;	——— Neutral purple.
		$\theta' > a'$;	——— Strong plum colour.
3d.	$\theta > a$;	$\theta' < a'$;	——— Light greenish blue.
		Intermediate,	——— White.
		$\theta' > a'$;	——— Strong red, or plum colour.

The rationale of these changes of colour will be more evident when we have announced the following law, which expresses one of the most general and distinguishing characters of polarized light.

833. *Law* 3. When a polarized ray (no matter how it acquired its polarization) is incident on a reflecting surface of a transparent, or other medium capable of completely polarizing light, in a plane perpendicular to that of the ray's polarization, and at an angle of incidence equal to the polarizing angle of the medium, no portion of the ray will be reflected. If the medium be of such a nature as to be capable only of incompletely polarizing light, a portion will be reflected, but much less intense than if the incident ray were unpolarized.

Law 3. Non-reflexibility of polarized light in certain, and what cases.

It is evident that this property may be employed to distinguish polarized from common light, as well as that of extinction by a plate of tourmaline. It is, however, much less convenient though better adapted for delicate inquiries.

834. The polarizing angle for white light is, in fact, the angle for the most luminous or mean yellow rays; and when the two reflexions, in planes at right angles to each other, are made at this angle, the yellow rays only totally escape reflexion, but a very small portion both of the red and blue end of the spectrum are reflected, and form a feeble purple beam, such as above described. The polarizing angle for red rays being less than for violet, it is evident that when either θ or θ' is equal to the polarizing angle for red, it will be less than that of yellow, and still less than that of blue and violet rays; thus, the red disappears most completely from the reflected beam in those cases when θ or θ' are less than a or a', leaving an excess of the green and blue rays, and *vice versâ* in the converse cases. Thus, too, if θ be $< a$, and at the same time $\theta' < a$, the colour produced will be a more intense green than if the incidences deviated opposite ways from the polarizing angles; and it is evident, that a compensation may arise from the effect of such opposite deviations giving an intermediate white ray, exactly as we see to have happened.

Explanation of the colours in the last experiment.

835. Some very remarkable consequences follow from the law announced by Dr. Brewster for finding the polarizing angle, which may be presented in the form of distinct propositions. Thus,

836. *Prop.* 1. When a ray is incident on a transparent surface, so that the reflected portion shall be completely polarized, the reflected and refracted portions make a right angle. For θ being the angle of incidence, we have $\tan \rho = \mu$ and ρ, being the angle of refraction, $\sin \rho = \frac{\sin \theta}{\mu} = \frac{\sin \theta}{\tan \theta} = \cos \theta$. Therefore $\rho = 90° - \theta$, but θ being the angle of incidence is also that of reflexion, and $\rho + \theta$ is therefore equal to the supplement of the angle between the reflected and refracted rays, which is therefore a right angle. Q. E. D.

Consequences of the law of polarization.

837. *Prop.* 2. When a beam of common light is incident at the polarizing angle on a parallel plate of a transparent medium, not only the portion reflected at the first surface, but also that reflected internally at the second, and the compound reflected ray, consisting of both united, are polarized.

Polarization by internal reflexion.

Since $\sin \rho = \cos \theta$, and since ρ is also the angle of incidence on the second surface, we shall have $\tan \rho = \text{cotan } \theta = \frac{1}{\tan \theta} = \frac{1}{\mu} =$ index of refraction *out* of the medium. Hence, ρ is the angle of polarization for rays internally incident, and therefore that portion of the beam which, having penetrated the first surface, falls on the second, being incident at its polarizing angle, the portion reflected *here* will also be polarized, and being again incident on the first surface, in the plane of its polarization, that part of it which is transmitted will (as we shall see hereafter) suffer no change in its plane of polarization, so that both it and the first reflected ray will come off polarized in the same plane. Q. E. D.

838. *Corol.* 1. Hence, to obtain a stronger polarized ray, we may dispense with roughening or blackening the posterior surface, provided we are sure that the surfaces are truly parallel.

839. If a series of parallel plates be laid one on the other so as to form a pile, the portions reflected from the several surfaces all come off polarized in the same plane, and by this means a very intense polarized ray may be obtained. It can never, however, for a reason we shall presently state, contain more than half the incident light, whatever be the number of plates employed.

Mode of obtaining an intense polarized beam.

840. For a great variety of optical experiments, a pile consisting of ten or a dozen panes of common window-glass set in a frame, is of great use and very convenient. Such a pile laid down before an open window affords a dispersed beam, each ray of which is polarized at the proper angle, and of great intensity and very proper for the exhibition of many of the phenomena hereafter to be described.

841. *Prop.* 3. If a ray be completely polarized by reflexion at the surface of one medium, and the reflected ray completely transmitted or absorbed at that of a second, Required the inclination of the two surfaces to each other?

Let a and a' be the polarizing angles of the respective media; then, since the planes of reflexion are at right angles to each other, and a, a' are the angles of incidence, if we call I the inclination required, we shall have by Art. 104, $\cos I = \cos a \,.\, \cos a'$. Now, if μ, μ' be the refractive indices of the media, we have $\tan a = \mu$, $\tan a' = \mu'$, and therefore

$$\tan I = \sqrt{\mu^2 + \mu'^2 + \mu^2 \mu'^2}.$$

842. *Corol.* 1. If the media be both alike,

$$\tan I = \mu \,.\, \sqrt{2 + \mu^2}; \text{ or } \cos I = \frac{1}{1+\mu^2}.$$

Thus, in the case of crown-glass, $\mu = 1.535$ and $I = 72^\circ\, 40'$, as in Art. 825.

843. Method of determining refractive indices by polarization. By the help of this law, connecting the angle of polarization with the refractive index, we may easily deduce the one from the other. This affords a valuable and ready resource in cases to which other methods can hardly be applied, for ascertaining the refractive powers of media, which are either opake, or in such small or irregularly shaped masses, that they cannot be used as prisms. For ascertaining the angle of polarization, only one polished surface, however small, is necessary, and we have only to receive a ray reflected from it on a blackened glass, or other similar medium of known refractive index, *at the polarizing angle*, and in a plane perpendicular to that at which it is reflected by the surface under examination. For this purpose it is convenient to have the glass plate (or, which is better, a polished plate of obsidian or dark coloured quartz) set in a tube diagonally, so as to reflect laterally the ray which traverses the axis of the tube. At the other end, the substance to be examined must be fixed on a revolving axis perpendicular to the axis of the tube, and having its plane adjusted so as to be parallel to the former, which must then be turned round till the dispersed light of the clouds, reflected by it, is entirely extinguished by the obsidian plate, and the inclination of the reflecting surface to the axis of the tube in this situation may be measured by a divided circle, connecting with the axis of rotation. By this means we may ascertain the polarizing angles, and therefore the refractive indices of the smallest crystals, or of polished stones, gems, &c., set in such a manner as not to admit of other modes of examination. To insure a fixed zero point on the graduated circle, the following mode (among many others) may be resorted to. Let a polished metallic reflector or small piece of looking-glass be permanently attached to the revolving axis, so that its plane shall be perpendicular to the axis of the tube, when the index of the divided circle marks $0^\circ\, 0'$. This adjustment being made once for all, let the surface to be examined be attached by wax or otherwise, not to the axis itself, but to a ring turning stiffly on it. Then, bringing the image of the sun, or any very distant object, sufficiently bright or well defined, seen in the reflector, to coincide with any other equally well defined, and also at a great distance, alter the attachment of the substance by pressure on the wax, and by turning round the ring, till a similar coincidence is obtained when the eye is transferred to *it*. Then we are assured that the two surfaces are parallel, and that therefore the reading off on the circle measures the true angle between the axis of the tube and the perpendicular, or the angle of reflexion, or at least differs from it only by a constant quantity, which may be ascertained at leisure, and applied as index error. (This mode of bringing a movable surface to a fixed position with respect to the divisions of an instrument, is applicable to a great variety of cases, and is at once convenient and delicate.)

844. Irregular polarization by glass surfaces. Dr. Brewster has remarked, that glass surfaces frequently exhibit remarkable, and apparently unaccountable, deviations from the general law; but on minute examination he found that this substance is liable to a superficial tarnish, or formation of infinitely thin films of a different refractive power from the mass of glass beneath. As the polarized ray never penetrates the surface, its angle of polarization is determined solely by this film, which is too thin to admit of any direct measure of its refractive index. When this tarnish has gone to a great extent, scales of glass detach themselves, as is seen in very old windows, (especially those of stables,) and even in green glass bottles which have long lain in damp situations, and which acquire a coat actually capable of being mistaken for gilding.

845. Action of metallic surfaces. In metallic or adamantine bodies, which polarize light but imperfectly, that angle at which the reflected beam approaches nearer in its character to those described as of polarized light, is to be taken for the angle of polarization, and from it the refractive power may still be found. The results deduced by this means for metallic bodies, agree with those obtained from the quantity of light reflected, in assigning very high refractive powers to them. Thus, for steel the polarizing angle is found to be above 71°, and for mercury $76\frac{1}{2}^\circ$, and their indices of refraction are, therefore, respectively 2.85 and 4.16. This latter result, indeed, differs greatly from that of Art. 594, but the observations are so uncertain, and the angle of greatest polarization so indefinitely marked, (not to mention the errors to which a determination of the reflective power itself is liable to,) that we cannot expect coincidence in such determinations. Perhaps 5.0 may be taken as a probable index.

846. The law of polarization announced by Dr. Brewster is general, and applies as well to the polarization of light at the separating surfaces of two media in contact, as at the external or internal surface of one and the same medium. He has attempted to deduce from it several theoretical conclusions, as to the extent and mode of action of the reflecting and refracting forces, for which we must refer the reader to his Paper on the subject. *Philosophical Transactions*, 1816.

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If a ray be reflected at an angle greater or less than the polarizing angle, it is *partially polarized*, that is to say, when received at the polarizing angle on another reflecting surface, which is made to revolve round the reflected ray without altering its inclination to it, the twice reflected ray never vanishes entirely, but undergoes alternations of brightness, and passes through states of maxima and minima which are more distinctly marked according as the angle of the first reflexion approaches more nearly to that of complete polarization. The same is observed when a ray so *partially* polarized is received on a tourmaline plate, revolving (as above described) in its own plane. It never undergoes complete extinction, but the transmitted portion passes through alternate maxima and minima of intensity, and the approach to complete extinction is the nearer the nearer the angle of reflexion has been to the polarizing angle. We may conceive a partially polarized ray to consist of two unequally intense portions; one completely polarized, the other not at all. It is evident that the former, periodically passing from evanescence to its total brightness, during the rotation of the tourmaline or reflector, while the latter remains constant in all positions, will give rise to the phenomenon in question. And all the other characters of *a partially polarized ray* agreeing with this explanation, we may receive it as a principle, that when a surface does not completely polarize a ray, its action is such as to leave a certain portion completely unchanged, and to impress on the remaining portion the character of complete polarization. Thus we must conceive polarization as a property or character not susceptible of *degree*, not capable of existing sometimes in a more, sometimes in a less, intense state. A single elementary ray is either wholly polarized or not at all. A *beam* composed of many *coincident rays* may be partially polarized, inasmuch as some of its component rays only may be polarized, and the rest not so. This distinction once understood, however, we shall continue to speak of a ray as wholly or partially polarized, in conformity with common language. We shall presently, however, obtain clearer notions on the subject of *unpolarized* light, and see reason for discarding the term altogether.

847. Partial polarization. How conceived.

848. Polarization by several reflexions in one plane.

If a ray be partially polarized by reflexion, Dr. Brewster has stated that a second reflexion *in the same plane* renders this polarization more complete, or diminishes the ratio of the unpolarized to the polarized light in the reflected beam; and that by repeating the reflexion, the ray may be completely polarized, although none of the angles of reflexion be the polarizing angle. Thus he found, that one reflexion from glass at 56° 45′ of incidence, two at incidences of 62° 30′ or at 50° 20′, three at 65° 33′ or at 46° 30′, four at 67° 33′ or 43° 51′, and so on, alike sufficed to operate the complete polarization of the ray finally reflected, provided all the reflexions were made in one plane. At angles above 82°, or below 18°, more than 100 reflexions were required to produce complete polarization.

§ IV. *Of the Laws of Reflexion of Polarized Light.*

849.

When polarized light is reflected at any surface, transparent or otherwise, the *direction* of the reflected portion is precisely the same as in the case of natural light, the angle of reflexion being equal to that of incidence; the laws we are now to consider are those of the intensity of the reflected light, and of the nature of its polarization after reflexion.

850. Intensity of reflection of a polarized ray incident at the polarizing angle in any plane.

One essential character of a polarized ray is, its insusceptibility of reflexion in a plane at right angles to that of its polarization when incident at a particular angle, *viz.* the polarizing angle of the reflecting surface. In this case, the intensity I of the reflected ray is 0. In all other cases it has a certain value, which we are now to inquire. Let us suppose, then, to begin with the simplest case, that the polarized ray fails on the reflecting surface at a constant angle of incidence, equal to its polarizing angle, and that the reflecting surface is turned round the incident ray as an axis, so that the plane of incidence shall make an angle ($= a$) of any variable magnitude with the plane of polarization. It is then observed, as we have seen, that when $a = 90°$ or $270°$, we have $I = 0$, and when $a = 0°$, or $180°$, I is a maximum. Hence, it is clear that I is a periodic function of a, and the simplest form which can be assigned to it (since negative values are inadmissible) is $I = A \,.\, (\cos a)^2$. This value, which was adopted by Malus on no other grounds than those we have stated, is however found to represent the variation of intensity throughout the quadrant, with as much precision as the nature of photometrical experiments admits, and we must therefore receive it as an empirical law at present, for which any good theory of polarization ought to be capable of assigning a reason *a priori*.

851. An unpolarized ray equivalent to two polarized ones.

A remarkable consequence follows from this law. It is that, so far as the intensity of the reflected ray is concerned, an ordinary or unpolarized ray may be regarded as composed of two polarized rays, of equal intensity, having their planes of polarization at right angles to each other. For such a compound ray being incident on a reflecting surface, as above supposed, if a be the inclination of the plane of polarization of one portion to that of incidence, $90 - a$ will be that of the other, and, therefore, since

$$A \,.\, (\cos a)^2 + A \,.\, (\cos \,.\, 90 - a)^2 = A, \qquad (a)$$

the reflected ray will be independent of a, and therefore no variation of intensity will be perceived on turning the reflecting surface round the incident ray as an axis, which is the distinguishing character of unpolarized light. *Any such pair of rays as here described are said to be oppositely polarized.*

852. Fresnel's general law for the intensity of a reflected ray.

When the polarized ray is not incident at the polarizing angle, but at any angle of incidence, the law of intensity of the reflected ray is more complicated. M. Fresnel has stated the following as the general expression for it. Let the intensity of the incident ray be represented by unity, and calling, as before, a the inclination of the plane of incidence to that of primitive polarization, and i the angle of incidence, i' the corresponding angle of refraction. Then will the intensity of the reflected ray be represented by

$$I = \frac{\sin^2 (i - i')}{\sin^2 (i + i')} \cdot \cos^2 a + \frac{\tan^2 (i - i')}{\tan^2 (i + i')} \cdot \sin^2 a. \qquad (b)$$

This formula is in some degree empirical, resulting partly from theoretical views, of which more hereafter, and being not yet verified, or indeed compared with experiment, except in particular cases, by M. Arago, whose results, so far as they go, are consonant with it.

853. Particular cases examined. It will be well to examine some of these. And first, then, when $a = 90°$, and $i =$ the polarizing angle of the reflecting surface, we have by (835 and 836) $i + i' = 90°$, and therefore $\tan (i + i') = \infty$, so that $I = 0$. In these circumstances, then, the reflected ray is completely extinguished, which agrees with fact.

854. Perpendicular incidence. 2dly. When the incidence is perpendicular, we have, in this case, both i and i' vanishing, and each term of I takes the form $\frac{0}{0}$. Now at the limit we have (μ being the refractive index) $i = \mu \cdot i'$, and very small arcs being equal to their sines or tangents, we have $\sin (i - i') = i' (\mu - 1)$; $\sin (i + i') = i' (\mu + 1)$, and so for the tangents. Consequently,

$$I = \left(\frac{\mu - 1}{\mu + 1}\right)^2 \cdot (\cos^2 a + \sin^2 a) = \left(\frac{\mu - 1}{\mu + 1}\right)^2,$$

which agrees with the expression deduced by Dr. Young and M. Poisson, (Art. 592,) for the intensity of the reflected ray in the case of unpolarized light. And if we regard the unpolarized ray as composed of two rays, each of the same intensity, ($= \frac{1}{2}$) polarized in opposite planes, the reason of the coincidence will be evident.

855. 3d. When $a = 0$, or the plane of polarization coincides with the plane of incidence, we have, in general,

$$I = \frac{\sin^2 (i - i')}{\sin^2 (i + i')}. \qquad (c)$$

856. 4th. When $a = 90°$, or when the plane of polarization is at right angles to the plane of incidence,

$$I = \frac{\tan^2 (i - i')}{\tan^2 (i + i')}. \qquad (d)$$

857. Intensity of reflexion of natural light 5th. When $a = 45$,

$$I = \frac{1}{2} \left\{ \frac{\sin^2 (i - i')}{\sin^2 (i + i')} + \frac{\tan^2 (i - i')}{\tan^2 (i + i')} \right\}. \qquad (e)$$

This last is the same result with that which would result from the supposition of two equal rays polarized, the one *in*, the other *perpendicularly to*, the plane of incidence, and each of half the intensity with the incident beam. It is therefore the general expression for the intensity of a ray of natural or unpolarized light reflected at an incidence $= i$ from the surface. The expressions in Art. 592 apply only to perpendicular incidences. We are thus furnished very unexpectedly with a solution of one of the most difficult and delicate problems of experimental Optics. Bouguer is the only one who has made any extensive series of photometrical experiments on the intensity of light reflected from polished surfaces at various angles, but his results are declared by M. Arago to be very erroneous, which is not surprising, as the polarization of light was unknown to him, and its laws might affect the circumstances of his experiments in a variety of ways.

858. Polarization of the light of the sky. One only need be mentioned, as every optical experimentalist ought to be aware of, and on his guard against it, it is that the light of clear, blue sky, is always partially polarized in a plane passing through the sun, and the part from which the light is received. The polarization is most complete in a small circle, having the sun for its pole, and its radius about 78°, (according to an experiment not very carefully made.) Now the semi-supplement of this (which is the polarizing angle) is 51°, which coincides nearly with the polarizing angle of water, (52° 45′.) Thus strongly corroborating Newton's theory of the blue colour of the sky, which he conceives to be the blue of the first order, reflected from particles of water suspended in the air. Dr. Brewster is the first, we believe, who noticed this curious fact. But to return to our subject.

859. Case of a ray partially polarized. When the incident ray is only partially polarized, we may regard it as consisting of two portions: the one, which we shall represent by a, completely polarized in a plane, making the angle a with that of incidence; the other $= 1 - a$ in its natural state, or, if we please, composed of two portions $\left(\frac{1 - a}{2}\right)$, one polarized in the plane of incidence, and one at right angles to it. The intensity of the reflected portion of the former is equal to

$$a \cdot \frac{\sin^2 (i - i')}{\sin^2 (i + i')} \cos^2 a + a \cdot \frac{\tan^2 (i - i')}{\tan^2 (i + i')} \cdot \sin^2 a,$$

and that of the latter will be represented by

$$\frac{1 - a}{2} \left\{ \frac{\sin^2 (i - i')}{\sin^2 (i + i')} + \frac{\tan^2 (i - i')}{\tan^2 (i + i')} \right\}$$

therefore, their sum, or the total reflected light, will be

$$\frac{\sin^2 (i - i')}{\sin^2 (i + i')} \cdot \frac{1 + a \cdot \cos 2a}{2} + \frac{\tan^2 (i - i')}{\tan^2 (i + i')} \cdot \frac{1 - a \cdot \cos 2a}{2}.$$

The above formulæ, it must be observed, apply only to the case of reflexion from the surfaces of uncrystallized media. The consideration of those where crystallized surfaces are concerned, cannot be introduced in this part of the subject.

Light. Part IV.

When the plane of reflexion coincides with that of the primitive polarization of the ray, the polarization is not changed by reflexion. Hence, at a perpendicular incidence it is unchanged. But in other relative situations of the two planes above-mentioned, the case is different, and it becomes necessary to inquire what change reflexion produces in the state and plane of polarization of the ray. Now it is found, as we have already seen, that when the reflection takes place in the plane of primitive polarization, if the incident ray be only partially polarized, the reflected one will be more so, in that plane. But if the incident ray be completely polarized, it retains this character after reflexion, (except in one remarkable case,) and only the *plane* of polarization is changed. Now, according to M. Fresnel, the new plane of polarization will make an angle with the plane of reflexion, represented by β, such that

860. Position of the plane of polarization of the reflected ray.

$$\tan \beta = \frac{\cos (i + i')}{\cos (i - i')} \cdot \tan a.$$

According to this formula, the plane of polarization coincides with the plane of incidence when $i + i' = 90°$. Now this is precisely the case when the ray falls at the polarizing angle on the reflecting surface. If $a = 90°$, or the ray before incidence be polarized in a plane perpendicular to the plane of incidence, it will continue to be so after reflexion, since in that case we have $\tan \beta = \infty$, or $\beta = 90°$.

861. The formula has been compared by M. Arago with experiment only in one intermediate case, *viz.* when $a = 45°$, and the coincidence of the results with experiment at a great variety of incidences, and over a range of values of β from $+ 38°$ to $- 44°$, both in the case of glass and water, is as satisfactory as can be desired. The particulars of this interesting comparison will be found in *Annales de Chimie*, xvii. p. 314. It may be observed also, that these results of M. Fresnel support one another, the latter being concluded from the former by considerations purely theoretical, so that every verification of the one is also a verification of the other.

862. Reflexion from crystallized surfaces.

When the polarized ray is reflected from a crystallized surface, the intensity of the reflected portion is no longer the same, but depends on the laws of double refraction, in a manner of which more hereafter. Whether, or how far, the laws above stated hold good for metallic surfaces, remains open to inquiry.

§ V. *Of the Polarization of Light by ordinary Refraction, and of the Laws of the Refraction of Polarized Light.*

863. Polarization by refraction.

When a ray of natural or unpolarized light is transmitted through a plate of glass at a perpendicular incidence, it exhibits at its emergence no signs of polarization; but if the plate be inclined to the incident ray, the transmitted ray is found to be partially polarized in a plane at right angles to the plane of refraction, and therefore at right angles to the plane of polarization of the portion of the reflected ray which has undergone that modification. The connection between the polarized portions of the reflected and refracted pencils is, however, still more intimate, since M. Arago has shown by a very elegant and ingenious experiment that these portions are always of equal intensity. This law may be stated thus: *When an unpolarized ray is partly reflected at, and partly transmitted through, a transparent surface, the reflected and transmitted pencils contain equal quantities of polarized light, and their planes of polarization are at right angles to each other.* Arago's law.

864. Hence it appears, that the transmitted ray contains a maximum of polarized light, when the light is incident at the polarizing angle of the medium, and this maximum is equal to the quantity of light the surface is capable of completely polarizing by reflexion. Now in all media known, this is much less than half the incident light, consequently the transmitted portion can never be wholly polarized by a single transmission.

865. When a ray is totally reflected at the inner surface of a medium, there is no transmitted portion, and it is a remarkable coincidence with the above law, that in this case the reflected beam contains no polarized portion whatever.

866. Polarization by several oblique transmissions.

With regard to the portion of light which has passed through the surface, and has not acquired polarization, M. Arago maintains that it remains in the state of natural or totally unpolarized light. Dr. Brewster, on the other hand, concludes from his experiments, that, although not polarized, it has undergone a physical change, rendering it more largely susceptible of polarization by subsequent transmission at the same angle. The question, in a theoretical point of view, is a material one, and *apparently* very easily decided. The facility, however, is only apparent, and as we have no title to decide it on the grounds of our own experience, we shall content ourselves with reasoning on the conclusions to which the two doctrines lead. Let 1 be the light incident on the first surface of a glass plate at the polarizing angle, and, after transmission through both surfaces, let $a + b$ be the intensity of the transmitted beam, (and of course $1 - a - b$ that of the reflected,) and let a be the polarized portion and b the unpolarized. When $a + b$ falls on another plate at the same angle, the portion a being polarized in a plane perpendicular to that of incidence, and incident at the polarizing angle, will be totally transmitted, and *its plane of polarization* (as may be proved by direct experiment) *in this case undergoes no change.* Hence the portion a will be transmitted (supposing no absorption) undiminished through any number of subsequent plates. With regard to the portion b, if this be to all intents and purposes similar to natural light, it will be divided by reflexion at the second plate into two portions, the first of which $= b \cdot (1 - a - b)$ being reflected wholly polarized, and the other $= b\,(a + b)$ will be transmitted. Of this, the portion $b\,a$ will be polarized in a plane at right angles to that of refraction, and will therefore be afterwards transmitted undiminished through all the subsequent plates. But the portion b^2 will be unpolarized light, and will be again divided by the third plate, and so on. Thus, there will be ultimately transmitted a pencil, consisting of a polarized portion

$= a + b\,a + b^2\,a + \ldots\ldots\ b^{n-1}\,a = a \cdot \frac{1 - b^n}{1 - b}$, and an unpolarized portion $= b^n$, so that no finite number of plates could ever *completely* polarize the whole transmitted beam.

867. Dr. Brewster's theory of imperfect polarization. Brewster's general law.

On the other hand, if the unpolarized portion b of the transmitted beam $a + b$ be more disposed than before, as Dr. Brewster conceives, to subsequent polarization, the progression above stated, instead of converging according to the law of a geometric progression, will converge more rapidly, or may even suddenly terminate under certain physical conditions. Now, Dr. Brewster states it as a general law, deduced from his own experiments, that *If a pencil of light be incident on a number of uncrystallized plates, inclined at the same or different angles, but all their surfaces being perpendicular to the plane of the first incidence, the total polarization of the transmitted pencil will commence when the sum of the tangents of the angles of incidence on each plate is equal to a certain "constant quantity due to the refractive power of the plates, and the intensity of the incident pencil."* This last phrase, which makes the number and position of the plates necessary to operate total polarization, depend on the intensity of the incident light, shows evidently that the *total polarization* here understood, is not mathematically, but only approximatively total. In fact, he states, this constant quantity for crown glass plates, and *for the flame of a wax candle at* 10 *feet distance*, to be equal to the number 41.84. In other words, the remainder of unpolarized light for this intensity of illumination, becomes insensible. Considered in this light, we regard Dr. Brewster's experiments as by no means incompatible with the law of decrease indicated by the geometric progression above-mentioned, and the contrary sense which has been put upon this expression by M. Arago, or his commentator, (*Encyclop. Brit. Supp.*, vol. vi. part 2, *Polarization of Light,*) appears to us strained beyond what strict criticism authorizes.

Conceiving, then, as we do, that no decided incompatibility in matter of fact exists between the statements of these distinguished philosophers, we cannot but regard as most simple, that doctrine which recognises no change of physical character in the unpolarized portion of either the transmitted or reflected beam. (See Art. 848.)

868. Internal reflexions between the plates.

In what has been above said of the polarization of the transmitted ray, we have not taken into consideration that part of the light reflected at each surface which is reflected back again, and traversing (partially at least) all the plates, mixes with the transmitted beam, and, being in an opposite plane, destroys a part of its polarization.

869. Phenomena of piles of plates exposed to polarized light.

If a pile of parallel glass plates be exposed to a polarized ray, so that the angle of incidence be equal to the polarizing angle, and then turned round the ray as an axis preserving the same inclination, the following phenomena take place:

1. When the plane of incidence is at right angles to that of the ray's polarization, *the whole of the incident light is transmitted*, (except what is destroyed by absorption within the substance of the glass, or lost by irregular reflexion from the inequalities in the surface arising from defective polish,) and this holds good whatever be the number of the plates. The polarization of the transmitted ray is unaltered.

2. As the pile revolves round the incident ray as an axis, a portion of the light is reflected, and this increases till the plane of incidence is coincident with the plane of primitive polarization, when the reflected light is a maximum. Now, M. Arago assures us, that the quantity of *polarized* light reflected from each plate is greater in proportion to the intensity of the incident beam than if natural light had been employed; and the same proportion holding good at each plate, the transmitted ray, however intense it may have been at first, will be weakened in geometrical progression with the number of plates, and at length will become insensible; so that in this situation the pile will present the phenomenon of an opaque body. In this reasoning, the light reflected backwards and forwards between the plates is neglected; but as it is all polarized in the same plane, and as in this situation the reflexions, however frequent, produce no change in its plane of polarization, all the reflected rays are in the same predicament; and, supposing the number of plates very great, the total extinction of the transmitted light will ultimately (though less rapidly) take place.

870. Phenomena of piles of plates, and of tourmaline plates compared.

Hence, a pile of a great number of glass plates inclined at an angle equal to the complement of the polarizing angle ($35° \pm$) to a polarized ray ought to present the same phenomenon with a plate of tourmaline cut parallel to the axis of its primitive rhomboid, alternately transmitting and extinguishing the whole of the light in the successive quadrants of its rotation, and being thus either opaque or transparent, according to its position. The analogy, however, cannot fairly be pushed farther, so as to deduce from this principle an explanation of the phenomena of the tourmaline; for, although it be true that a plate of tourmaline so cut, *is* composed of laminæ inclined to its surface, these laminæ are in optical contact; and, moreover, their position with respect to the surface is not the same in plates cut in all directions around the axis, because although an infinite number of plates may be cut containing *the axis* of a rhomboid in their planes, only three can have the same relation to its several *faces*, parallel to which the component laminæ must be supposed to lie. Moreover, the phenomena are not produced, unless the tourmaline be coloured. The analogy between piles of glass plates and laminæ of agate (of which more presently) is also, we are inclined to think, more apparent than real.

871. Further analogy.

A pile of plates such as described above presents, moreover, the same difference of phenomena when exposed to polarized and unpolarized light, that a plate of tourmaline does; since in the latter case, supposing the pile sufficiently numerous, one half the incident light is transmitted, completely polarized in a plane perpendicular to that of incidence.

872.

The laws which regulate the polarization of a pencil transmitted by a transparent surface, inclined at any proposed angle to the incident ray, and in any plane to that of its primitive polarization (supposing it polarized) remain open to experimental investigation.

§ VI. *Of the Polarization of Light by Double Refraction.*

873. Light polarized by double refraction oppositely in the two pencils. Experiments in proof thereof.

When a ray of natural light is divided into two by double refraction, in such a manner that the two pencils at their final emergence remain distinct and susceptible of separate examination, they *are both found completely polarized, in different planes*, exactly, or nearly, at right angles to each other. To show this, take a pretty thick rhomboid of Iceland spar, and, covering one side of it with a blackened card, or other opaque thin substance, having a small pinhole through it, hold it against the direct light of a window or a candle, with the covered surface from the eye. Two images of the pinhole will then be seen: one, undeviated from the line joining the eye and the real hole, by the ordinarily refracted rays; and the other, deviating from that line, in a plane parallel to the principal section of the surface of incidence, by the extraordinary. These images will appear, to the naked eye, of equal brightness; but, if we interpose a plate of tourmaline, (as already described,) and turn the latter about in its own plane, they will be rendered unequal, and will appear and vanish alternately at every quarter revolution of the tourmaline; the *ordinary* image being always at its maximum of brightness, and the *extraordinary* one extinct, when the axis of the tourmaline plate is perpendicular to the principal section of the surface of incidence, and *vice versâ* when parallel to it.

874. Experiment varied.

The same thing happens, when, instead of examining the two images through a tourmaline plate, we receive their light on a glass plate inclined at the polarizing angle to it, and turn this plate round the ordinary ray as an axis. The images will appear and disappear alternately, as the reflector performs successive quadrants of its revolution.

875.

Hence, we see that the two pencils are *completely* and oppositely polarized; the ordinary pencil in a plane passing through the axis of the rhomboid; the extraordinary one in a plane at right angles to it.

876. Another form of the experiment.

The same phenomenon is much better seen by using a *prism* of any double refracting crystal, having such a refracting angle as to give two distinctly separated images of a distant object, (as a candle.) These appear and disappear alternately at quarter revolutions of a tourmaline plate or glass reflector, and are of equal brightness at the intermediate half-quarters.

877. Transmission of polarized light through doubly refracting media.

Double refraction, then, polarizes the two refracted pencils oppositely, into which an unpolarized incident ray is separated. Let us now see what happens to a *polarized ray*. For this purpose let a plate of glass be laid down before an open window, so as to polarize the reflected light, and hold the rhomboid of Iceland spar (covered as before) with the covered side from the eye, not (as in the former experiment) against the direct light, but inclined downwards, against the reflected light from the glass. Then, generally speaking, two images of the pinhole will be seen, but of *unequal* intensities; and, *if we turn round the rhomboid, in the plane of the covered side, these images will be seen to vary perpetually in their relative brightness, the one increasing to a maximum, while the other vanishes entirely, and so on reciprocally.* When the principal section of the rhomboid is in the plane of reflexion (*i. e.* of polarization) of the incident ray, the ordinary image is a maximum; the extraordinary is extinct, and *vice versâ* when these two planes make a right angle. The experiment may be advantageously varied by using a doubly refracting prism; and, while looking through it at the polarized image of a candle, turning it round slowly in the plane bisecting its refracting angle.

878. Unequal division of the light between the two refracted pencils.

This experiment leads us to the following remarkable law, *viz.* that if a ray, at its incidence on a doubly refracting surface, be polarized in the plane parallel to the principal section, it will not suffer bifurcation, but will pass wholly into the ordinary image; if, on the other hand, its plane of primitive polarization be perpendicular to the principal section, it will pass entirely into the extraordinary image. In intermediate positions of the plane of primitive polarization, bifurcation takes place, and the ray is *unequally* divided between the two refracted pencils, in every case except when the plane of primitive polarization makes an angle of 45° with the principal section. In general, if a be the angle last mentioned, and A the incident light, (supposing none lost by reflexion,) $A \cdot \cos^2 a$ will be the intensity of the ordinary, and $A \cdot \sin^2 a$ of the extraordinary pencil, their sum being A.

879. Huygens's experiment

All these changes and combinations are exhibited in the following remarkable experiment of Huygens, which, reasoned on by himself and Newton, first gave rise to the conception of a polarity, or distinction of *sides*, in the rays of light when modified by certain processes. Take two pretty thick rhomboids of Iceland spar, (which should be very transparent, as they are easily procured,) and lay them down one upon the other, so as to have their homologous sides parallel, or so that the molecules of each shall have the same relations of situation as if the two rhomboids were contiguous parts of one larger crystal. They should be laid on a sheet of white paper having a small, very distinct, and well-defined black spot on it. This spot then will be seen double through the combined crystals, as if they were one, (*a*, fig. 173,) and the line joining the images will be parallel to the principal section of either. Now, let the upper crystal be turned slowly round in a horizontal plane on the lower, and two new images will make their appearance between the two first seen, which, at first, are very faint, as at *b*, fig. 173, and form a very elongated rhombus with the two former. They increase, however, in intensity, while the other pair diminishes, till the angle of rotation of the upper crystal is 45°, where the appearance of the images is as at c. Continuing the rotation, the rhomb approaches to a square, as at *d*, and the two original images have become extremely faint; and when the rotation is just 90°, they will have disappeared altogether, leaving the others diagonally placed, as at *e*. As the rotation still proceeds, they reappear and increase in brightness, till the angle of revolution $= 90° + 45° = 135°$, when the images are all equal, as at *f*; after which the original images still increasing, and the others diminishing, the appearance *g* is produced, which, on the completion of a precise half revolution, passes into *h* by the union of both the original images into one, and the total evanes- Fig. 137.

cence of the other pair. In this case only single refraction (apparently) happens; or, rather, the double refractions of the two rhomboids taking place in opposite directions, and being equal in amount, compensate each other. Unless, however, the rhomboids be of exactly equal thickness, this precise compensation will not take place, and the images will remain distinct, though at a minimum of distance. We may express the four images thus:

O *o*, the image ordinarily refracted by both rhomboids.
O *e*, the image refracted ordinarily by the first, and extraordinarily by the second.
E *o*, the image refracted extraordinarily by the first, and ordinarily by the second.
E *e*, the image refracted extraordinarily by both.

Then, if A be the intensity of the incident light, supposing none lost by reflexion or absorption,

$$O\,o = \tfrac{1}{2}A \cdot \cos^2 a = E\,e; \quad O\,e = \tfrac{1}{2}A \cdot \sin^2 a = E\,o,$$

and the sum of all the four images = A.

880. The same phenomena (with some unimportant variations) take place when we apply two doubly refracting prisms one behind the other close to the eye, and view a distant object through them, turning one round on the other. The rationale of these phenomena follows so evidently from the laws stated in Art. 875 and 878, that it will not be necessary to enlarge on it.

881. *Use of an achromatic double refracting prism,*

The property of a double refraction, in virtue of which a polarized ray is unequally divided between the two images, furnishes us with a most convenient and useful instrument for the detection of polarization in a beam of light, and for a variety of optical experiments. It is nothing more than a prism of a doubly refracting medium rendered achromatic by one of glass, or still better, by another prism of the same medium properly disposed, so as to increase the separation of the two pencils. The former method is simple; and, when large refracting angles are not wanted, the uncorrected colour in one of the images is so small as not to be troublesome. It is most convenient to make the refracting angle such as to produce an angular separation of about 2° between the images. Thus, in fig. 174, let A B C G F be a prism of Iceland spar, cut in such a manner (we will at present suppose) that the refracting edge C G shall contain the axis of the crystal; and let it be achromatized as much as possible by a prism of glass C D E F G. Then, if Q be a small, colourless, luminous circle of about a degree or two in apparent diameter, as seen by an eye at O, the interposition of the combined prisms will divide it into two, Q and *q*. Now, if the light of Q be completely unpolarized, these two will remain exactly of equal intensity while the prism A B C G is turned round in a plane at right angles to the line of vision. But if any polarity exist in the original light, the two images Q, *q* will, in turning round the prism, appear alternately more and less bright one than the other; and being always seen immediately side by side, the least inequality, and consequently the least admixture of polarized light in the incident beam, will be detected.

Fig. 174. *First achromatized by glass.*

882. *Dr. Wollaston's mode of doubling the separation of images.* Fig. 175.

Iceland spar, from its very great double refraction, is commonly used for these prisms; but it is so soft, and its structure so lamellar, as to be difficult to polish, and still more so to preserve polished. We have found quartz and limpid topaz to answer extremely well. The following ingenious mode of rendering available the low double refraction of the former, due to Dr. Wollaston, is here eminently useful. Let A B C D *a b c d* and E F G H *e f g h* (fig. 175) be two halves of a hexagonal prism of quartz (the form it affects) produced by a section parallel to two of the sides. In the vertical face A D *d a* draw any line L K parallel to the sides, and therefore to the axis of the prism, (which is also that of double refraction,) and join C L, *c k*. Then a plane C L *k c* will cut off a prism C L K *d c* D, having L *k*, D *d*, or C *c*, for its refracting edges, either of which is parallel to the axis. Again, in the other half of the prism join E *f* and H *g*, and cut the prism by a plane passing through these lines; then, regarding either portion as a double refracting prism, having for refracting edges the lines E H, *f g*, *these* will have the axis of double refraction perpendicular to their refracting edges; and, in particular, the axis will lie *in* the faces H E *e h*, or F G *g f* at right angles to H E or *f g*. If, then, we take care to make the refracting angle C L D of the prism C L K *d c* D equal to that of the edge H E of the prism H E *e f g h*; and if we make these two prisms act in opposition to each other, placing the edge H E opposite to D *d*, and the edge *h e* opposite to K L; and having thus brought the two surfaces D L *k d* and H E *e h* in contact, cement them together with mastic, or Canada balsam, it is evident, that their principal sections will be at right angles to each other; and therefore only two images will be formed, the whole of the extraordinary ray of the one prism passing into the ordinary image of the other, and *vice versâ*. Now, to see how this acts to double the separation of the images, let us conceive *m n* to be a luminous line viewed through one of the prisms with its edge downwards and horizontal. It will be separated into two images, *e* and *o*, the one more raised than the other. Suppose the ordinary image to be most refracted. Then, if we interpose the other prism with its edge upwards, both these images will be refracted downwards; but the ordinary image *o*, which was before *most* raised, now undergoing extraordinary refraction, is *least depressed*, and comes into the position *o e*, while the extraordinary one *e*, which was before *least raised* is now *most depressed*, and comes into the situation *e o*; and it is evident that (the refracting angles being equal, and the double refraction of the two prisms the same) the line *o e* will fall as far short of the original line *m n*, as *e o* surpasses it, *viz.* by a quantity equal to the distance between the two first images *o* and *e*; so that the distance between the twice refracted images is double that of those which have undergone only one refraction. We have found this combination extremely advantageous, as quartz takes a very perfect polish, and from its hardness is not liable to injury from scratches.

883. *Action of crystals possessing no double refraction.*

Crystals which have no double refraction may be regarded as limits of those which have, or as crystals in which the two rays are propagated with equal velocity, and therefore undergo no bifurcation; or, in other words, in which the images formed coincide. In this case we should expect to find no polarization of the emergent light, because the two pencils, being polarized at right angles to each other, form together a single ray having the characters of unpolarized light. This is verified by experiment. The light transmitted by fluor spar, for

instance, exhibits no signs of polarization, unless so far as the ordinary action of the surface goes. We are aware of no experiments indicating how far the action of the *surfaces* of feebly double refracting crystals may modify their polarizing forces, or rather their effects on a ray which has penetrated below the surface; or, in other words, how far piles of crystallized laminæ may have an analogous or different action from those of uncrystallized. Dr. Brewster, indeed, found piles of mica films to polarize light by transmission, like glass piles, but the subject is open to further inquiry.

§ VII. *Of the Colours exhibited by Crystallized Plates when exposed to Polarized Light, and of the Polarized Rings which surround their Optic Axes.*

884. This splendid department of Optics is entirely of modern and, indeed, of recent origin. The first account of the colours of crystallized plates was communicated by M. Arago to the French Institute in 1811, since which period, by the researches of himself, Dr. Brewster, M. Biot, M. Fresnel, and, latterly, also of M. Mitscherlich, and others, it has acquired a developement placing it among the most important as well as the most complete and systematic branches of optical knowledge. As might be expected, under such circumstances, as well as from the state of political relations, and the consequent limited intercourse between Britain and the Continent at the period mentioned, an immense variety of results could not but be obtained independently, and simultaneously, or nearly simultaneously, on both sides of the channel. To the lover of knowledge, for its own sake,—the *philosopher*, in the strict original sense of the word,—this ought to be matter of pure congratulation; but to such as are fond of discussing rival claims, and settling points of scientific precedence, such a rapid succession of interesting discoveries must, of course, afford a welcome and ample supply of critical points, the seeds of an abundant harvest of dispute and recrimination. Regarding, as we do, all such discussions, when carried on in a spirit of rivalry or nationality, as utterly derogatory to the interests and dignity of science, and as little short, indeed, of sacrilegious profanation of regions which we have always been accustomed to regard only as a delightful and honourable refuge from the miserable turmoils and contentions of interested life, we shall avoid taking any part in them; and, taking up the subject (to the best of our abilities and knowledge) as it is, and avoiding, as far as possible, all reference to misconceived facts and over-hasty generalizations, which in this as in all other departments of science, have not failed (like mists at daybreak) to spread a temporary obscurity over a subject imperfectly understood, shall make it our aim to state, in as condensed a form as is consistent with distinctness, such general facts and laws as seem well enough established to run no hazard of being overset by further inquiry, however they may merge hereafter in others yet more general;—a consummation devoutly to be wished.

885. First method of exhibiting the colours of crystallized plates. Instanced in mica.

The general phenomenon of the coloured appearances to which this section is devoted, may be most readily and familiarly shown as follows. Place a polished surface of considerable extent (such as a smooth mahogany table, or, what is much better, a pile of ten or a dozen large panes of glass laid horizontally) close to a large open window, from which a full and uninterrupted view of the sky is obtained; and having procured a plate of mica, of moderate thickness, (about a thirtieth of an inch, such as may easily be obtained, being sold in considerable quantity for the manufacture of lanterns,) hold it between the eye and the table, or pile, so as to receive and transmit the light reflected from the latter as nearly as may be judged at the polarizing angle. In this situation of things, nothing remarkable will be perceived, however the plate of mica be inclined; but if instead of the naked eye we look through a tourmaline plate, having its axis vertical, the case will be very different. When the mica plate is away, the tourmaline will destroy the reflected beam, and the surface of the table, or pile, will appear dark and non-reflective; at least in one point, on which we will suppose the eye to be kept steadfastly fixed. No sooner is the mica interposed, however, than the reflective power of the surface appears to be suddenly restored; and on inclining the mica at various angles, and turning it about in its own plane, positions will readily be found in which it becomes illuminated with the most vivid and magnificent colours, which shift their tints at the least change of position of the mica, passing rapidly from the most gorgeous reds to the richest greens, blues, and purples. If the mica plate be held perpendicular to the reflected beam, and turned about in its own plane, two positions will be found in which all colour and light disappears; and the reflected ray is extinguished, as if no mica was interposed. Now, if we draw on the plate with a steel point two lines corresponding to the intersection of the mica with a vertical plane passing through the eye in either of these two positions, we shall find that they make an exact right angle. For the moment, let us call these lines A and B; and let a plane drawn through the *line* A, perpendicular to the plate, be called *the section* A; and one similarly drawn through the line B, *the section* B. Then we shall observe further, that when we turn the plate from either of these positions, 45° round, in its own plane, so that the sections A and B shall make angles of 45° with the plane of reflexion, (*i. e.* of polarization of the incident ray,) the transmitted light will be a maximum.

Two remarkable sections of the crystallized plate.

886. Law of the tints exhibited at a perpendicular incidence.

If the thickness of the mica do not exceed $\frac{1}{30}$th of an inch, it will be coloured in this position; if materially greater, colourless; and if less, more and more vividly coloured, and with tints following closely the succession of the reflected series of the colours of thin plates, and, like them, rising in the scale, or approaching the central tint (black) as the thickness is less. The analogy in this respect, in short, is complete, with the exception of the enormous difference of thickness between the mica plate producing the tints in question, and those required to produce the Newtonian rings. It appears by measures made in the manner hereafter to be described, that the tint exhibited by a plate of mica exposed perpendicularly to the reflected ray, as above described, is the same with that reflected by a plate of air of $\frac{1}{4000}$th part of the thickness of the mica employed.

If the mica (still exposed perpendicularly to the ray) be turned round in its own plane, the tint does not change, but only diminishes in intensity as its section A or B approaches the plane of polarization of the incident light. When, however, the plate is not exposed perpendicularly, this invariability no longer obtains; and the changes of tint appear in the last degree capricious and irreducible to regular laws. In two situations, however, the phenomena admit a simple view. These are when the sections A and B are both 45° from the plane of polarization, and the mica plate is inclined backwards and forwards *in the plane* of one or the other of these sections. This condition is easily attained by first holding the plate perpendicularly to the reflected ray; then turning it in its own plane till the lines A, B are each 45° inclined to the vertical plane, then finally causing it to revolve about either of these lines as an axis. It will then be seen that when made to revolve *round* one of them (as A) or *in the plane* of the section B, the tint, if white, will continue white at all angles of inclination; but if coloured, will *descend* in the scale of the coloured rings, growing continually less highly coloured, till it passes, after more or fewer alternations, into white; after which, further inclination of the plate will produce no change. On the other hand, if made to revolve *round* B, or *in the plane of* A, the tints will *rise* in the scale of the rings; and when the mica plate is inclined either way, so as to make the angle of incidence about 35° 3′, will have attained its maximum, corresponding to the black spot in the centre of Newton's rings. In this position of the plate, the reflected beam is totally extinguished by the tourmaline, as if the sections A or B had been vertical. But if the angle of incidence be still further increased the colours reappear, and *descend* again in the scale of the rings, passing through their whole series to final whiteness. We take no notice here of a slight deviation from the strict succession of the Newtonian colours, which is observed in the higher orders of the tints, as we shall have more to say respecting it hereafter.

887. Tints exhibited in the two sections above mentioned.

888. Characters of the two most remarkable sections.

We see, then, that the sections A and B, though agreeing in their characters in the case of a perpendicular exposure of the mica, yet differ entirely in the phenomena they exhibit at oblique incidences. If the incidence take place in the plane of the section B, the tint descends, on both sides of the perpendicular, *ad infinitum*. While, if the incidence be in the section A, it rises to the central black, which it attains at equal incidences on either side of the perpendicular (35° 3′), and then descends again *ad infinitum*, or to the composite white at the other extreme of the scale.

889. The principal section defined. Contains the two optic axes. Characters of these axes.

The section A, then, (which, for this reason, we will call the *principal section* of the mica plate,) is characterised by containing two remarkable lines inclined at equal angles to the surface of the plate, along either of which, if a polarized ray be incident, its polarization will not be disturbed by the action of the plate. To satisfy ourselves of this, we have only to fix the mica to the extremity of a tube, so as to have the axis of the tube inclined at an angle of 35° 3′ to the perpendicular (or 54° 57′ to the plate) in the plane of the section A; then directing the axis of the tube to the centre of the dark spot, or the reflecting surface, it will be seen to continue dark, and remain so while the tube makes a complete revolution on its axis. Now, this could not be if the mica exercised any disturbing power on the plane of polarization. Hence, we conclude, that the two lines in question possess this remarkable property, *viz.* that *whatever be the plane of polarization of a ray incident along either of them, it remains unaltered after transmission.* For, although in the experiment above described, the plane of polarization remained fixed, and that of incidence was made to revolve, it is obvious that the reverse process would come to the very same thing.

890. Now, this character belongs to no other lines, however chosen, with respect to the plate. If we fix the plate on the end of the tube at any other angle, or in any other plane with respect to the axis of the latter, although *two* positions in the rotation of the tube will always be found where the disappearance of the transmitted ray takes place, in no other case but that of the two lines in question will this disappearance be total, or nearly so, in *all* points of its revolution.

891. Position of the optic axes in mi

The refracting index of mica being 1.500, an angle of incidence of 35° 3′ corresponds to one of refraction = 22° 31′. Hence, the position of the lines *within* the mica corresponding to these external lines is $22\frac{1}{2}°$ inclined to the perpendicular, and the angle included between them 45°. These, then, are *axes* within the crystal, bearing a determinate relation to its molecules. Dr. Brewster has termed them axes of no polarization, a long name. M. Fresnel, and others, have used the phrase optic axes, to which we shall adhere. As this term has before been applied to the "axes of no double refraction," we must anticipate so far as to advertise the reader that these, and the "axes of no polarization," are in all cases identical.

892 The polarized rings about the optic axes. General description of their phenomena. Fig. 176.

Having, by the criteria above described, determined the principal section, and ascertained the situation of the optic axes of the mica plate under examination, let the plate be inclined to the polarized beam, so that the latter shall be transmitted along the optic axes, the principal section A making an angle of 45° with the plane of polarization; and let the eye (still armed with the tourmaline plate, with its axis vertical) be applied close to the mica. A splendid phenomenon will then be seen. The black point corresponding to the direction of the optic axis will be seen to be surrounded with a set of broad, vivid, coloured rings, of an elliptic, or, at least, oval form, divided into two unequal portions by a black band somewhat curved, as represented in fig. 176. This band passes through the *pole*, or angular situation of the optic axis, about which the rings are formed as a centre. Its convexity is turned towards the direction of the other axis, and on that side the rings are also broader. If, now, the other axis be brought into a similar position, a phenomenon exactly similar will be seen surrounding its place, as a pole. If the mica plate be very thick, these two systems of rings appear wholly detached from, and independent of, each other, and the rings themselves are narrow and close; but if thin (as a 30th or 40th of an inch) the individual rings are much broader, and especially so in the interval between the poles, so as to unite and run together, losing altogether their elliptic appearance, and dilating towards the middle (or in the direction of a perpendicular to the plate) into a broad coloured space, beyond which the rings are no longer formed about each pole separately, but assume the form of reentering curves, embracing and including both poles. Their nature will presently be stated more at large.

If, preserving the same inclination of the mica plate to the visual ray, it be turned about it as an axis, the black band passing through the pole will shift its place, and revolve as it were on the pole as a centre with double the angular velocity, so as to obliterate in succession every part of the rings. When the plate has made 45° of its revolution, so as to bring its principal section into the plane of polarization of the incident beam, this band also coincides in direction with that plane, and is then visibly prolonged, so as to meet that belonging to the set of rings about the other pole; and is crossed at the middle point between the poles by another dark space perpendicular to it, or in the plane of the section B, presenting the appearance in fig. 177. 893. Further particulars. Fig. 177.

These phenomena, if a tourmaline be not at hand, may be viewed, (somewhat less commodiously, unless the mica plate be of considerable size,) by using in its place the reflector figured in fig. 170, or by a pile of glass plates interposed obliquely between the eye and the mica. In this manner of observing them, the colours are surprisingly vivid, no part of the red and violet rays being absorbed more than the rest; whereas the tourmalines generally exert a considerable absorbing energy on these rays in preference to the rest, and thus the contrast of colours is materially impaired. On the other hand, however, from the greater homogeneity of the transmitted light, the rings are more numerous and better defined; and in this respect the phenomenon is greatly improved by the use of homogeneous light. 894. Other modes of exhibiting these phenomena.

We have taken mica as being a crystallized body very easily obtained of large size, and presenting its axes readily, and without the necessity of artificial sections. It is thus admirably adapted for obtaining a general rough view of the phenomena, preparatory to a nicer examination. From the wide interval between its axes, however, and the considerable breadth of its rings, it is less adapted, when employed as above stated, to give a clear conception of the complicated changes which the rings undergo, on a variation of circumstances. For this reason we shall now describe another and much more commodious mode of examining the systems of polarized rings presented by crystals in general, which has the advantage of bringing the laws of their phenomena so evidently under our eyes as to make their investigation almost a matter of inspection. 895.

It is evident, that when we apply the eye close to, or very near a plate of mica, or other body, and view, beyond it, a considerable extent of illuminated surface, each point of that surface will be seen by means of a ray which has penetrated the plate in a different direction with respect to the axes of its molecules; so that we may consider the eye as in the centre of a spherical surface from all points of which rays are sent to it, modified according to the state of primitive polarization, and the influence of the peculiar energies of the medium, corresponding to the direction in which they traverse it, and the thickness of the plate *in that direction.* 896. General principle of methods of viewing the rings.

Any means, therefore, by which we can admit into the eye through the plate and tourmaline a cone of rays nearly or completely polarized in one general direction, or according to any regular law, will afford a sight of the rings; and therefore exhibit, at a single view, a synopsis, as it were, of the modifications impressed on an infinite number of rays so polarized traversing the plate in all directions. The property of the tourmaline so often referred to puts it in our power to perform this in a very elegant and convenient manner, by the aid of the little apparatus of which fig. 178 is a section. A B C D is a short cylinder of brass tube, the end of which, A C, is terminated by a brass plate, having an aperture *a b*, into which is set a tourmaline plate cut parallel to the axis: *h g i k* is another similar brass cylinder, provided with a similar aperture and a similar tourmaline plate G, and fitted into the former so as to allow of the one being freely turned round within the other by the milled edges B D, *h k*. A lens H of short focus, set in a proper cell, is screwed on in front of the tourmaline G, so as to have its focus a little behind its posterior surface, (that next the eye, O.) Between the two surfaces A C, *g i* is another short cylinder of thin tube *c d*, carrying a brass plate with an aperture somewhat narrower than those in which the tourmalines are set, and on which any crystallized plate F to be examined may be cemented with a little wax. This, with the cylinder to which it is fixed, is capable of being turned smoothly round within the cylinder A B C D by means of a small pin *e* passing through a slit *f* made in the side, and extended round so as to occupy about 120° of the circumference; by which a rotation to that extent may be communicated to the crystallized plate F in its own plane between the tourmaline plates. The pin *e* should screw into the ring *c d*, that it may be easily detached, and admit the ring and plate to be taken out for the convenience of fixing on it other crystals at pleasure. Periscopic tourmaline apparatus described. Fig. 178.

The use of the lens H is to disperse the incident light, and thus equalize the field of view when illuminated by any source of light, whether natural or artificial, as well as to prevent external objects being distinctly seen through it, which would distract the attention and otherwise interfere with the phenomena. The rays converged by the lens to a focus within the crystallized plate F, afterwards diverge and fall on the eye O, after traversing the plate in all directions within the limit of the field of view. As by this contrivance they pass through a very small portion of the crystal, there is the less chance of accidental irregularities in its structure disturbing the regular formation of the rings, since we have it in our power to select the most uniform portion of a large crystal. The rays, after passing through the lens, are all polarized by the tourmaline G, in planes parallel to its axis; and passing through the eye in this state, if the crystal F be not interposed, the rays will, or will not, penetrate the second tourmaline, according as its axis is parallel or perpendicular to that of the first. In consequence, when the cylinder carrying the former is turned round within that carrying the latter, the field of view is seen alternately bright and dark. 897. Mode of action of this apparatus.

When the crystallized substance F is interposed, provided it be so disposed that one or other of its optic axes is situated any where in the cone of rays refracted by the lens, so that one of them shall reach the eye by traversing the axis, the polarized rings are seen. If both the axes of the crystal (supposing it to have more than one) fall within the field, a set of rings will be seen round both, and may be studied at leisure. In order to bring the whole of their phenomena distinctly under view, it is requisite to select such crystals as have their axes not much inclined to each other, so as to allow the rings about both to be seen without the necessity of looking very obliquely into the apparatus. In mica the axes are rather too far removed for this. The best crystal we can select for the purpose is nitre. 898. Selection of crystals.

899. Nitre, method of preparing and polishing it.

Nitre crystallizes in long, six-sided prisms, whose section, perpendicular to their sides, is the regular hexagon. They are generally very much interrupted in their structure; but by turning over a considerable quantity of the ordinary saltpetre of the shops, specimens are readily found which have perfectly transparent portions of some extent. Selecting one of these, cut it with a knife into a plate above a quarter of an inch thick, directly across the axis of the prism, and then grind it down on a broad, wet file, till it is reduced to about $\frac{1}{5}$th or $\frac{1}{6}$th inch in thickness; smooth the surfaces on a wet piece of emeried glass, and polish them on a piece of silk strained very tight over a strip of plate glass, and rubbed with a mixture of tallow and colcothar of vitriol. This operation requires practice. It cannot be effected unless the nitre be applied *wet*, and rubbed till quite dry, increasing the rapidity of the friction as the moisture evaporates. It must be performed in gloves, as the vapour from the fingers, as well as the slightest breath, dims the polished surface effectually. With these precautions a perfect vitreous polish is easily obtained. We may here remark, that hardly any two salts can be polished by the same process. Thus, Rochelle salt must be finished wet on the silk, and instantly transferred to soft bibulous linen, and rapidly rubbed dry. Experience alone can teach these peculiarities, and the contrivances (sometimes very strange ones) it is necessary to resort to for the purpose of obtaining good polished sections of soft crystals, especially of those easily soluble in water.

900. Rings exhibited by nitre. Fig. 179. Fig. 180. Fig 181. Fig. 182.

The nitre thus polished on both its surfaces (which should be brought as near as possible to exact parallelism) is to be placed on the plate at F; and the tourmaline plates being then brought to have their axes at right angles to each other (which position should be marked by an index line on the cylinders) the eye applied at O, and the whole held up to a clear light, a double system of interrupted rings of the utmost neatness and beauty will be seen, as represented in fig. 179. If the crystallized plate be made to revolve in its own plane between the tourmalines (which both remain unmoved) the phenomena pass through a certain series of changes periodically, returning, at every 90° of rotation, to their original state. Fig. 180 represents their appearance when the rotation is just commenced; fig. 181, when the angle of rotation is $22\frac{1}{2}°$, or $67\frac{1}{2}°$; and fig. 182, when it equals 45°. When the tourmalines are also made to revolve on each other, other more complicated appearances are produced, of which more presently. We shall now, however, suppose them retained in the situation above mentioned, *i.e.* with their axes crossed at right angles, and proceed to study the following particulars:

1. The form and situation of the rings.
2. Their magnitudes in the same and different plates.
3. Their colours.
4. The intensity of the illumination in different parts of their periphery.

901. Situation of the axes in the crystal.

The situation of the rings is determined by the position of the principal section of the crystal, or by that of the optic axes within its substance. These in nitre lie in a plane parallel to the axis of the prisms, and perpendicular to one or other of its sides. It is no unusual thing to find crystals of this salt whose transverse section consists of distinct portions, in which the principal sections make angles of 60° with each other; indicating a composite or mackled structure in the crystal itself. These portions are divided from each other by thin films, which exhibit the most singular phenomena by internal reflexion, on which this is not the place to enlarge. In an uninterrupted portion, however, the forms of the rings are as represented in the figures above referred to, their poles subtending at the eye an angle of about 8°. Now, it is to be remarked, that as the plate is turned round between the tourmalines, although the black hyperbolic curves passing through the poles shift their places upon the coloured lines, and in succession obliterate every part of them; forming, first, the black cross in fig. 179, by their union; then breaking up and separating laterally, as in fig. 180, and so on. Yet the rings themselves retain the same form and disposition about their poles, and, except in point of intensity, remain perfectly unaltered; their whole system turning uniformly round as the crystallized plate revolves, so as to preserve the same relations to the axes of its molecules. Hence we conclude, that the coloured rings are related to the optic axes of the crystal, according to laws dependent only on the nature of the crystal, and not at all on external circumstances, such as the plane of polarization of the incident light, &c.

902. Form of the rings, that of lemniscates. Fig. 183.

The general form of the rings, abstraction made of the black cross, is as represented in fig. 183. If we regard them all as varieties of one and the same geometrical curve, arising from the variation of a parameter in its equation, it will be evident that this equation must, in its most general form, represent a re-entering symmetrical oval, which at first is uniformly concave, and surrounds both poles, as A; then flattens at the sides, and acquires points of contrary flexure, as B; then acquires a multiple point, as C; after which it breaks into two conjugate ovals D D, each surrounding one pole. This variation of form, as well as the general figure of the curves, bears a perfect resemblance to what obtains in the curve well known to geometers under the name of the *lemniscate*, whose general equation is

$$(x^2 + y^2 + a^2)^2 = a^2 (b^2 + 4 x^2),$$

when the parameter b gradually diminishes from infinity to zero; $2a$ representing the constant distance between the poles.

903. Verified by experiment.

The apparatus just described affords a ready and very accurate method of comparing the real form of the rings with this or any other proposed hypothesis. If fixed against an opening in the shutter of a darkened room, with the lens H outwards, and a beam of solar light be thrown on the latter, parallel to the axis of the apparatus, the whole system of rings will be seen finely projected against a screen held at a moderate distance from E. Now, if this screen be of good smooth paper tightly stretched on a frame, the outlines of the several rings may easily be traced with a pencil on it, and the poles being in like manner marked, we have a faithful representation of the rings, which may be compared at leisure with a system of lemniscates, or any other curve graphically constructed, so as to pass through points in them chosen where the tint is most decided. This has

accordingly been done, and it has been found that lemniscates so constructed coincide throughout their whole extent, to minute precision, with the outlines of the rings so traced, the points graphically laid down falling on the pencilled outlines. The graphical construction of these curves is rendered easy by the well-known property of the lemniscate, in which the rectangle under two lines P A × P′ A drawn from the poles to any point A in the periphery is invariable throughout the whole curve. This is easily shown from the above equation, and the value of this constant rectangle in any one curve is represented by $a \times b$.

904. Variation of the parameter b in arithmetic progression from ring to ring.

When we shift from one ring to another, a remains the same, because the poles are the same for all. To determine the variation of b, let the rings be illuminated with homogeneous light, (or viewed through a red glass,) and outlined by projection, as above. Then, if we determine the actual value of $a\,b$ by measuring the lengths of two lines P A, P′ A drawn from P, P′ to any point of each curve; and, calculating their product, (to which $a\,b$ is equal,) it will be found that this product, and therefore the parameter b, *increases in the arithmetical progression* 0, 1, 2, 3, 4, *&c. for the several dark intervals of the rings beginning at the pole*, and in the progression $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. for the brightest intermediate spaces. To ensure accuracy, the mean of a number of values of P A × P′ A, at different points of the periphery, may be taken to obviate the effect of any imperfection in the crystal.

905. Effect of varying the thickness of the plate.

This, then, is the law of the magnitudes of the successive rings formed by one and the same plate. But if we determine the value of the same product for plates of nitre similarly cut, but of different thicknesses, or of the same reduced in thickness by grinding, it will be found to vary inversely as the thickness of the plate, *cæteris paribus*.

906. The colours of the rings.

The colours of the polarized rings bear a great analogy to those reflected by thin plates of air, and in most crystals would be precisely similar to them but for a cause presently to be noticed. In the situation of the tourmaline plates here supposed (crossed at right angles) they are those of the reflected rings, beginning with a black centre, at the pole. If examined in the situation of fig. 179, and traced in a line from either pole cutting across the whole system, at right angles to the line joining the poles, they will almost precisely follow the Newtonian scale of tints. For the present we will suppose that they do so in all directions. It is evident, then, that each particular tint (as the bright green of the third order, for instance) will be disposed in the form of a lemniscate, and will have its own particular value of the product $a\,b$. The tint, then, may be said to be corresponding to,—dependent on,—or, if we will, *measured by* $a\,b$. In conformity with this language the coloured curves have been termed, and not inaptly, *isochromatic lines*. Now, in the colours of thin plates, we have seen that these tints arise from a law of periodicity to which each homogeneous ray is subject; and that (without entering at this moment into the cause of such periods) the successive maxima and minima of each particular coloured ray passed through, in the scale of tints, correspond to successive multiples by $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, &c. of the period peculiar to that colour. In the colours of thin plates, the quantity which determines the number of periods is the thickness of the plate of air, or other medium traversed; and the number of times a certain standard thickness peculiar to each ray is contained therein, determines the number of periods, or parts of a period, passed through. In the colours, and in the case now under consideration, the number of periods is proportional to the product $(\theta \times \theta')$ of the distances from either pole, for one and the same thickness of plate,—and for different plates to t the thickness,—and, therefore, generally, to $\theta \times \theta' \times t$, provided we neglect the effect of the inclination of the ray in increasing the length of the path of the rays within the crystal, or regard the whole system of rings as confined within very narrow limits of incidence.

Numerical measure of the tint of any isochromatic line.

Law of periodicity.

907. Transition from nitre to other crystals whose axes are farther asunder.

This condition obtains in the case here considered, because of the proximity of the axes in nitre to each other and to the perpendicular to the surfaces of the plate. But in crystals such as mica, or others where they are still wider asunder, it is not so; and the projection of the isochromatic curves on a plane surface will deviate materially from their true form, which ought to be regarded as delineated on a sphere having the eye, or rather a point within the crystal, for a centre. In such a case, it might be expected that the usual transition from the arc to its sine should take place; and that, instead of supposing the tint, or value of $a\,b$, to be proportional simply to $\theta \times \theta' \times t$, (putting $\theta = \text{A P}$, and $\theta' = \text{A P}'$,) we ought to have it proportional to $\sin\theta \times \sin\theta' \times$ length of the path of the ray within the crystal. Now (putting ρ for the angle of refraction, and t for the thickness of the plate) we have $t \,.\, \sec\rho =$ length of the ray's path within the crystal. If, then, we put n for the number of periods corresponding to the tint $a\,b$ for the ray in question, and suppose $h = \frac{a\,b}{n}$, or the unit whose multiples determine the order of the rings, we shall have

General expression for the tint polarized by any crystallized plate.

$$n = \frac{a\,b}{h} = \frac{t}{h} \,.\, \sin\theta \,.\, \sin\theta' \,.\, \sec\rho, \qquad (a)$$

and

$$h = \frac{t}{n \,.\, \cos\rho} \,.\, \sin\theta \,.\, \sin\theta'. \qquad (b)$$

If, then, the suppositions made be correct, we ought to have the function on the right hand side of this last equation invariable, in whatever direction the ray penetrates the crystallized plate, and whatever be the order of the tint denoted by n. We shall here relate only one experiment, to show how very precisely the agreement of this conclusion with fact is sustained.

908. Experiment verifying this law.

A ray of light was polarized by reflexion at a plate of perfectly plane glass, and transmitted through a plate of mica, having its principal section 45° inclined to thè plane of primitive polarization, and the mica plate made to revolve in the plane of its principal section about an axis at right angles thereto, (or about the axis B, Art. 885.) In this state of things, if viewed through a tourmaline as above described, or by other more refined

Light. means presently to be noticed, the succession of tints exhibited by the mica was that of a section of the rings in fig. 182, made by a line drawn through both the poles. To render the observation definite, a red glass was interposed so as to reduce the rings to a succession of red and black bands, and the angles of incidence corresponding to the maxima and minima of the several rings very accurately measured. These are set down in *Col.* 2 of the following table. *Col.* 1 contains the values of n, 0 corresponding to the pole, $\frac{1}{2}$ to the first maximum, 1 to the first minimum, $1\frac{1}{2}$ to the second maximum, and so on. The third column contains the angles of refraction computed for an index 1.500; the fourth and fifth, those of θ and θ'; the sixth, those of h deduced from the above equation, and which ought to be constant. The excesses above the mean are stated in the last column, and show how very closely that equation represents the fact. The thickness of the mica was 0.023078 inches $= t$. Part IV.

Values of n.	Angles of incidence.	Angles of refraction $= \rho$.	Values of θ.	Values of θ'.	Values of h.	Excesses above the mean.
0.0	35° 3′ 30″	22°31′ 0″	0° 0′ 0″	45° 2′ 0″		
0.5	32 55 20	21 14 40	1 16 20	43 45 40	0.032952	− 0.000195
1.0	30 34 40	19 49 30	2 41 30	42 20 30	0.033622	+ 0.000475
1.5	28 15 40	18 24 0	4 7 0	40 55 0	0.033035	− 0.000112
2.0	25 34 20	16 43 30	5 47 30	39 14 30	0.033327	+ 0.000180
2.5	22 46 20	14 57 15	7 33 45	37 28 15	0.033148	+ 0.000001
3.0	19 35 40	12 55 10	9 35 50	35 26 10	0.033058	− 0.000089
3.5	15 48 40	10 27 50	12 3 10	32 58 50	0.033026	− 0.000121
4.0	10 48 50	7 11 10	15 19 50	29 42 10	0.033010	− 0.000137

909. General establishment of the law. Proceeding thus, and measuring across the system of rings in all directions for plates of various crystals and of all thicknesses, it has been ascertained, as a general fact, that in all substances which possess the property of developing periodical colours by exposure to polarized light in the manner described, the tint (n), or rather the number of periods and parts of a period corresponding, in the case of a ray of given refrangibility, to a thickness t, an angle of refraction ρ, and a position within the crystal, making angles θ and θ' with the optic axes, is represented by the equation

$$n = \frac{t \,.\, \sec \rho}{h} \times \sin \theta \,.\, \sin \theta',$$

Case of a crystal formed into a sphere. h being a constant depending only on the nature of the crystal and the ray. Were the crystal of a spherical form, instead of a parallel plate, $t \,.\, \sec \rho$, which represents the path traversed by the ray within it, must be replaced by a constant equal to the diameter of the sphere, and in that case the tint would be simply proportional to the product of the sines of θ and θ'. This elegant law is due to M. Biot, though it is to Dr. Brewster's indefatigable and widely extended research that we owe the general developement of the splendid phenomena of the polarized rings in biaxal crystals. It appears, then, from this, that if, on the surface of a sphere formed of any crystal, curves analogous to the lemniscate, or having $\sin \theta \times \sin \theta'$ constant for each curve, and varying in arithmetical progression from curve to curve, be described,—then, if the sphere be turned about its centre in a polarized beam, as above described, the tint polarized at every point of each curve will be the same, and in passing from curve to curve will obey the law of periodicity proper to the crystal.

910. Methods of viewing the rings at great obliquities. There is hardly any character in which crystals differ more widely than in the angular separation of their optic axes, as the table annexed to the end of this article will show. This, while it affords most valuable criteria to the chemist and mineralogist, in discriminating substances and pointing out differences of structure and composition which would otherwise have passed unnoticed, renders the investigation of their phenomena difficult, since it is frequently impossible, by any contrivance, to bring both the axes under view at once; and necessitates a variety of artifices to obtain a sight of the rings about both. It is often very easy to cut and polish crystallized bodies in some directions, and very difficult in others. However, by immersing plates of them in oil, and turning them round on different axes, or by cementing on their opposite sides prisms of equal refracting angles oppositely placed, as in fig. 184, we may look through them at much greater obliquities than without such aid; and thus, by increasing the range of vision to nearly a hemisphere, avoid in most instances the necessity of cutting them in different directions. Fig. 184.

911. Rings in crystals with one axis. Fig. 185. When the two axes coalesce, or the crystal becomes uniaxal, the lemniscates become circles; and the black hyperbolic lines, passing through the poles, resolve themselves into straight lines at right angles to each other, forming a black cross passing through the centre of the rings, as in fig. 185. In this case the tint is represented by $t \,.\, \sin \theta^2$; and in the case of plates, where t, the thickness, is considerable, or where, from the otherwise peculiar nature of the substance the rings are of small dimensions, θ is small, and therefore proportional to its sine; so that in passing from ring to ring θ^2 increases in arithmetical progression. Hence the diameters of the rings are as the square roots of the numbers 0, 1, 2, 3, &c.; and therefore their system is similar, with the exception of the black cross, to the rings seen between object-glasses. Carbonate of lime cut into a plate at right angles to the axis of its primitive rhomboid, exhibits this phenomenon with the utmost beauty. The most familiar instance, however, may be found in a sheet of clear ice about an inch thick frozen in still weather. A pane of window-glass, or a polished table to polarize the light, a sheet of ice freshly taken up in winter

Familiarly shown in ice.

Light. produce the rings, and a broken fragment of plate glass to place near the eye as a reflector, are all the apparatus required to produce one of the most splendid of optical exhibitions. Part IV.

912. Analogy between the colours of the polarized rings and those produced by the law of interference

If θ be not very small, the measure of the tint, instead of $t \cdot \sin \theta^2$, is $t \cdot \sec \rho \cdot \sin \theta^2$. We have seen that in uniaxal crystals, $\sin \theta^2$ is proportional to the difference of the squares of the velocities v and v' of the ordinary and extraordinary ray, or to $v'^2 - v^2$. Now, if we denote by τ and τ' the times taken by these two rays to traverse the plate, we have $v = \frac{t \cdot \sec \rho}{\tau}$ and $v' = \frac{t \cdot \sec \rho}{\tau'}$; therefore $t \cdot \sec \cdot \rho \sin \theta^2$ is proportional to

$$(t \cdot \sec \rho)^3 \times \left(\frac{1}{\tau'^2} - \frac{1}{\tau^2} \right), \text{ that is, to } \frac{(\tau + \tau')(\tau - \tau')}{(\tau \tau')^2} \cdot (t \sec \rho)^3,$$

or (which is the same thing) to $(v + v') \cdot v v' (\tau - \tau')$. But, neglecting the squares of very small quantities, of the order $v' - v$ and $\tau - \tau'$, for such they are in the immediate neighbourhood of the axis, the factors $v + v'$ and $v v'$ are constant; so that the tint is simply proportional to $\tau - \tau'$, the difference of times occupied by the two rays in traversing the plate; or *the interval of retardation* of the slower ray on the quicker. This very remarkable analogy between the tints in question and those arising from the law of interferences, was first perceived by Dr. Young; and, assisted by a property of polarized light soon to be mentioned, discovered by Messrs. Arago and Fresnel, leads to a simple and beautiful explanation of all the phenomena which form the subject of this section, and of which more in its proper place.

913. Circumstances which distort the rings.

The forms of the rings are such as we have described, only in regular and perfect crystals; every thing which disturbs this regularity, distorts their form. Some crystals are very liable to such disturbances, either arising from an imperfect state of equilibrium, or a state of strain in which the molecules are retained, or to actual interruptions in their structure. Thus, specimens of quartz and beryl are occasionally met with, in which the single axis usually seen is very distinctly separated into two, the rings instead of circles have oval forms, and the black cross (which in cases of a well developed single axis remains quite unchanged during the rotation of the crystallized plate in its own plane) breaks into curves convex towards each other, but almost in contact at their vertices, at every quarter revolution. Cases of interruption occur in carbonate of lime very commonly, and in muriacite perpetually; and the effects produced by them on the configurations of the rings rank among the most curious and beautiful of optical phenomena. They have not, however, been anywhere described, and our limits will not allow us to make this article a vehicle for their description.

914. Colours of the rays.

The form of the rings being, then, considered, let us next inquire more minutely into their colours. These being all composite, and arising from the superposition on each other of systems of rings formed by each homogeneous ray, we can obtain a knowledge of their constitution only by examining the rings in homogeneous light. This is easy, for we have only to illuminate the apparatus described above by homogeneous light of all degrees of refrangibility from red to violet, by passing a prismatic spectrum from one end to the other over the illuminating lens H, the eye being applied as usual at O, and observe the changes which take place in the rings, in passing from one coloured illumination to another; and, if necessary, measure their dimensions. This is readily done, either by projecting them on a screen in a darkened room, as described in Art. 903, or by detaching the lens H, fig. 178, and simply looking through the apparatus at a sheet of white paper strongly illuminated by the rays of a prismatic spectrum, where the rings will appear as if depicted on the paper, and their outlines easily marked, or their diameters measured. The following are the general facts which may thus be readily verified.

915. In crystals with one axis. Deviations from Newton's scale. In the apophyllite.

First, in the case of crystals with a single axis, the rings remain circular, and their centres are coincident for all the coloured rays, but their dimensions vary. In the generality of such crystals, their diameters for different refrangibilities follow nearly the law of the Newtonian rings, when viewed in similar illuminations; their squares (or rather the squares of their sines) being proportional, or nearly so, to the lengths of the fits, or of the undulations of the rays forming them. This law, however, is very far from universal; and in certain crystals is altogether subverted. Thus, in the most common variety of apophyllite, (from Cipit, in the Tyrol,—not from Fassa, as is commonly stated,) the diameters of the rings are nearly alike for all colours, those of the green rings being a very little less; those formed by rays at the confines of the blue and indigo exactly equal, and those of violet rays a little greater than the red rings. It is obvious, that were the rings of all colours exactly equal, the system resulting from their superposition would be simple alternations of perfect black and white, continued *ad infinitum*. In the case in question, so near an approach to equality subsists, that the rings in a tourmaline apparatus appear merely black and white, and are extremely numerous, no less than thirty-five having been counted, and many of those too close for counting being visible in a thick specimen.

916. When examined more delicately, colours are, however, distinguished, and are in perfect conformity with the law stated, being for the first four orders as follow:

First order. Black, greenish white, bright white, purplish white, sombre violet blue.

Second order. Violet almost black, pale yellow green, greenish white, white, purplish white, obscure indigo inclining to purple.

Third order. Sombre violet, tolerable yellow green, yellowish white, white, pale purple, sombre indigo.

Fourth order. Sombre violet, livid grey, yellow green, pale yellowish white, white, purple, very sombre indigo, &c.

917. Carbonate of lime, beryl, ice, and tourmaline (when limpid) are instances of uniaxal crystals, in whose rings the Newtonian scale of tints is almost exactly imitated; and, consequently, the intervals of retardation of the ordinary and extraordinary rays of any colour on one another, are proportional to the lengths of their undulations. On the other hand, in the hyposulphate of lime, we are furnished with an instance of more rapid

Light. In hyposulphate of lime.

degradation of tints, and therefore of a more rapid variation of the interval just mentioned. The following was the scale of colour of the rings observed in this remarkable crystal: Part IV.

First order. Black, very faint sky blue, pretty strong sky blue, very light bluish white, white, yellowish white, bright straw colour, yellow, orange yellow, fine pink, sombre pink.

Second order. Purple, blue, bright greenish blue, splendid green, light green, greenish white, ruddy white, pink, fine rose red.

Third order. Dull purple, pale blue, green blue, white, pink.

Fourth order. Very pale purple, very light blue, white, almost imperceptible pink.

After which the succession of colours was no longer distinguishable.

918. Other remarkable cases of deviation.

A degradation still more rapid has been observed in certain rare varieties of uniaxal apophyllite, accompanied with remarkable and instructive phenomena. In these, the diameters of the rings (instead of diminishing as the refrangibility of the light of which they are formed increases) increase with great rapidity, and actually become infinite for rays of intermediate refrangibility; after which they again become finite, and continue to contract up to the violet end of the spectrum, where, however, they are still considerably larger than in the red rays. In consequence of this singularity, their colours when illuminated with white light furnish examples of a complete inversion of Newton's scale of tints. The following were the tints exhibited by two varieties of the mineral in question, in one of which the critical point where the rings become infinite took place in the indigo, and in the other in the yellow rays. In the former they were

First order. Black, sombre red, orange, yellow, green, greenish blue, sombre and dirty blue.

Second order. Dull purple, pink, ruddy pink, pink yellow, pale yellow (almost white,) bluish green, dull pale blue.

Third order. Very dilute purple, pale pink, white, very pale blue.

In the latter variety, the tints were

First and only order. Black, sombre indigo, indigo inclining to purple, pale lilac purple, very pale reddish purple, pale rose red, white, white with a hardly perceptible tinge of green.

919. Relation between the diameters of the rings and the doubly refractive energy.

The *doubly refracting energy* of a crystal may be not improperly measured by the difference of the squares of the velocities of an ordinary and extraordinary ray similarly situated with respect to the axes; but as this difference, for rays variously situated in one and the same crystal, is proportional to $\sin \theta^2$, or in biaxal crystals to $\sin \theta \,.\, \sin \theta'$, the *intrinsic double refractive energy* of any crystal may be represented by

$$e = \frac{v^2 - v'^2}{\sin \theta \,.\, \sin \theta'}; \qquad (c)$$

regarding this henceforth as the *definition* of this energy, we have, in uniaxal crystals, $e = \frac{v^2 - v'^2}{\sin \theta^2}$, and this will evidently measure the actual amount of separation of two such rays when emergent from the crystal.

If in this we put for v and v' their equals $\frac{t \,.\, \sec \rho}{\tau}$ and $\frac{t \,.\, \sec \rho}{\tau'}$, we shall have, after reduction,

$$v^2 - v'^2 = v\, v' \,(v + v') \,.\, \frac{\tau' - \tau}{t \,.\, \sec \rho}. \qquad (c)$$

In a parallel plate, perpendicular to the axis and in the immediate vicinity of the axis, v' and $\sec \rho$ may be regarded as constant, and $v^2 - v'^2$ is proportional to $\tau' - \tau$, the interval of retardation of one ray on the other, to which the tint in white light and the number of periods and parts of a period in homogeneous light (to which, for brevity, we will continue to extend the term *tint*) are proportional. We see, then, that in such cases the intrinsic double refracting energy is directly as the tint polarized, and inversely as $\sin \theta^2$, and therefore also inversely as the squares of the diameters of the rings. As the rings increase in magnitude, then, *cæteris paribus*, the double refractive energy diminishes; and hence a very curious consequence follows, *viz.* that in the two cases last mentioned it vanishes altogether for those colours where the rings are infinite; in other words, that although the crystal be doubly refractive for all the other coloured rays, there is one particular ray in the spectrum (*viz.* the indigo in the former, and the yellow in the latter case) with respect to which its refraction is single. In the passage through infinity, there is generally a change of sign. In the instances in question this change takes place in the value of e or $v^2 - v'^2$, which passes from negative to positive. And the spheroid of double refraction changes its character accordingly from oblate to prolate, passing through the sphere as its intermediate state. The manner in which this may be recognised, without actually measuring, or even perceiving its double refraction, will be explained further on.

Case of crystals at once attractive, repulsive, and neutral.

920. Application to biaxal crystals.

For crystals with two axes we have only, at present, the ground of analogy to go upon in applying the above formula and phraseology to their phenomena. The general fact of an intimate connection of the double refracting energy with the dimensions of the rings, is indeed easily made out; for it is a fact easily verified by experiment, that *all crystals, whether with one or two axes, in which the rings or lemniscates formed are of small magnitude in respect of the thickness of the plate producing them, are powerfully double refractive, and vice versâ;* and that, generally speaking, the separation of the ordinary and extraordinary pencils is, *cæteris paribus*, greater in proportion as the rings are more close and crowded round their poles. In uniaxal crystals, in which the laws of double refraction are comparatively simple, there is little difficulty in submitting the point to the test of direct experiment and exact measurement, and it is found to be completely verified. In biaxal, however, such precise and direct comparison is more difficult, and calls for a knowledge of the general laws of double refraction. The analogy, however, supported by the general coincidence above mentioned, is too strong to be refused; and, as we advance, will be found to gain strength with every step.

Light. Part IV.

In biaxal crystals, similar deviations from exact proportionality between the lengths of the periods of the several coloured rays and those of their undulations, or fits, exist; but their effect in disturbing the colours of the rings is interfered with, and frequently masked by, another cause, which has no existence in uniaxal crystals, *viz.* that *the optic axes differ in situation, within one and the same crystal for the differently refrangible homogeneous rays;* and, therefore, that the elementary lemniscates, whose superposition forms the composite fringes seen in a white illumination, differ not only in magnitude but in the places of their poles and the interval between them. To make this evident to ocular inspection, take a crystal of Rochelle salt, (tartrate of soda and potash,) and having cut it into a plate perpendicular to one of its optic axes, or nearly so, and placed it in a tourmaline apparatus, let the lens H be illuminated with the rays of a prismatic spectrum, in succession, beginning with the red and passing gradually to the violet. The eye being all the time fixed on the rings, they will appear for each colour of perfect regularity of form, remarkably well defined, and contracting rapidly in size as the illumination is made with more refrangible light; but in addition to this, it will be observed, that the whole system appears to shift its place bodily, and advance regularly in one direction as the illumination changes; and if it be alternately altered from red to violet, and back again, the pole, with the rings about it, will also move backwards and forwards, vibrating, as it were, over a considerable space. If homogeneous rays of two colours be thrown at once on the lens, two sets of rings will be seen, having their centres more or less distant, and their magnitudes more or less different, according to the difference of refrangibility of the two species of light employed.

921. Separation of the optic axes of differently refrangible rays in biaxal crystals.

Since the plate in this experiment is supposed to have its surfaces perpendicular to the mean position of the optic axis, the cause of these appearances cannot be found in a mere apparent displacement of the rings by refraction at the surface, existing to a greater extent for the violet than the red rays, add to which, that the angle which their poles describe, is neither the same in magnitude nor direction for different crystals. In some, the optic axes approach each other in violet light, and recede in red; while in others the reverse is the case. In all, however, so far as we are aware, *the optic axes for all the coloured rays lie in one plane, viz. the principal section of the crystal.* This is rendered matter of inspection by cutting any crystal so that both axes shall be visible in the same plate, and placing it with its principal section in the plane of primitive polarization. In this state of things the first ring about each pole, as in fig. 179, is seen divided into two halves, and puts on, if the plate be pretty thick, the appearance of two semi-elliptic spots, one on each side of the principal section. These spots are observed to be differently coloured at their two extremities: in some crystals the ends of the spots, as well as the segments of the rings adjacent to them, which are turned towards each other, being coloured red, and the other, or more distant ends, with blue; and in others, the reverse. In some crystals this coloration is slight, and in a very few, imperceptible; but in others it is so great, that the spots are drawn out into long spectra, or tails of red, green, and violet light; and the ends of the rings are in like manner distorted and highly coloured, presenting the appearance in fig. 186. This is the case with Rochelle salt, above mentioned. If these spectra be examined with coloured glasses, or with homogeneous light, they will be seen to be composed as in fig. 187, by the superposition of well defined spots of the several simple colours arranged in lines on each side of the principal section. In the case of Rochelle salt, the angular extent of these spectra, *within the medium,* which measures the interval between the optic axes for violet and red rays, amounts to no less than 10°.

922. All the axes lie in the plane of the principal section.

Fig. 186.

Fig. 187.

Dr. Brewster has given the following list of crystals presenting these phenomena, which he has divided into two classes, according to his peculiar and ingenious views.

923. Dr. Brewster's list of crystals exhibiting deviations of tint from this cause.

Class I.

Nitre.
Sulphate of baryta.
Sulphate of strontia.
Phosphate of soda.
Tartrate of potash and soda.
Supertartrate of potash and soda.
Arragonite.
Carbonate of lead. (?)
Sulphato-carbonate of lead.

Class II.

Topaz.
Mica.
Anhydrite.
Native borax.
Sulphate of magnesia.

Unclassed.

Chromate of lead.
Muriate of mercury.
Muriate of copper.
Oxynitrate of silver.
Sugar.
Crystallized Cheltenham salts.
Nitrate of mercury.
Nitrate of zinc.
Nitrate of lime.
Superoxalate of potash.
Oxalic acid.
Sulphate of iron.
Carbonate of lead. (?)
Cymophane.
Felspar.
Benzoic acid.
Chromic acid.
Nadelstein (Faröe.)

To which list a great many more might be added. Bicarbonate of ammonia, indeed, is the only biaxal crystal we have examined in which the optic axes for all colours appear to be strictly coincident.

924. Phenomena of the virtual poles explained.

This separation of the axes of different colours explains a remarkable appearance presented by the rings of all biaxal crystals, when placed with their principal section 45° inclined to the plane of polarization of the incident light. It is universally observed that, in traversing the whole system of rings in the plane of the principal

section, the nearest approximation to Newton's scale of colours is obtained by assuming, for the origin of the scale, not the poles themselves, but other points (which have been called *virtual poles*, though improperly) lying either between or beyond them, according to the crystal examined, and at a distance from them, invariable for each species of crystal, whatever be the thickness of the plate. In consequence, the poles themselves are not absolutely black, but tinged with colour; and their tint descends in the scale as the thickness of the plate increases, and as, in consequence, one, two, or more orders of rings intervene between them and the points from which the scale originates. These points are observed to lie between the poles in all crystals which have the blue axes nearer than the red, such as Rochelle salt, borax, mica, sulphate of magnesia, topaz; and beyond them for those in which the red axes include a less angle than the blue, as sulphate of baryta, nitre, arragonite, sugar, hyposulphite of strontia; and this fact, as well as the constancy of their distance from the poles when the thickness of the plate is varied, renders their origin evident. In fact, since the violet rings are smaller than the red, if the centre about which the former are described, instead of being coincident with that of the latter, be shifted in either direction, carrying its rings with it, some one of the violet rings will necessarily be brought up to, and fall upon a red ring of the same order; and the same holding good with the intermediate rays, provided the law which determines the separation of the different coloured axes be not very different from that which regulates the dimensions of the rings of corresponding colours, the point of coincidence of a red and violet ring of the same order will be nearly that of a red and green, or any intermediate colour. The tint, then, at this point will be either absolutely black, (if they be dark rings which are thus brought to coincidence,) or white, if bright; and from this point the tints will reckon either way with more or less exactness, according to the same scale which would have held good had the points of coincidence been the poles themselves. Should, however, the two laws above mentioned differ very widely, an uncorrected colour will be left at the point of nearest compensation, just as happens when two prisms whose scales of dispersion are dissimilar are employed to achromatise each other. To what an extent the disturbance of the Newtonian scale of tints may be carried by this and the other causes already explained, the reader may see by turning to the table of tints exhibited by Rochelle salt in *Phil. Trans.* 1820, part i.

925. Two suppositions as to the mode of action of crystals in forming the rings.

We come next to consider the law of the intensity of the illumination of the rings in different parts of their periphery; but this part of their theory will require us to enter more fundamentally into the mode in which their formation is effected, and to examine what modifications the polarized ray incident on the crystallized plate undergoes in its passage through it, so as to present phenomena so totally different from those which it would have offered without such intervention. It is evident then, first, that since the ray, if not acted on by the plate, would have been entirely stopped by the second tourmaline, but, when so acted on, is partially transmitted so as to exhibit coloured appearances of certain regular forms; that the crystallized plate must have either destroyed altogether the polarization of that part of the light which is thereby enabled to penetrate the second tourmaline, or, if not, must have altered its plane of polarization, so as to allow of a partial transmission. Between these two suppositions it is not difficult to decide. Were the portion of light which passes through the second tourmaline and forms the rings wholly *depolarized*, that is, restored to its original state of natural light, since the remainder, its complement to unity, which continues to be stopped by the tourmaline, retains its state of polarization unaltered, it is evident, that each ray at leaving the crystallized plate would be composed of two portions, one unpolarized ($= A$), the other ($= 1 - A$) polarized. Of these, the half only of the first ($\frac{1}{2} A$) would be transmitted by the second tourmaline. Now, suppose this to be turned round in its own plane through any angle ($= a$) from its original position, then the unpolarized portion will continue to be half transmitted; and the polarized, being now partially also transmitted, (in the ratio of $\sin^2 a : 1$,) will mix with it, so that the compound beam will be represented by

Doctrine of polarization considered.

$$\tfrac{1}{2} A + (1 - A) . \sin^2 a = \sin^2 a + \frac{A}{2} . \cos 2a.$$

Now, if we suppose a to pass in succession through the values 0, 45°, 90°, 135°, 180°, &c., this will become respectively $\frac{1}{2} A$, $\frac{1}{2}$, $1 - \frac{1}{2} A$, $\frac{1}{2}$, $\frac{1}{2} A$, &c. Hence, at every quarter revolution the tints ought to change from those of the reflected rings to those of the transmitted, the complements of the former to white light; and at every half quarter revolution no rings at all should be seen, but merely an uniformly bright field illuminated with half the intensity of light which would be seen were the second tourmaline altogether removed.

926. Phenomena of the complementary rings. Fig. 188.

But the phenomena which actually take place are very different. At the alternate quadrants, it is true, the complementary rings are produced, and the appearance is as represented in fig. 188. The black cross is seen changed into a white one; the dark parts of the rings are become the bright ones; the green is changed into red, and the red into green, &c.; so that if we were to examine no farther, the fact would appear to agree with the hypothesis. But in the intermediate half quadrants, this agreement no longer subsists. Instead of a uniformly illuminated field, a compound set of rings, consisting of eight compartments, alternately occupied by the primary and complementary set, is seen, presenting the appearance of fig. 191, and which is further described in Art. 935.

927. Hypothesis of a change of polarization.

The phenomena then are incompatible with the idea of *depolarization*. It remains to examine what account can be given of them on the supposition of a *change* of polarization operated by the plate; and here we must remark *in limine*, that this cause is what in Newton's language would be termed a *vera causa*, a cause actually in existence; for we have already seen that every ray, whether polarized or not, traversing a double refracting medium in any direction, except precisely along its axis, is resolved into two, polarized in opposite planes. When the incident ray is polarized, these portions (generally speaking) differ in intensity, and though, owing to the parallelism of the plate, they emerge superposed, their polarization is not the less real, and either of them may be suppressed, and the other suffered to pass, by receiving them on a tourmaline properly situated. This is so far

Light. agreeable to the observed fact, when the tourmaline plate next the eye is removed, the rays of which the two sets of rings consist, coexist in the transmitted cone of rays whose apex is the eye, but, being complementary to each other, produce whiteness. This may be made matter of ocular demonstration, by employing, instead of a tourmaline, which absorbs one image, a doubly refracting achromatic prism, of sufficiently large refracting angle to separate the two pencils by an angle greater than the apparent diameter of the system of rings, when the primary set will appear in one image, and its complementary set in the other; meanwhile, to return to our tourmalines, since the two sets of rings seen in the two positions of the posterior tourmaline are complementary, it follows, that *all the* rays suppressed in one position are transmitted in that at right angles to it, and *vice versâ*; and, as a necessary consequence, that every pair of corresponding rays in the primary and complementary set are polarized in opposite planes. Part IV. Both sets of rings shown at once.

928. M. Biot's doctrine of movable polarization.

The only thing, then, which appears mysterious in the phenomena thus conceived, is the production of *colour*. A doubly refracting crystal, which receives a polarized ray of whatever colour, divides it between its two pencils, according to a ratio dependent only on the situation of the planes of polarization and of incidence, and of the axes of the crystal, and not at all on its refrangibility. How then happens it, that at certain angles of incidence the red rays pass wholly into one image, and the green or violet into the other, while at other incidences the reverse takes place: whence, in short, arises the law of periodicity observed. To answer this question, M. Biot imagined his theory of alternate, or as he terms it *movable polarization*, according to which, as soon as a polarized ray enters into a thin crystallized lamina, its plane of polarization commences a series of oscillations, or rather alternate assumptions *per saltum* of two different positions, one in its original plane, the other in a plane making with that plane double the angle which the principal section of the crystal makes with it. These alternations he supposes to be more frequent for the more refrangible rays, and to recur periodically, like Newton's fits of easy reflexion and transmission, at equal intervals all the time the ray is traversing the crystal, which intervals are shorter the more inclined its path is to the axis or axes. This theory is remarkably ingenious in its details; and in its application to the phenomena of the rings, though open (as stated by its author) to certain obvious criticisms, is yet, we conceive, capable of being regarded as a faithful representation of most of their leading features. There is, however, one objection against it of too formidable a nature to allow of its being received unless explained away, if any other can be devised not open to the same or greater. It is, that it requires us to consider the action of a *thin* crystal on light as totally different, not merely in degree, but in kind, from that of a *thick* one, while yet it marks no limit by which we are to determine where its action as a *thin* crystal ceases, and that proper to a *thick* one commences, nor establishes any gradations by which one mode of action passes into the other. A thick crystal, as we know, polarizes the rays ultimately emergent from it in two planes, dependent only on the position of the crystal and that of the ray, while M. Biot's theory makes the position of the plane of polarization of the incident ray an element in determining their ultimate polarization by a *thin* one. Nor are we in this theory to regard as thin crystals only *films* or delicate laminæ. A plate of a tenth of an inch thick or more may be a thin plate in some cases of feebly polarizing bodies, such as apophyllite, &c. Objection against it.

929. M. Biot's general apparatus described. Fig. 189, 190.

As the apparatus employed by M. Biot for studying the phenomena of the colours of thin crystallized plates offers great conveniences for the measurement of the angles at which different tints are produced, and for their exhibition in their state of greatest purity and contrast, we shall here describe it, and state some of the chief results at which he has arrived. A (fig. 189) is a plane glass blackened at the posterior surface, or a plate of obsidian inclined at the polarizing angle to the axis of a tube A B, so as to reflect along it a polarized ray; (if greater intensity be required, we may use a pile of glass plates, taking care that they be of truly parallel surfaces, and placed exactly parallel to each other.) B C is a tube, stiffly movable round A B as an axis, having a graduated ring at B, read off by a vernier attached to the tube A B, and carrying two arms, G and H, through which the axis of a swing frame E passes, which can thus be inclined at any angle to the common axis of the tubes, its inclination, or the angle of incidence of the ray reflected along the axis on the plane of the frame being read off by an index on the divided lateral circle D. In this frame is an aperture F, in which turns a circular plate of brass having a hole in its centre, over which is fastened with wax the crystallized plate to be examined, and which can thus be turned round in its own plane, independently of any motion of the rest of the apparatus, so as to place its principal section in any *azimuth* with respect to the plane of incidence. We have found it convenient to have this part of the apparatus constructed as in fig. 190, where *a* is the square plate of the frame; *b* a divided circle movable in it and read off by an index; *c*, *d* is a circular plate movable within the divided circle to admit of adjustment, after which it is fastened in its place by a little clamp, so as to turn with the circle; this carries in its centre another swinging circle *e*, moving stiffly on its axis, and having in the middle an aperture, over which the crystal is cemented, thus giving room for an adjustment of the plane of the surface of incidence, in case it be not exactly at right angles to the principal section of the crystal, an adjustment very useful when artificial surfaces are under examination, which it is hardly possible to cut and polish with perfect precision. It is also convenient for some experiments to have a second frame similar to the first, placed on the prolongation of the arms G, H. M is a doubly refracting prism, rendered achromatic either by a prism of flint glass, or, still better, by another prism of the same doubly refracting medium. Two prisms of quartz, arranged as in Art. 882, are very convenient. Their angles should be such, that when placed at M the two images of a small aperture P, in a diaphragm near the end of the tube, should appear almost in contact. The prisms so adjusted are mounted on a stand N, independent of the other apparatus, and capable of being turned round by an arm K, carrying a vernier, by whose aid the angle of rotation, or position of the plane in which the double refraction takes place, can be read off on a divided circle L. The prism should be so adjusted in its cell, that when the vernier reads off zero, the extraordinary image should be extinguished; and when 90°, the ordinary. Occasionally a tourmaline plate or a glass reflector may be substituted for the prism.

930. To use this apparatus, the crystallized lamina (which we will at present suppose to be a parallel plate of any

Use of this apparatus.

uniaxal crystal, having its axis perpendicular to the plane of the plate,) is to be placed on the swing frame across the aperture, and being adjusted so as to have its axis directed precisely along the axis of the tube when the vernier of D reads off zero, which is readily performed by the various adjustments belonging to the frame, as above described, the instrument is ready for use. The attainment of this condition may be known by turning the tube C on the tube A B as an axis, when the extraordinary image of the aperture P, seen through a doubly refracting prism, ought to vanish in the zero position of the vernier K, and not be restored in any part of the rotation of the tube; for it is manifest, that the axis is the only line to which this property belongs, or to which all the rings are symmetrical. It is then evident, that, however the parts of the apparatus be disposed, 1st, the reading off of the vernier D will give the angle of incidence on the plate; 2d, that of the vernier B, the angle made by the *plane of incidence* with the *plane of primitive polarization;* 3d, that of the vernier *c* will indicate the angle included by any assumed section of the crystallized plate perpendicular to its plane with the plane of incidence; and, lastly, that the reading of the vernier K will give the angle between the plane of primitive polarization and the principal section of the doubly refracting prism.

931. Its application to the phenomena of the rings of one axis. Fig. 188.

Suppose now we adjust the vernier B to zero, it will then be found, that however the plate E be situated, or whatever be the incidence of the ray, only the ordinary image will be seen (being white,) the extraordinary being extinguished (or black.) In this case we traverse the system of rings in the direction of the vertical arm of the black cross, fig. 185, of the primary, and the white one of the complementary set, see fig. 188. The phenomena are the same if we set the vernier B to 90°, and then turn the frame E on its axis, thus varying the incidence in a plane at right angles to that of primitive polarization, or, which comes to the same thing, traversing the rings along the horizontal arm of the black and white crosses. In intermediate positions of the vernier B, we traverse the rings along a diameter, making an angle with vertical equal to the reading of the vernier. In this case the two images of P are both visible, and finely coloured; the *extraordinary* image presenting the tint of the primary rings due to the particular angle of incidence indicated by the vernier D; the *ordinary*, that of the complementary system corresponding to the same angle. The colours of the two images are thus seen in circumstances the most favourable, being finely contrasted and brought side by side, so as to be capable of the nicest comparison. It is when the vernier D reads 45°, or the plane of incidence is 45°, inclined to that of primitive polarization, that the contrast of the two images is at its maximum, the tints in the extraordinary image being then most vivid, and those in the ordinary free from any mixture of white light. In general, if A represent the light of the extraordinary image in the position above mentioned, and a the angle read off on the vernier B, in any other position of the plane of incidence, the two images in this new position (for the same angle of incidence) will be represented respectively by

$$A \,.\, (\sin 2a)^2, \text{ and } 1 - A\,(\sin 2a)^2$$

that is, by

$$A \,.\, (\sin 2a)^2, \text{ and } (\cos 2a)^2 + (1 - A) \,.\, (\sin 2a)^2.$$

The former of these expressions indicates a ray whose tint is represented by A, and its intensity by $(\sin 2a)^2$; the latter, a complementary tint $1 - A$ of the same intensity, diluted with a quantity of white light, whose intensity is represented by $(\cos 2a)^2$.

932. Agreement of the formulæ with M. Biot's hypothesis.

These expressions represent with great fidelity the tints of both images, the intensity of the extraordinary, and the apparent degree of dilution of the ordinary one; and since a ray A polarized in a plane making an angle $2a$ with the principal section of the doubly refracting prism, would be divided between the extraordinary and ordinary image in the ratio of $(\sin 2a)^2 : (\cos 2a)^2$, it follows, that if we regard the pencil at its emergence from the crystallized plate as composed of two portions, one ($= A$) polarized in the above named plane, the other ($= 1 - A$) preserving its primitive polarization, the two pencils formed by the doubly refracting prism will be composed as follows:

	Extraordinary image.	Ordinary image.
1st. From the pencil A.....	$A\,(\sin 2a)^2$	$A \,.\, (\cos 2a)^2$
2d. From the pencil $(1 - A)$....	0	$1 - A$
Sum	$A\,(\sin 2a)^2$	$1 - A + A \,.\, \cos 2a^2$ $= 1 - A \,.\, (\sin 2a)^2$

Office of the doubly refracting prism or tourmaline.

which are identical with those above. Thus we see, that the facts are so far perfectly conformable to M. Biot's hypothesis of movable polarization, and that we are even necessitated to admit it, *provided we take it for granted, that the rings exist actually formed and superposed in the pencil emergent from the crystallized lamina, and that the office of the doubly refracting prism is merely to analyze the emergent pencil, and separate the two sets from each other.* But if the objection mentioned above against that doctrine be really well founded, this assumption cannot be correct, and we are then driven to conclude, that the doubly refracting prism, or tourmaline, or glass reflector, interposed between the eye and the crystallized plate, performs a more important office than merely to separate the tints already formed; and that, in fact, they are actually produced by its action,—the crystallized plate only preparing the rays for the process they are here finally to undergo.

933. To explain how this may be conceived to happen will form the object of another Section. Meanwhile we will here only add, that the transition from uniaxal to biaxal crystals is readily made. We have only to consider, that by varying the angle of incidence, (the line bisecting the angle between the optic axes being supposed perpendicular to the plane of the plate,) we cross the rings in a line passing through their centre of symmetry O, fig. 183, and making an angle with their principal diameter PP′, equal to the angle read off on the vernier B, and that by turning the plate in its own plane, or varying the angle read off by the vernier *c*, we in effect make the system traversed pass through the successive states represented in fig. 179, 180, 181, 182, changing, not the tint, but the intensity of the extraordinary image.

When the doubly refracting prism is turned in its cell, the tints grow more dilute, and when placed in an azimuth a, that is, when its principal section is placed in the plane of incidence, both images are colourless, but of unequal brightness. This accords with M. Biot's doctrine of movable polarization; for if we grant that the pencil A is polarized in a plane making an angle $2a$ with that of primitive polarization, it will make, now, an angle $= a$ with that of the principal section of the prism, and A . $(\sin a)^2$ will be that part of the extraordinary image arising from the pencil A; on the other hand, the pencil $1 - A$ retaining its original polarization, $(1 - A) . \sin a^2$ will be the portion of the extraordinary image produced by it in the new position of the prism, and the sum, or the whole image, will be simply $1 \times \sin a^2$, which being independent of A, or of the tint, indicates that the image is colourless. In the same manner it may be shown, that the ordinary image will equal $1 \times \cos a^2$, and their intensities will, therefore, be to each other as $\sin a^2$ to $\cos a^2$, and will be equal at 45° of azimuth; all which is conformable to fact.

934. Effect of turning the prism in its cell.

The motion of the prism in its cell corresponds to a rotation of the posterior tourmaline in its own plane in the tourmaline apparatus. The general appearance presented by the rings of a single axis, when this rotation is not a precise quadrant, is represented in fig. 191, and the succession of changes being as follows: At the first commencement of the rotation the arms of the black cross appear to dilate; they grow at the same time fainter, and segments of the complementary rings appear in them, whose bright intervals correspond to the dark ones of the primary set, their red to the green portions of that set, and *vice versâ*. The junction of the two sets is marked by a faint white or undecided tint. As the rotation proceeds, the primary segments contract in extent, and become more diluted with white, while the secondary extend, and grow more decided; at the same time the centre of the system grows gradually bright, and when the rotation has attained 90°, the whole has assumed the appearance in fig. 188. The phenomena are precisely analogous in the rings of biaxal crystals. The least deviation from exact rectangularity in the tourmalines gives rise to complementary segments in the dark hyperbolic curves answering to the arms of the black cross, and to a corresponding dilution and contraction of the primary segments, which at last disappear altogether in the undistinguishable whiteness of a pair of white hyperbolas precisely similar to the black ones of the primary rings in their perfect state.

935. Effect of turning the tourmalines about on each other. Fig. 191.

Hitherto we have considered the rings as so narrowed by the thickness of the plate, as to be all contracted within a compass round the poles which the eye can take in at once; but if the thickness be greatly diminished, this will no longer be the case; and, instead of rings of a distinguishable form, we shall see only broad bands of colour extending to great distances from the poles, and even visible when the axes themselves are so much inclined to the surfaces of the plate as to be quite out of sight; or even when the axes actually lie in the plane of the plate. This is the case with the laminæ into which sulphate of lime readily splits; the axes lie *in their plane*, so that to see the rings in them, we must form artificial surfaces perpendicular to the lamina, a difficult and troublesome operation, from the extreme softness and fissile nature of the substance. The phenomena of the colours of this crystal were early studied, and almost of necessity misconceived, till Dr. Brewster, by exhibiting the real axes, showed that they form only a particular case of the general phenomenon we have already dwelt on.

936. Tints produced by very thin plates at great distances from the axes.

Adhering to the denominations employed in Art. 885—888, let us call the plane containing the two axes, *the section* A; that perpendicular to it, and passing through the line which bisects their *lesser* included angle, *the section* B; and that which similarly passes through the line bisecting their *greater* included angle, and is perpendicular to both the others, *the section* C. If the crystal have but one axis, the sections A and B pass through it, and C is at right angles to it. Then if the lamina contains both axes, its plane will be that of the section A, and the other two sections will intersect it in two lines (B and C) at right angles to each other. Conceive, now, a polarized ray to pass through such a lamina at a perpendicular incidence. Then if the plane of polarization coincide with either of the sections B and C, its polarization will be undisturbed, and the whole of the transmitted light will pass into the ordinary image. But if the plate be turned round in its own plane, the extraordinary image will reappear and become a maximum at every 45° of the plate's rotation; and if it be sufficiently thin, will exhibit some one of the colours of the rings, and the tints will descend regularly in the scale as the thickness is increased, the thickness being a measure of the tint, conformably to the general law in Art. 907, of which this is only a particular case.

937. Phenomena of a single thin plate.

When two such plates are laid together, with their sections B and C corresponding, it is evident that they are in the same relation as if they formed part of one and the same crystal; and we might therefore expect to find what really happens, *viz.* that such a compound plate polarizes the same tint that a single plate equal to the sum of the thicknesses would do. But if they be crossed, *i. e.* laid so together that the section B of the one shall coincide with the section C of the other, M. Biot has shown that the tint polarized is that due to the difference of their thicknesses. If, therefore, this difference be exactly nothing, the crossed plates will be exactly neutralized, at least at a perpendicular incidence, and that whatever be their thickness. (To procure two plates of exactly the same thickness, we have only to choose a clear and truly parallel plate terminated by fresh surfaces of fissure, and break it across.)

938. Phenomena of crossed plates at a perpendicular incidence.

When, however, the incidence is not perpendicular, such a compound plate as described will still exhibit colours which vary in, apparently, a very irregular manner as the incidence changes, and with different degrees of rapidity in different planes. The tourmaline apparatus here renders signal service in rendering the law of these tints, at first sight extremely puzzling, a matter of inspection. When such a crossed plate is placed between the tourmalines, crossed at right angles, it exhibits the singularly beautiful and striking phenomenon represented in fig. 192, in which the tints are those of the reflected scale of Newton, the origin being in the black cross. If the tourmalines be parallel, the complementary colours are produced with equal regularity, as in fig. 193. If the compound crystal be turned round in its own plane, the figures turn with it, but undergo no change other than an alternation of intensity, being at a maximum of brightness when the arms of the cross are parallel and

939. Phenomena at oblique incidences. Fig. 192. Fig. 193.

perpendicular to the plane of original polarization, and vanishing altogether when they make angles of 45° with that plane. If the plates be not crossed exactly at right angles, or be not precisely of equal thickness, other phenomena arise which it is easier for the reader to produce for himself than to read a detailed account of. The same may be said of the very splendid but complicated phenomena produced by crossing two equally thick plates of biaxal crystals, such as mica, topaz, &c. having the section A at right angles to their surfaces.

940. Law of tints formed by the superposition of similar plates.

Regarding, however, at present only the tint produced at a perpendicular incidence, it is found that when any number of plates of one and the same medium, of any thicknesses, are superposed with their homologous sections corresponding, the tint polarized is that due to the sum of their thicknesses; but when any one or more of them have their sections B and C at right angles to the homologous sections of the others, the tint is that due to the sum of the thicknesses of those placed one way, *minus* the sum of those of the plates placed the other way. In algebraical language, if we call t, t', t'', &c. the thicknesses, and *regard as negative* those of the plates laid crosswise, the tint T polarized by the system will be that due to the thickness $t + t' + t'' +$ &c.

941. Law of tints produced by dissimilar plates

When the ray is made to traverse a plate of quartz, zircon, carbonate of lime, or any other uniaxal crystal cut so as to contain the axis of double refraction, the same law of the tints holds good, the tint T being proportional to the thickness t of the plate, and for any given plate we have $T = k\,t$, k being a constant depending on the nature of the plate. Now, if several plates of different uniaxal crystals be superposed, of which t, t', &c. are the thicknesses, and if a negative value of t be supposed to denote a transverse position of the axis of the plate, the resultant tint will be represented by

$$T = k\,t + k'\,t' + k''\,t'' + \text{\&c.}$$

942. Opposite action of plates of positive and negative crystals.

In this equation, if the plates be all of one substance, k, k', &c. are all alike; but if they be different, k is to be regarded as a negative quantity for all such crystals as belong to M. Biot's repulsive class, (Art. 803,) such as carbonate of lime; and positive for all such (quartz, for instance) which belong to his positive class. Thus, each term in the above equation may change its sign from two causes, either from a change in the nature of the crystal, or from a change of 90° in its azimuth.

943. General law.

The above is only a particular case of a more general law which may be thus announced,—*The tint ultimately produced is proportional to the interval of acceleration or retardation of the ordinary ray on the extraordinary, after traversing the whole system; the partial acceleration or retardation in each plate being proportional to the length of the path described within the plate, multiplied by the square of the sine of the angle which the transmitted ray makes, internally, with the optic axis of the plate, if it have but one axis, or to the product of the sines of its inclination to either, if it have two;* and this law holds good for all positions of the plates, and all arrangements of them one among the other. Thus (to instance its application) in the case of two similar and equal plates crossed at right angles; by the laws of polarization, the ray which, after its transmission through the first plate is ordinary, is refracted extraordinarily by the second, and *vice versâ;* thus the two rays, on entering the second plate exchange velocities; and, therefore, when finally emergent, since the thickness of the second is equal to that of the first, the one ray will have lost ground on the other in its second transmission just as much as it gained it in its first; and thus the interval of retardation and the tint will be reduced to nothing.

944. Superposition of plates cut at right angles to their axes.

From this it appears, that if two uniaxal plates cut at right angles to the axis be superposed, and adjusted so as to have their axes precisely coincident, the system of rings will have their diameters diminished if the plates be both attractive or both repulsive; but enlarged, if their characters be opposite. The experiment is rather delicate; but if made with care, placing the plates on one another with soft wax, and adjusting their surfaces by pressure to the exact position, it succeeded perfectly in the hands of Dr. Brewster.

945. Method of ascertaining whether a crystal be positive or negative.

This affords a means, independent of any measurement of the separation of the ordinary and extraordinary pencils, of ascertaining whether an uniaxal crystal be attractive or repulsive; for if its rings be *dilated* by combining it with a thin plate of carbonate of lime, cut at right angles to the axis, it is positive; if contracted, negative. A simpler and readier method still is to fasten on a plate of the substance under examination, so cut as to show the rings, a plate of sulphate of lime of moderate thickness, and then, interposing it between the tourmalines, to turn it about in its own plane. A position will be found where the rings are unaltered. In this situation the section B or C of the sulphate of lime is in the plane of primitive polarization. If the compound plate be turned 45° from this situation, it will now be observed (if the thicknesses of the two plates be properly proportioned) that the rings in two opposite quadrants are entirely obliterated; and that in the other two they are removed to a much greater distance from the centre, forming segments of larger circles, much closer together; and in which the tints, instead of commencing from the centre, commence from a black interval between two adjacent white rings in the midst of the system, and thence descend in the scale both inwards and outwards. In this state of things, the position of the sulphate of lime, with respect to the tourmalines, must be carefully noted; and the crystallized plate being detached, a plate of carbonate of lime, (perpendicular to its axis,) or of any other known uniaxal crystal, must be substituted for it; and the sulphate of lime replaced *in the same position.* If, then, it be found, that the same two quadrants of the rings are obliterated in this, as in the former case, and the new set of rings in the other quadrants be also similarly situated,—then the crystal examined is of the same character as the carbonate of lime, or other crystal used as a standard of comparison; but if, on the other hand, the quadrants where the rings were obliterated in the former case be those where the new rings are formed in the latter, then the characters of the two substances are opposite. If the crystallized plate be too thin, or of too feeble polarizing power to exhibit these phenomena with necessary distinctness, we must place it in azimuth 45° on the divided apparatus described in a former article (929;) and, fixing conveniently in the polarized beam a *very* thin plate of sulphate of lime also in azimuth 45°, ascertain, by making the crystal revolve, whether its tints have been raised or depressed in this plane by the action of the sulphate; then, removing the crystal, replace it with a standard one, and repeat the observation without touching

the sulphate. If both crystals have their tints raised, or both depressed, their characters are similar; if they be contrarily affected, dissimilar. An analogous mode of observation applies to biaxal crystals.

§ VIII. *On the Interferences of Polarized Rays.*

946. Origin of the subject.

In repeating the experiments of Dr. Young on the law of interference it occurred to M. Arago, that it would be worth while to examine whether the state of polarization of the interfering rays would cause any modification in the phenomena. The experiment was easy in the case where both rays had the same polarization, being, in fact, the ordinary case; but when the interfering rays were required to have a different state of polarization, it will easily be conceived that it must be a matter of great delicacy and difficulty to superadd this condition to the others called for by the nature of the case, which requires that the interfering rays should emanate *at the same instant* from a common origin, and should have executed the same precise number of undulations or periods (within a very few units) between their origin and the point where their interference is observed. For it is not possible to change the state of polarization of a ray without either altering its course, or transmitting it through some medium in which more or fewer undulations are executed in the same space. The joint ingenuity of himself and M. Fresnel, who was associated with him in this interesting inquiry, however, soon found means of obviating the difficulties and delicacies of the subject, and the results of their experiments have been embodied by them in the following laws:

947. Laws of interference of polarized light.

1. *That two rays polarized in one and the same plane act on or interfere with each other just as natural rays, so that the phenomena of interference in the two species of light are absolutely the same.*

2. *That two rays polarized in opposite planes* (i. e. *at right angles to each other*) *have no appreciable action on each other, in the very same circumstances where rays of natural light would interfere so as to destroy each other.*

948.

949.

3. *That two rays primitively polarized in opposite planes may be afterwards reduced to the same plane of polarization, without acquiring thereby the power of interfering with each other.*

950.

4. *That two rays polarized in opposite planes, and then reduced to similar states of polarization, interfere like natural rays, provided they belong to a pencil the whole of which was primitively polarized in one and the same plane.*

951.

5. *In the phenomena of interference produced by rays which have undergone double refraction, the place of the coloured fringes is not alone determined by the difference of routes or velocities, but that in certain circumstances a difference of half an undulation must be allowed for.*

952.

Such are the laws of interference of polarized pencils, as stated by Messrs. Arago and Fresnel. We use in their enunciation, and indeed throughout the sequel of this part of the doctrine of Light, the language of the undulatory system, as really the most natural, and adapting itself with the least violence and obscurity to the facts. The reader may, if he please, substitute that of the corpuscular hypothesis and the Newtonian fits, superadding that of a rotation of the luminous molecules about their axes, with M. Biot; or simply content himself with a bare enunciation of facts, and with general terms expressive of the existing conditions of periodicity, without much trouble, and only a little circumlocution, but with a great sacrifice of clearness of conception. With respect to the laws themselves, the first is easily verified; we have only to repeat any of the experiments on the interference of rays emanating from a common origin, described in our section on that subject, substituting polarized instead of natural light, and the results will be precisely similar, and that in whatever plane the light be polarized. Rays, then, polarized in the same plane, interfere as natural rays under similar circumstances.

Experimental verification of the first law.

953. Difficulties peculiar to the inquiry.

The verification of the second law is more difficult and delicate. The conditions of the production of colours by interference require that the interfering rays should emanate simultaneously from a common origin, or form parts of one and the same wave proceeding therefrom as a centre; and should have performed, at the point where their interference is examined, the same number of undulations in their respective routes, within a very few units. Now at their leaving their origin they *could not be otherwise* than in the same state of polarization; and as they are required to arrive at the point of interference in opposite states, a change of polarization must be operated on one or both rays, either by reflexion, transmission, or double refraction, after leaving their origin, and that without altering, more than by a few undulations, the difference of their routes. Now, when we consider how minute a quantity an undulation is, it is easy to conceive the delicacy required in adjusting the parts of any apparatus constructed for this purpose, or the peculiar contrivances which must be resorted to to render such extreme and almost impracticable nicety unnecessary.

954. Verification of the second law.

Several ingenious and elegant methods of making the experiment have been devised by the authors last named, of which we shall content ourselves with stating one or two of the easiest and most satisfactory. And, first, the origin of the interfering rays being the image of the sun at the focus of a small lens, as we shall suppose it throughout this section, (unless the contrary be expressly said,) it is evident that if we interpose between the eye and this image a rhomboid of Iceland spar, there will be formed two images separated from each other by a space which will be greater the thicker is the rhomboid; but which will always (unless extremely thick rhomboids be used) be very small; so that the single luminous point will now be resolved into two, very near each other, and which, by the laws of polarization, send to the eye rays polarized in opposite planes. But in this disposition of things, the condition of near equality of routes is subverted; for the ordinary and extraordinary pencils pursue different paths within the crystal, and with very different velocities; so that a difference will thus arise in the total number of undulations executed by each, sufficient to destroy all evidence

M. Fresnel's experiment with a bisected rhomboid.

of interference by the production of coloured fringes. To obviate this difficulty, M. Fresnel sawed in half a rhomboid of Iceland spar, the two halves of which must of necessity have, at their line of separation and its immediate confines, precisely equal thicknesses. These halves he placed one on the other, only turning one 90° round in azimuth, so as to have their principal sections at right angles. In this state, a pencil entering them nearly at the intersection of the planes of separation would at its final emergence be divided, not into four, but into two only, (see Art. 879,) the ray ordinarily refracted in the first half having undergone extraordinary refraction in the second, and *vice versâ*. The two rays, therefore, have exchanged velocities and directions, in the second transmission; and, therefore, when emergent, will have described exactly equal paths with equal velocities in each respectively, and will differ only in their states of polarization, which will be at right angles to each other. We have here, then, a case in which pencils diverge from two points side by side, and in a state in all other respects proper for interfering; nevertheless, when we look for the fringes which ought to be formed under such circumstances, (and which with natural light would be seen, see Art. 735 and 736,) none are visible. Their absence, then, must be owing to the opposite state of polarization of the interfering rays.

955. M. Arago's experiments with mica piles.

M. Arago, to make the same experiment, employed a process independent of double refraction. Two fine slits were made in a thin plate of copper, through which rays from the common origin were transmitted, and formed fringes (in their natural state) when viewed by an eye lens in the manner described, (Art. 709.) He now prepared two piles of pieces of very thin mica, or films of blown glass laid one on the other, fifteen in number, and then divided this compound plate in half by a sharp instrument, so that the halves, in the immediate neighbourhood of the line of division, could not be otherwise than of almost exactly equal thickness. These piles, when exposed at an incidence of 30° to a ray, were found to polarize the portion transmitted almost completely. They were then placed before the slits so as to receive and transmit the rays from the luminous point at precisely that incidence, and through spots which were very near each other in the undivided state of the pile. They were, moreover, so arranged, (being set on revolving frames,) that the plane of incidence could be varied (and therefore that of polarization) by turning either round in azimuth without altering its inclination to the ray, or varying the spot through which the ray passed. And it was then found, that when both piles were placed so as to polarize the rays in parallel planes, as, for instance, when both were inclined directly downwards, or one directly down and the other directly up—the fringes were formed as if the piles were away; but where one of the piles was turned round the incident ray as an axis through 90°, and so placed as to polarize the rays transmitted by it at right angles to the other, the fringes totally disappeared, nor could they be restored by inclining either pile a little more or less to the incident ray in the plane of incidence, the effect of which would be to alter gradually the length of the ray's path within the pile without changing its polarization, and thus, to compensate any slight inequality which might still subsist in their thicknesses. In intermediate positions the fringes appeared, but always the more vividly the nearer the planes of polarization approached to exact parallelism, thus attaining their maximum, and undergoing total obliteration at each quadrant of the rotation of either pile, (the other being at rest.)

956. Tourmaline plates substituted for the piles.

A plate of tourmaline carefully worked to exact parallelism, and bisected, would answer equally well with the transparent piles to polarize the rays; but the tourmaline should be selected of very homogeneous texture, such are not easy to meet with, though they may be found; and in this manner the experiment is perfectly easy and satisfactory. One half the tourmaline is fixed over one aperture, the other movable in a cell in its own plane over the other. The same phenomena will then be observed by turning round the movable tourmaline as with the oblique pile in the last experiment.

957. M. Fresnel's fundamental experiment. Analysis of the polarized tints.

An experiment still more simple, and equally conclusive, is the following, of M. Fresnel. He placed before the sheet of copper (having, as before, two narrow slits in it very near each other) a *single* thin parallel lamina of sulphate of lime. Now, as this body possesses double refraction, each pencil would be divided into two—an ordinary and an extraordinary one—which, according as they emanate from the right or left hand slit, we will term R *o*, R *e*, and L *o*, L *e*. If natural light be used to illuminate the slits, these pencils will be of equal intensity, but those marked *e* will be polarized oppositely from those marked *o*. We may then form four combinations: 1. R *o* may interfere with L *o*; 2. R *e* may interfere with L *e*; 3. R *o* with L *e*; 4. R *e* with L *o*. Now of these, R *o* and L *o* are similarly polarized, and they have described equal paths with equal velocities; therefore, supposing them capable of interference, they will give rise to a set of fringes corresponding exactly to the middle of the line joining the two slits, or, as we may express it, in the axis of the apparatus. The same may be said of R *e* and L *e*. These two sets of fringes will therefore be superposed, and appear as one of double intensity. Again, R *o* may be combined with L *e*; but as these two rays have traversed the sulphate in different directions and with different velocities, those rays of each pencil which meet *in the axis* will differ by too many undulations to produce colour; and if the pencils interfere, the place of the fringes will, instead of the axis, be shifted towards the side where the pencil has the greatest velocity, (Art. 737,) and that the more, the thicker the lamina of sulphate, so that if taken of a proper thickness, this set of fringes may be removed entirely out of the reach of the middle set, and should be seen independent of it. In like manner, the pencil R *e* may interfere with L *o*, and give rise to another set of lateral fringes; but as the ray which in the former combination was the swifter, in this is the slower, this set will lie on the opposite side of the middle set, supposing it produced at all; and thus there should be seen three sets of fringes, one bright, in the middle, and two fainter on either side. But, in fact, only one set is seen, *viz.* the middle set. Therefore the combination of the rays R *o* and L *e*, L *o* and R *e*, which are polarized oppositely, produce no fringes, *i. e.* they do not interfere.

But if we cut the lamina in half, and turn one half a quadrant round in its own plane, these rays are *then* reduced to the same polarization; and the rays R *o* and L *o*, R *e* and L *e*, which in the former case gave rise to

Light. the central fringes, are now placed in opposite states of polarization; and it is accordingly found that the central fringes have disappeared entirely, and that two lateral sets formed respectively by R *o* and L*e*, R *e* and L *o*, have started into existence. If we turn the lamina slowly round, these will gradually fade away, and the central reappear and become brighter, and so on alternately; thus affording a convincing proof of the truth of the second of the laws above enunciated. Part IV Experiment varied.

959. Verification of the third law.

The experiment related by Messrs. Arago and Fresnel in support of their third law is as follows: Resuming the arrangement of Art. 955 or 956, and placing the piles or tourmalines so as to polarize the two pencils oppositely, let a doubly refracting crystal be placed *between the eye and the sheet of copper*, with its principal section 45° inclined to either of the planes of polarization of the interfering rays. Each pencil will then divide itself by double refraction into two of equal intensity, and polarized in two planes at right angles, one of which is the principal section itself. We ought, therefore, to expect to see two systems of fringes, one produced by the interference of the ordinary ray from the right hand aperture (R *o*) with that of the left (L *o*,) and the other by that of R *e* with L *e*; yet no fringes are seen. The experiment may be varied by substituting for the doubly refracting prism a tourmaline, or pile, with its principal section in azimuth 45°. This must reduce to a common polarization all the rays which traverse it, *viz.* the half of each pencil, yet no fringes are seen, and therefore no interference takes place. See Errata

960. Experiments in proof of the fourth and fifth laws.

The following experiment is adduced in the Memoir cited in support of the fourth and fifth of the above laws. A lamina of sulphate of lime is perpendicularly exposed to a polarized pencil diverging from a minute point, and immediately behind it is placed a plate of brass pierced with two very small holes near together. The principal section of the lamina is to be placed at an angle of 45° with the plane of primitive polarization. In consequence, from each of the holes (right, R,—and left, L) will emerge a ray composed of two equal rays, R *o* and R *e*, and L *o*, L *e* oppositely polarized, *viz.* at angles + 45° and − 45° with the plane of primitive polarization, which we will suppose vertical. In this situation of things a rhomboid of Iceland spar is placed between the two holes, and the focus of the eye lens employed to view the fringes, with its principal section vertical, *i. e.* making again with that of the lamina angles of 45° either way. Each of the four rays then above mentioned will be divided into two equal rays, an ordinary and an extraordinary, thus giving rise in all to the eight rays

R *o o*, R *e o*; L *o o*, L *e o*; R *o e*, R *e e*; L *o e*, L *e e*.

These rays are received on the eye lens, and conveyed into the eye. Let us now examine their respective routes and states of polarization.

961.

First, then, the rays R *o* and R *e*, after quitting the lamina, are parallel; and by reason of the very small thickness of it, may be regarded as superposed, being undistinguishable from each other; but they have described within the lamina different paths by different velocities, so that on emerging they will differ in phase, by an interval of retardation proportioned to the thickness of the lamina, and which we will call d, so that x being the phase of the ray R *o*, $x + d$ will be that of R *e*. The very same may be said of L *o* and L *e*. Moreover, the two rays of either of these pairs respectively are oppositely polarized, *viz.* in planes + 45° and − 45° from the vertical. This we may represent at once thus:

Ray.	Phase.	Plane of Polarization.
R *o*	x	+ 45°
R *e*	$x + d$	− 45°
L *o*	x	+ 45°
L *e*	$x + d$	− 45°

962.

Next, the portions into which either of these rays is subdivided, in traversing the rhomboid, follow in their passage through it different paths, and have different velocities; but all which are refracted *ordinarily* have one common direction and velocity; and so of those refracted extraordinarily; hence, between the ordinary and extraordinary rays here produced, will arise a difference of phase which we shall call δ, so that if x be the phase of any ordinary ray, $x + \delta$ will be that of the corresponding extraordinary one; and their planes of polarization will be opposed, and will form angles respectively = 0 and 90° with the vertical. Thus the circumstances will stand thus:

A.			B.		
Ray.	Phase.	Plane of Polarization.	Ray.	Phase.	Plane of Polarization.
R *o o*	x	0°	R *o e*	$x + \delta$	90°
R *e o*	$x + d$	0°	R *e e*	$x + d + \delta$	90°
L *o o*	x	0°	L *o e*	$x + \delta$	90°
L *e o*	$x + d$	0°	L *e e*	$x + d + \delta$	90°

963.

These eight pencils are all equal in intensity, and all those contained in the first set (marked A) will meet in one part of the field of view, while those marked B (on account of the thickness of the rhomboid, which we here suppose considerable, so as to produce a sensible, and even a large separation of the ordinary and extraordinary pencils) will meet in another, distant from the point of concourse of (A) by an interval proportional to the thickness of the rhomboid, and which we will here suppose so large as to throw the fringes (if any) there produced, entirely out of the way of mixing with those produced at the concourse of A. Let us then consider separately, the pencils of rays of the parcel A, and see what interferences can take place. And first, R *o o* may

combine with L *o o*, and since their difference of phase is zero, they will interfere in the axis of the apparatus; and their planes of polarization being coincident, there is no reason why fringes should not there be produced by their concourse. The same holds good of the combination R *e o* and L *e o*, and, consequently, there will be superposed on each other in the axis two sets of fringes, producing one of double brilliancy.

964. Next, R *o o* may interfere with L *e o;* but there being a constant difference of phases *d* in favour of the latter, the fringes produced by their concourse will lie to the left of the axis, by an interval proportional to the thickness of the lamina of sulphate, and will be seen separately. Similarly, the concourse of the pencils R *e o* and L *o o* will determine the production of another set of lateral fringes; but the difference of phases *d* being in this case in favour of the right hand pencil, this system will be situated as much to the right of the axis as the other was to the left.

965. Thus in the ordinary image three sets of fringes ought to be seen, and in the extraordinary, by a similar reasoning, as many. Now, in fact, this is the case, and the phenomena are seen on making the experiment precisely as here described. But it is evident that the rays which form the lateral fringes, by their interferences, are precisely those which, at their leaving the sulphate, had opposite polarizations, but have been afterwards reduced to similar polarization by the action of the rhomboid.

966. Variation of the experiment. If instead of a rhomboid of sensible double refraction we substitute a plate of sulphate of lime, or of rock crystal, so thin as to produce no visible separation of the pencils, the fringes produced by the pencils B will be superposed on those arising from the interference of the pencils A, and we should expect therefore, instead of six, to see three sets of fringes, the middle one being still the brightest. But, in fact, we see but one set, and the lateral fringes vanish altogether. This remarkable result proves that the colours resulting from the concourse of the rays ordinarily refracted by the rhomboid, are complementary to those resulting from that of the extraordinary rays; and therefore that we must allow half an undulation to be gained or lost when we would pass from one set to the other, precisely as in the phenomena of the reflected and transmitted colours of thin plates.

Allowance of half an undulation.

967. Application to the colours of crystallized laminæ. One of the most important consequences of these laws, is that they supply the defective link in the chain which connects the doctrine of undulations with the colours of crystallized laminæ as described in the last section. It had been already remarked (as we have seen) by Dr. Young, that the passage of the ordinary and extraordinary rays with different velocities through the crystallized plate, would give rise to that difference of physical condition of the rays at their emergence which would lead to the production of colours; but the difficulty remained to explain, not why colours were produced in certain circumstances, but why they were not produced in all, in short, what share the polarization of the incident, and the analysis of the emergent rays, had in the production of the phenomena.

968. Why colours are not seen through a thin crystallized plate alone. To see the nature of this difficulty more clearly, imagine a wave proceeding from a distant radiant point to be incident on a very thin crystallized lamina. It will be subdivided into two, each traversing the plate in a different direction and with its own proper velocity, and each of them emerging parallel to its original direction. The incident wave will, therefore, after emergence be resolved into two parallel to each other, but separated by a small interval equal to the interval of retardation. Now the hindmost of these ought, according to the law of interferences, to interfere with a subsequent wave of the system to which the foremost belongs, and thus periodical colours should arise on merely looking against the sky through such a lamina without any other apparatus. Why then are none seen? To this the law of Messrs. Arago and Fresnel afford a satisfactory answer. The two systems of waves into which the incident system is resolved are *oppositely polarized*, and therefore, though all other conditions be satisfied, incapable of interfering.

969. Explanation of the colours of the polarized rings. Fig. 194. To understand how the colours of the polarized rings must be conceived to be produced by interference, let us take the simplest case when a polarized ray, A B, fig. 194, is incident on any thin crystallized plate B, whose principal section is 45° inclined to the plane of primitive polarization. Let A be the system of waves which constitutes the incident ray; then in its passage through the crystallized lamina it will be divided into systems O and E of equal intensities, polarized in planes + 45° and − 45° inclined to that of primitive polarization, and the one lagging a few undulations behind the other, so as to interfere, as represented in the figure, and constituting the parallel rays C F and D G. Let these now be received on, and transmitted through, a doubly refracting prism F G H L placed with its principal section in the plane of primitive polarization, or 45° inclined to that of the crystallized lamina. Then will each of the incident rays be again subdivided, C F into H M and I P, and D G into K N and L Q, all of equal intensity. Of these, H M and K N emerge parallel, as also K N and L Q respectively. Now the systems of waves O and E which follow each other at a certain interval *d* will continue to do so in both the refracted rays, as if they formed one compound system; so that each of the pencils H M K N and I P L Q will consist of a double system of waves O *e* and E *e*, O *o* and E *o* respectively. The former pair following each other at the interval *d*, and the latter at the interval $d \pm \frac{1}{2}$ undulation, (by reason of the demonstrated fact, that in passing from the ordinary to the extraordinary system half an undulation must be allowed. See Art. 966.) Now as each ray of these pairs respectively have similar polarizations, *viz.* those of the pair ordinarily refracted (O *o* and E *o*) in the plane of the principal section of the prism, and those of the extraordinary pair O *e* and E *e* in a plane at right angles to it, there is no reason why interference should not take place, and the consequence must be, the production of complementary colours in the two pencils finally emergent corresponding to the intervals of retardation *d* and $d + \frac{\lambda}{2}$, which is just what really happens.

970. Explanation of the complementary tints. Conceive now another ray incident on B in the direction A B, but polarized in a plane at right angles to that of the ray considered in the last paragraph. Then this will undergo precisely the same series of divisions and subdivisions as the former. But the intervals of retardation will be different; for its plane of polarization when incident on B being *now* related to the plane of ordinary refraction, as that of the other ray at its incidence was

Light. Part IV.

to the extraordinary, and *vice versâ*, a difference of half an undulation must (as already explained) be admitted in the relative position of the two systems of waves O, E, at their emergence, from this cause, independent of the interval of retardation within the plate; so that if d were the interval in the former case, $d - \frac{1}{2}\lambda$ will be the difference now, and, after passing through the prism, we shall have for the intervals of retardation in the two binary pencils, instead of d and $d + \frac{1}{2}\lambda$ which they were before, $d - \frac{1}{2}\lambda$ and d. Hence the two pencils will exchange colours when the polarization of the incident light is varied by a quadrant, and this is also conformable to fact. If this reasoning be not thought conclusive, the reader is referred forwards to Art. 983 and 984.

971. Why colours are not produced by unpolarized light.

Next, let the incident ray be unpolarized. This case, as we have seen Art. 851, is the same with that of a ray consisting of two equal rays oppositely polarized, and therefore in each pencil will coexist, superposed on each other, the primary and complementary colour arising from either portion, which being of equal intensity will neutralize each other's colours and the emergent pencils will be white, and each of half the intensity of the incident beams. This then is the reason (on this doctrine) why we see no colours when the light originally incident on the crystallized plate is unpolarized.

972. M. Fresnel's general rule for determining how to allow for the half undulation gained or lost.

Thus, the theory of interferences, modified by the principles above stated, affords, as we see, an explanation of the colours of crystallized plates totally distinct from that of movable polarization. The only delicacy in its application to all cases, lies in the determination which of the emergent pencils must be regarded as having its interval of retardation increased by half an undulation. M. Fresnel gives the following rule for this essential point. (Note on M. Arago's Report to the Institute on a Memoir of M. Fresnel relative to the colours of doubly refracting laminæ, *Annales de Chimie*, vol. xvii. p. 80.*) *The image whose tint corresponds precisely to the difference of routes, is that in which the planes of polarization of its constituent pencils after having been separated from each other, are brought together by a contrary motion, while, on the other hand, the pencils whose planes of polarization are brought to coincidence by a continuance of the same motion by which they were separated, produce by their reunion the complementary image.* To understand this better, let P C be the plane of primitive polarization projected on that of the paper, to which let us suppose the ray perpendicular, C O that of the principal section of the crystallized lamina, and C S that of the principal section of the doubly refracting prism; then the incident pencil polarized in the plane P P′ will after penetrating the lamina be divided into two, one O polarized in the plane C O, the other E in the plane C E perpendicular to it. Now, C O may always be so taken as to make an angle not greater than a right angle with C P, and C E so as to have C P *between* C E and C O; so that the plane C P may be conceived to open or unfold itself like the covers of a book, into C O and C E, one on either side. Again, C S may always be regarded as making an angle not greater than a right angle with C O, and when the ray O resolves itself into two (O *o* and O *e*) by refraction at the prism, its plane of polarization C O may be conceived to open out into the two C S and C T at right angles to each other, including C O between them; and in like manner the ray E will resolve itself into two E *o* and E *e*, and its plane of polarization C E will open out into the two C S and C T′, having C E between them in the case of fig. 195 (*a*), and into C S′ and C E in that of fig. 195 (*b*): in the former case C T′ is a prolongation of C T, in the latter C S′ is a prolongation of C S. The rays O *o* and E *o* then which make up the ordinary pencil, have, in the case of fig. (*a*), been each brought to a coincident plane of polarization C S by two motions in contrary directions, as represented by the arrows, and the extraordinary ones O *e* and E *e* have been separated and brought back to a coincident plane by motions continued in the same direction for each respectively. The reverse is the case in fig. *b*. In the case then of fig. *a* the colours of the ordinary pencil O *o* + E *o* will be those which correspond precisely to the difference of routes, and those of the extraordinary one O *e* + E *e* will correspond to that difference *plus* half an undulation, while in that of fig. *b* the reverse happens. This rule is empirical, *i. e.* is merely a result of observation. It is clear that the principle of the conservation of the *vis viva* in this, as in the colours of uncrystallized plates, requires that the two images should be complementary to each other, and therefore half an undulation *must* be gained or lost by one or the other pencil, but which of the two is to be so modified we have no means of knowing *a priori*.

Fig. 195.

973.

This once determined, however, we have no difficulty in deducing the formulæ of intensity and other circumstances of the phenomena when the azimuth of the crystallized plate is arbitrary, instead of being, as we have hitherto supposed, limited to 45°. The analytical expressions of the intensity of the pencils we must reserve for our next section.

§ IX. *Of the application of the Undulatory Doctrine to the explanation of the phenomena of Polarized Light and of Double Refraction.*

974. Newton's objections against the undulatory theory.

The phenomena of double refraction and polarization, as exhibited in the experiments of Huygens on Iceland spar, were regarded by Newton and his followers as insuperable objections to the undulatory doctrine, inasmuch as it appeared to them impossible, by reason of the *quaquâversum* pressure of an elastic fluid, to conceive an undulation as having a different relation to different regions of space, or as possessing *sides*. "Are not," says Newton, "all hypotheses erroneous in which light is supposed to consist in pressure or motion propagated

* This Memoir was read to the Institute, Oct. 7, 1816. A Supplement was received Jan. 19, 1818. M. Arago's report on it was read June 4, 1821. And while every optical philosopher in Europe has been impatiently expecting its appearance for seven years, it lies as yet unpublished, and is known to us only by meagre notices in a periodical Journal.

Light. through a fluid medium?".....“ for pressures or motions propagated from a shining body through an uniform medium, must be on all sides alike, whereas it appears that the rays of light have different properties in their different sides.".....“ To me, this seems inexplicable, if light be nothing else than pressure or motion propagated through ether.” *Opticks*, book iii. quest. 28. And, again, quest. 29 ; “ Are not rays of light very small bodies emitted from shining substances?".....“ The unusual refraction of Iceland crystal looks very much as if it were performed by some kind of attractive virtue lodged in certain sides both of the rays and of the particles of the crystal.".....“ I do not say this virtue is magnetical.—It seems to be of another kind. I only say, that, whatever it be, it is difficult to conceive how the rays of light, unless they be bodies, can have a permanent virtue in two of their sides which is not in their other sides, and this, without any regard to their position as to the space or medium through which they pass.” Part IV.

975. Examined. Although we have no knowledge of the intimate constitution of elastic media, or the manner in which their contiguous particles are related to each other and affect each other's motion, yet it is certain that the mode and laws of the propagation of motion through them by undulation cannot but depend very materially on this connection. The only analogies we have to guide us into any inquiry into these laws, are those of the propagation of sound in air or water, and of tremors through elastic solids, and along tended chords and surfaces; and such is the extreme difficulty of the subject when taken up in a purely mathematical point of view, that we are forced to have recourse to these analogies, and, dismissing in the present state of science the vain hope of embracing the whole subject in analytical formulæ, suffer ourselves to be instructed by experience, as to what modifications the peculiar constitution of vibrating media may produce in the propagation of motion through them. Now, when sound is propagated through air or water, in which the molecules are at least *supposed* to have no mutual connection but to be capable of moving with equal facility, and to be restored to their places with equal elastic forces, in whatever direction they are displaced, and in which, moreover, it is (at least theoretically) taken for granted, that the motion of any molecule has an equal tendency to set in motion those adjacent to it, in whatever direction these may be situated with respect to it; it is difficult to conceive that the motion of a molecule in the surface of a wave, at some distance from the centre whence the sound emanates, can be performed otherwise than in the direction of the radius, or at right angles to the surface of the wave; so that in this case the motion of the vibrating molecules must coincide with the direction of the rays of sound, and there appears, therefore, no reason why such rays should bear different relations to the different regions of space surrounding them, whether right or left, above or below; for the ray being regarded as an axis, all parts of the sphere round it are similarly related to it.

976. But if we conceive a connection of any kind, such as may possibly be established by repulsive and attractive forces, or magnetic or other polarities subsisting between the molecules of the vibrating medium, the case is altered. It will no longer then follow of necessity, that the individual motion of each molecule is performed in the direction in which the general wave advances, but it may be conceived to form any angle with that direction, even a right angle. A familiar instance of such a mode of propagation may be seen in the wave which runs along a long stretched cord, struck, shaken, or otherwise disturbed at one end. The direction of the wave is the length of the cord, and that of the motion of each molecule lies in a plane perpendicular to it. Now this is precisely the kind of propagation which M. Fresnel conceives to obtain in the case of light. He supposes the eye to be affected only by such vibrating motions of the ethereal molecules as are performed in planes perpendicular to the directions of the rays. According to this doctrine, a polarized ray is one in which the vibration is constantly performed in one plane, owing either to a regular motion originally impressed on the luminous molecule, or to some subsequent cause acting on the waves themselves, which disposes the planes of vibration of their molecules all one way. An unpolarized ray may be regarded as one in which the plane of vibration is perpetually varying, or in which the vibrating molecules of the luminary are perpetually shifting their planes of motion, and in which no cause has subsequently acted to bring the vibrations thus excited in the ether to coincident planes.

Fresnel's theory of transverse vibrations.

977. Propagation of light assimilated to that of waves along a stretched cord. The analogy of the tended cord (which appears to have suggested itself to Dr. Young on considering the optical properties of biaxal crystals in 1818) will help our conception greatly. Suppose such a cord of indefinite length, stretched horizontally, and one end of it being held in the hand, let it be agitated to and fro with a motion perpendicular to the length of the cord. Then will a wave or succession of waves be propagated along it, and every molecule of the cord will, after the lapse of a time proportional to its distance from the hand, begin to describe a line or curve similar and similarly situated to that described by the extremity at which the agitation originates. If the original agitation be regularly repeated and constantly confined to one plane, the same will be true of the motion of each molecule, and the whole extent of the cord will be thrown into the form of an undulating curve lying in one plane, so far as the motion has reached. In this case it will represent a polarized ray or system of waves. If, after a few vibrations in one plane, the extremity be made to execute a few in another, and then again in another, and so on, so that the plane of vibration shall assume in rapid succession all possible situations, since each molecule obeys exactly the same law of motion with the extremity, the curve will consist of portions lying in all possible planes, and since by reason of the propagation of the undulation along it, every point of it is in succession agitated by the motion of every other, all these varied vibrations will run through any given point of it, and were a sentient organ like the human retina stationed there, the impression it would receive would be analogous to that excited in the eye by an unpolarized ray of light.

978. Objections considered. It may be objected to this mode of conceiving the luminiferous undulations, that the molecules of the ether, if it be a fluid, such as we have hitherto all along regarded it, cannot be supposed connected in strings, or chains like those of a tended cord, but must exist separate and independent of each other. But it is sufficient for our purpose to admit such a degree of lateral adhesion (we hesitate to term it *viscosity*) as may enable each molecule in its motion not merely to *push before it* those which lie directly in the line of its motion, but to *drag along*

with it those which lie on either side, in the same direction with itself. Or, acknowledging at once the difficulty, since light is a real phenomenon, we are not to expect it to be produced without a mechanism adequate to so wonderful an effect. We do not hesitate to attribute to the fluids which are imagined to account for the phenomena of heat, electricity, magnetism, &c. properties altogether repugnant to our ordinary notions of fluids, and why should we deny ourselves the same latitude when light is to be accounted for. It is true the properties we must attribute to the ether appear characteristic of a solid than of a fluid, and may be regarded as reviving the antiquated doctrine of a plenum. But if the phenomena can be thereby accounted for, *i. e.* reduced to uniform and general principles, we see no reason why that, or any still wilder doctrine, should not be admitted, not indeed to all the privileges of a demonstrated fact, but to those of its representative, or *locum tenens*, till the real truth shall be discovered. Assuming it, then, with M. Fresnel, as a postulatum, that the vibrations of the ethereal molecules which constitute light are performed in planes at right angles to the direction of the ray's progress, let us see what account can be given of the phenomena of polarized light.

979. Explanation of the phenomena of interference on this doctrine.

And first, then, of the interference of two polarized rays, whether polarized in the same, or different planes. The plane of polarization in this doctrine may be assumed to be either that in which the vibrations are executed, (*i. e.* a plane passing through the direction of the ray and the line described by each of the vibrating molecules in its excursion,) or one perpendicular to it, which we please. Reasons, presently to be stated, render the latter preferable, but at present it is a matter of indifference which we assume. Now, in § 3, Part III. we have investigated at length, with a view to the present inquiry, the modes of vibration which result from the combination of any assigned vibrations, whether executed in the same or different planes; and it follows from the purely mechanical principles there laid down, 1st, That the combination of two vibrations executed in the same plane, produces a resultant vibration in the same plane, which may be of any degree of intensity from the sum to the difference of the intensities of its component vibrations, according to the difference of their phases. Now, each of these systems of vibration represents a polarized ray; so that rays polarized in the same plane ought, on these principles, to be capable of destroying or reinforcing each other by interference, as we see they do. But the case is otherwise when the component vibrations are executed in different planes, for in that case it is obvious that they never can destroy each other completely so as to produce rest. The general case of non-coincident planes of vibration is analyzed in Art. 618; and in Art. 621 we see, that even when each of the component vibrations is rectilinear, the resultant is elliptic; so that each molecule of the ether performs continual gyrations in one direction, and never can be totally quiescent.

980. Analogy of the stretched cord.

Thus we see that the interference of rays similarly polarized, and the non-interference of those dissimilarly, is a necessary consequence of the hypothesis we are considering; and indeed was the phenomenon which first suggested it. It may be familiarly explained by the analogy of our tended cord. Conceive such a cord to have its extremity agitated at equal regular intervals with a vibratory motion performed in one plane, then it will be thrown, as we have seen, into an undulatory curve, all lying in the same plane. Now, if we superadd to this motion another, similar and equal, but commencing exactly half an undulation later, it is evident that the direct motion every molecule would assume, in consequence of the first system, will at every instant be exactly neutralized by the retrograde motion it would take in virtue of the other; and, therefore, each molecule will remain at rest, and the cord itself be quiescent. But if the second system of motions be performed in a plane at right angles to the first, the effect will evidently only be to distort the figure of the cord into a curve of double curvature, which, in the general case, will be an elliptic helix, and will pass into the ordinary circular one when the two component vibrations differ in phase by a quarter of an undulation, or 90°. (See Art. 627. *Corol.*)

981. Case of a rotatory or spiral motion.

In this case the extremity of the cord describes a circle with a continuous motion, and this motion is imitated by each molecule along its whole length. It is easy to make this a matter of experiment; we have only to hold in our hands the end of a long stretched cord, or grasp it firmly in any part of its extent, and work the part held round and round, with a regular circular motion, and we shall see the cord thrown into a helicoidal curve, each portion of which circulates in imitation of the original source of the motion

982 Resultant of two rays oppositely polarized investigated

But experience shows, not merely that two equal rays polarized at right angles do not destroy each other for any assignable difference of origins, but, that whatever be this difference, the intensity of the resultant ray remains absolutely the same. Now this is also a necessary consequence of the theory of transverse vibrations. To show this, we need only refer to the expressions for A, B, C in equation (7,) Art. 619, resuming at the same time the notation and reasoning of that article. The intensity of the impression made on the eye by any ray being proportional to the *vis viva*, is represented by the sum of the several *vires vivæ* in the three rectangular directions, or by $A^2 + B^2 + C^2$, that is, by

$$a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2 + 2\,a\,a' \cdot \cos(p - p') + 2\,b\,b' \cdot \cos(q - q') + 2\,c\,c' \cdot \cos(r - r').$$

Now if we assume the directions of the coordinates x and y to be those transverse to that of the ray, and the one in the plane of polarization of one ray, the other in that of the other, at right angles to it, and that of z in the direction of the ray itself, we have

$$a' = 0, \quad b = 0 \quad c = 0, \quad c' = 0;$$

and therefore the above expression for the intensity becomes

$$A^2 \quad B^2 + C^2 = a^2 + b'^2,$$

which is independent of $p - p'$, $q - q'$, $r - r'$, the difference of phases, and is equal to the sum of the inten-

sities of the separate rays. And we may remark, by the way, that no other supposable mode of vibration but that in question, in which c and c', the amplitudes of vibration in the direction of the ray vanish, could produce the same result. (Fresnel's *Considerations Théoriques sur la Polarization de la Lumière. Bulletin de la Société Philomatique*, October, 1824.)

983. Let us now consider what will happen when a ray polarized in any plane is resolved into two polarized in any other two planes at right angles to each other, and these again reduced to two others also at right angles to each other, by a second resolution. Suppose C, (fig. 195, *a*) to be the course of a ray projected on a plane perpendicular to its direction, (that of the paper,) and in which, consequently, the vibrations of the molecule C are performed. Let P C P′ be the line of vibration of this molecule, and therefore (according to the hypothesis assumed) at right angles to the plane of primitive polarization. When this ray is divided into two others oppositely polarized, the vibrations are of course resolved into two others performed in planes at right angles to each other. Let C O and C E be the projections of these planes, which are therefore perpendicular to the planes of polarization of the two new rays respectively. Suppose that at any instant the molecule C of the primitive ray is moving *from* C in the direction C P; then this motion, if resolved into two, will give rise to two motions, one in the direction *from* C *towards* O, the other *from* C *towards* E. If each of these motions be again resolved into two, in planes whose projections are S C S′ and T C T′, at right angles to each other, that in the direction C O will produce two motions, one in the direction C S, and the other in the direction C T; and on the other hand the motion in the direction C E will produce one in the direction C S, and the other (in the case of fig. 195, *a*) in the direction C T′ opposite to C T. Thus the two resolved motions in the plane S S′ will conspire, but those in the plane T T′ will oppose, each other. In the case of fig. 195, *b*, the reverse will happen; the motions in the plane T, T′ conspiring, and those in the plane S S′ opposing, each other. For simplicity of conception, however, we will confine ourselves to the former case. If, now, we pass from the consideration of the vibrations to that of the rays, it will appear that we have, in fact, resolved the original ray polarized in the plane P P′ into two, polarized in planes perpendicular respectively to C O and C E; and these again, finally, each into two, *viz.* one polarized in the perpendicular to S S′, and one perpendicular to T T′. The two portions polarized perpendicular to S S′ form one ray, and those perpendicular to T T′ another; but in the former, the component portions tend to strengthen,—in the latter, to destroy each other. Hence, if we consider the two former portions as having a common origin, we must regard the latter as differing by half an undulation.

Rationale of the rule for allowing half an undulation.

Fig. 195.

984. Hitherto we have supposed the second resolution of the rays to take place at the same point C in the course of the ray as the first, but this may not be the case, and several cases may be imagined; first, we may suppose the two portions into which the ray is first resolved to run on in the same line with equal velocities; and after describing any given space, to be then resolved, at another point C′ (whose projection in the figure will coincide with C) into the final rays S S′ and T T′. It is evident that this will make no difference in the result, for the phases in which each ray arrives at C′ will be alike; and after the second resolution the conspiring vibrations in the direction S S′ will still be in the same phase, and the opposing ones in the plane T T′ must still be regarded as in opposite phases, *i. e.* as differing by half an undulation. Or, secondly, we may suppose, that, owing to any cause, the two resolved rays do not travel with equal velocity, (as in the case where the resolution is performed by double refraction.) In this case, if i be the interval of retardation of the one ray on the other when they arrive at C′, i will represent the difference of phases of the two rays at the instant of their second resolution. Consequently, when resolved, the final ray, whose vibrations are performed in S S′, will be the *sum;* and that whose vibrations are performed in T T′, the *difference* of two rays, one in a certain phase (θ), the other in the phase $\theta + i$; or, which is the same thing, the former will be the *sum* of two components in the phases θ and $\theta + i$; the latter, the *sum* of two in the phases θ and $\theta + i + 180°$, so that still the difference of half an undulation is to be applied. In the case of fig. 195, *b*, if we pursue the same reasoning, it will appear that this difference still subsists, but must be applied conversely, *viz.* to the compound ray whose vibrations are performed in C S.

985. We have here, then, the theoretical origin of the allowance of half an undulation, in those cases where it is required to account for the polarized tints, Art. 966, and of the rule laid down in Art. 972 for its correct application. However arbitrary the assumption may have appeared as there presented, and however singular it may have seemed to make the affections of a ray at one point of its course dependent on those which it had at a former instant, we now see that the whole is a direct and very simple consequence of the ordinary elementary rules for the composition and resolution of motions. It is worthy of notice, that the fact was ascertained *before* the theory of transverse vibrations was devised, so that this theory has the merit of affording an *a priori* explanation of what had previously all the appearance of a mere gratuitous hypothesis.

986. In conceiving the resolution of a ray into two others polarized in different planes, we may be aided by the analogy of the tended cord, which we have before had occasion to refer to. In fig. 196 let A B be a stretched cord, branching at B into the two B C and B D, making a small angle with each other at B, and having either equal or unequal tensions. Suppose the plane in which the two branches lie to be (for illustration's sake) horizontal, and let the extremity A of the single cord be made to vibrate regularly in a vertical plane; or, at least, let the vibrations of the cord, before arriving at B, be reduced to a vertical plane by means of a small polished vertical guide I K, against which the cord shall press lightly, and on which it may slide freely without friction. Beyond the point of bifurcation B, and at such a distance that the excursions of the molecule B shall subtend no sensible angle from them, let two other such polished *guiding planes* be placed, inclined at different angles to the horizon, and making a right angle with each other. Suppose now B to make any excursion from its point of rest, then were the plane E F parallel to I K, the molecule of the branch B C contiguous to E F would slide on E F through a space equal to the whole excursion of B; but since it is inclined to I K at an angle

Application of the analogy of the stretched cord.

Fig. 196.

($= \theta$) a part only of the motion of B will be employed in causing this molecule to glide on E F, and the remainder will cause the cord to bend over and press on the obstacle; but by reason of the minuteness of the excursions of B, this bending and the resistance of the obstacle and consequent loss of force will be very minute and may be neglected. Now, since the pressure of the obstacle removes the cord from the position it would have taken had no obstacle existed, in a direction perpendicular to its surface, it is easy to see that the amplitude of excursion of the contiguous molecule *on the plane* E F must be to that of B as cos θ to radius; and, therefore, calling a the amplitude of B's excursions, a . cos θ will be that of the molecule contiguous to E F, and of course that of every subsequent molecule of the branch B C. Here the part of B's motion, which is perpendicular to E F, is not expended or destroyed in bending the cord B C over the obstacle, but remains in activity, and exerts itself on the branch B D, causing it to glide on the plane G H; and the amplitude of the excursions of the molecule in contact with this plane will in like manner be represented by a . cos (inclination of G H to I K,) that is, by a . cos $(90 - \theta)$, or by a . sin θ. The *vis viva*, then, in each of these respective planes is represented by a^2 . cos θ^2 and a^2 . sin θ^2, whose sum is equal to a^2, the initial *vis viva*.

987. Rationale of Malus's rule for the intensity of the complementary rays.

If we decompose, in like manner, the maximum velocity a of the ethereal molecule C (fig. 195) in the direction C P into two in the respective directions C O and C E, we get a . cos θ and a . sin θ for the elementary velocities; and since the amplitudes, *cæteris paribus*, are as the velocities, (Art. 610,) the amplitudes of the component rays will be respectively a . cos θ and a . sin θ; and their intensities, which are as the squares of the amplitudes, (Art. 605,) will be a^2 . cos θ^2 and a^2 . sin θ^2. Now this is the very law propounded by Malus for the intensities of the two portions into which a polarized ray is divided by double refraction, and of which the theory of transverse vibrations gives, as we see, a simple and rational *à priori* account, thus raising it from a mere empirical law to the rank of a legitimate theoretical deduction.

988. Case of the two resolved undulations propagated with different velocities.

We have not done with the analogy of the tended cord. What we have shown in Art. 986 is independent of the tensions of the branches into which the cord is divided, and relates only to the amplitudes of their excursions from rest when thrown into vibration. But the velocity with which the waves, once produced, will be propagated along either branch depends solely on its tension. Nothing, however, prevents the tensions of the two branches from being very different; for, whatever be the ratio of two forces applied in the directions B C and B D, they may be balanced at B by a proper force applied along any other line as B A. Hence the waves will run along B C and B D with different velocities. Similarly, if we conceive, that owing to the peculiar constitution of crystallized bodies, and the relation of their particles to the ether which pervades them, its molecules are more easily displaced, or yield to a less force in certain planes than in others; or, in other words, that it possesses different elasticities in different directions; then will the planes of polarization assumed by the resolved portions of the rays determine the elasticities brought into action, and, by consequence, the velocities of their propagation. Now we have, in a former section, shown that the bending of a ray at the confines of a medium depends essentially on its velocity within as compared with that without, by the analytical relations deduced from the "principle of swiftest propagation." A difference of velocity, therefore, draws with it, as a necessary consequence, a diversity of path; and thus the bifurcation, or double refraction of a ray incident on a crystallized surface, presents no longer any difficulty in theory, provided we can find an adequate reason for the resolution of its vibrations into two determinate planes at the moment of its entering the crystal.

989. Explanation of the phenomena of double refraction in crystals with one axis.

Let us take (with M. Fresnel, *Annales de Chimie*, xvii. p. 179 *et seq.*) the case of a crystal with one axis. We may regard this, or rather the ether within it, modified in its action by the molecular forces of the crystal, as an elastic medium in which the elasticity in a direction perpendicular to the axis is different from that in a direction parallel to it, that is, in which the molecules are more easily compressible in the one than in the other direction; but, equally so in all directions perpendicularly to the axis, on whatever side the pressure be applied. To aid our conceptions in imagining such a property, we may assimilate an uniformly elastic medium to an assemblage of thin, elastic, hollow, spherical shells in contact; and such a medium as we are considering, to a similar assemblage of oblate or prolate hollow ellipsoids, arranged with all their axes parallel to one common direction, which is that of the axis of the crystal.* It is evident that the resistance of the spherical assemblage to pressure must be the same in all directions, but that of the spheroidal must differ according as the pressure is applied perpendicularly or parallel to the axis. Thus, it is easy to crush an egg by a force applied in the direction of its shorter diameter, which will yet sustain a violent pressure applied at the extremities of its longer. It is, moreover, evident, if any molecule of such an assemblage were disturbed, so as to throw it into vibration, that, provided always the amplitude of its excursions were extremely small compared to the diameter of each ellipsoid, the immediate tendency of the vibration will be to communicate motion to two strata only of molecules, *viz.* that in which the axis and equator of the disturbed molecule lie respectively, since it is only at the poles and equator that they touch, and therefore only through these points that motion can be communicated from one to the other. Consequently, any motion communicated to a molecule of such a mass could only be propagated by vibrations performed in planes parallel and perpendicular to the axis. Hence, if a vibratory motion in any plane be propagated into such an assemblage of particles from without, it will immediately, on its reaching it,

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* The idea of spheroidal *molecules* in Iceland spar suggested itself to Huygens (rather fancifully, perhaps) as a means by which spheroidal *undulations* might be propagated through it, (*Op. Reliq.* tom. i. *Tractatus de Lumine*, p. 70, cited by Wollaston, *Phil. Trans.* ciii. p. 58;) and the last-named eminent Philosopher, in the Bakerian Lecture for 1813, has most ingeniously shown how such molecules may be combined to build up crystals, having the primitive forms and cleavages of acute and obtuse rhomboids. It is true, that in all this there is much hypothesis; and it should be observed, too, that the crystallographic structure would require *oblate* spheroids, where in the text we have employed *prolate*, and *vice versâ*. But we intend there only an *analogy*, not a *theory*. It would be easy to devise hypothetical modes of action where these forms might be reversed if needful.

be resolved into two, in the planes above named; and these, by reason of the different elasticities, will be propagated with different velocities.

990. Bifurcation of the refracted ray explained.

The reader must not suppose that this is intended for an account of the real mechanism of crystallized bodies. It is merely intended to show that it is not absurd, or contradictory to sound mechanical principles, to assume that such *may* be their constitution, that vibrations can only be propagated through them by molecular excursions executed in planes parallel and perpendicular to their axes. Assuming, then, that such is the case, the vibrations of a ray incident on such a crystal will be resolved into two, performed in these respective planes, and their velocities of propagation being different, the rays so arising will follow different courses when bent by refraction. Let us first consider that whose vibrations are executed in planes perpendicular to the axis. Since the crystal is symmetrical with respect to its axis, and equally elastic in all directions perpendicular to it, the velocity of propagation of this portion will be the same in all directions. Its index of refraction, therefore, will be constant, and the refraction of this portion will follow the ordinary law. Moreover, its plane of polarization being that *perpendicular* to which the vibrations are performed, will necessarily pass through the axis, in which respect it also agrees with the ordinary ray, as actually observed.

Properties of the ordinary ray.

991. Properties of the extraordinary ray explained.

The extraordinary ray arises from the other resolved portion of the original vibration, which is performed in a plane parallel to the axis. By the principle of transverse vibrations, it is also performed in a plane perpendicular to the ray. If, then, we suppose a plane to pass through the extraordinary ray and the axis, it will cut a plane perpendicular to the ray in a straight line, which will be the direction of the vibratory motion. This direction, then, is inclined to the axis in an angle equal to the complement of that made by the extraordinary ray with the latter line, and therefore, when the extraordinary ray is parallel to the axis, the line of vibration is perpendicular to it, and *vice versâ*. In the former case, the elastic force resisting the displacement of the molecules is the same as in the case of the ordinary ray, and therefore the velocities of both rays are equal, and their directions coincide, and thus along the axis there is no separation of the rays. In the latter, the elasticity is that parallel to the axis, and therefore differing from the former by the greatest possible quantity. Here, then, the difference of velocities, and therefore of directions is at its maximum. In intermediate situations of the extraordinary ray, the elasticity developed is intermediate, and therefore also the velocity and double refraction. Thus we see, that according to this doctrine the difference of velocities, and consequent separation of the pencils should be nothing in the axis, and go on increasing till the extraordinary ray is at right angles to it, which is conformable to fact. Lastly, the plane of polarization of the extraordinary ray being at right angles to the plane of vibration, must also be at right angles to a plane passing through the ray and the axis, which is also conformable to fact.

992. The theory of M. Fresnel gives then, as we see, at least a plausible account of the phenomena of double refraction in the case of uniaxal crystals; and when we consider the profound mystery which, on every other hypothesis, was admitted to hang over this part of the subject, we must allow that this is a great and important step. But the same principles are equally applicable to biaxal crystals with proper modifications, and (which is a strong argument for their reality) lead, when so applied, to conclusions which, though totally at variance with all that had been taken for granted before, on the grounds of imperfect analogy and insufficient experiment, have been since verified by accurate and careful experiments, and have thus opened a new and curious field of optical inquiry. Nothing stronger can be said in favour of an hypothesis, than that it enables us to anticipate the results of experiment, and to predict facts opposed to received notions, and mistaken or imperfect experience.

993. Explanation of the phenomena of movable polarization.

But before we enter on this, it may be right to show how the phenomenon on which the theory of movable polarization is founded, is accounted for by the doctrine of transverse vibrations. According to this theory, as soon as a polarized ray enters a crystal, it commences a series of alternate assumptions of one or other of two planes of polarization, in the azimuths $0°$ and $2i$, i being the inclination of the principal section to the plane of primitive polarization: the plane assumed being in azimuth $0°$, when the thickness traversed is such as to render the interval of retardation of the ordinary on the extraordinary ray 0, or any whole number of undulations, and in azimuth $2i$ when it is any whole odd number of semi-undulations. Suppose a ray polarized in the azimuth 0 to be incident perpendicularly on a crystallized lamina, having its principal section in the azimuth i, then it will be resolved into two, the vibrations of which are respectively performed in the principal section, and perpendicular to it. Consequently, if we represent by unity the amplitude of the original vibrations, those of the two resolved vibrations will be equal respectively to $\sin i$ and $\cos i$. Now, the thickness of the plate being first supposed such as to render the interval of retardation an exact number of undulations, these rays will emerge from the lamina in exact accordance, and being parallel, the systems of waves of which they consist will run on together. Being polarized, however, in opposite planes they will neither destroy each other, nor produce a compound ray equal to their sum, but their resultant must be determined as in Art. 623. For we have here the case of rectilinear vibrations, in complete accordance, of given amplitudes, and making a given angle ($90°$,) so that the result there obtained is immediately applicable to this case, and the resultant vibration will be, first, rectilinear, so that the compound ray will appear wholly polarized in one plane: and, secondly, its amplitude will be, both in quantity and direction, the diagonal of a parallelogram whose sides are the amplitudes of the component vibrations. Consequently, it will be identical with that by whose resolution these were produced, and therefore the resultant, or emergent compound ray will be, in respect both of its polarization and intensity, precisely similar to the original incident one.

Case of complete accordance.

994. Fig. 197. Case of complete discordance.

When the difference of paths within the crystal is an exact odd multiple of half an undulation, the waves at their egress from the posterior surface will be in complete discordance. But their resultant may still be determined by the same rule, regarding either of the rays as negative, *i. e.* as having its vibrations executed in the opposite direction. For suppose the molecule C moving in the direction C P, with the velocity C P

Light. (fig. 197) at the entry of the ray, then the resolved velocities in the planes C O and C E will be represented in quantity and direction by C O and C E. But at their egress, the vibrations in the direction C E having gained or lost a half undulation on those in C O, if C O represent the quantity and direction of motion of the molecule C in that plane, C E′ equal and opposite to C E will represent its motion in the other plane, and this, combined with C O will compose, not the original motion C P, as in the former case, but C Q, making an equal angle with C O on the other side. The resultant ray, then, instead of being polarized in the plane of the incident one, (*i. e.* perpendicular to C P) will be polarized in a plane perpendicular to C Q, making an angle equal to P C Q (= 2 P C O = 2 *i*) with C O. Part IV.

995 When the difference of routes is neither an exact number of whole, or half undulations, the vibrations of the resultant ray (by Art. 621) will no longer be rectilinear, but elliptic; and in the particular case when the interval of retardation is a quarter or an odd number of quarter undulations, it will be circular. In this case, the emergent ray, varying its plane of vibration every instant, will appear wholly depolarized, so as to give two equal images by double refraction in all positions of the analysing prism.

996. Experiment exhibiting these several cases of interference

These several consequences may be rendered strikingly evident by a delicate and curious experiment related by M. Arago. Let a polarized pencil, emanating from a single radiant point, be incident on a double rhomboid of Iceland spar, composed of two halves of one and the same rhomboid, superposed so as to have their principal sections at right angles to each other. Then the emergent rays will emanate as if from two points (see Art. 879) near each other, and polarized in opposite planes. Let these two cones of rays be received on an emeried glass, or in the focus of an eye-lens, so that the glass or field of view shall be illuminated at once by the light of both, which being oppositely polarized will exhibit no fringes or coloured phenomena, but merely a uniform illumination; and let all the light but that which falls on a single very small point of the field of view be stopped by a plate of metal, with a small hole in it, so as to allow of examining the state of polarization of the compound ray illuminating this point, separately from all the rest. Then it will be seen, on analysing its light by a tourmaline or double refracting prism, that, when the spot examined is distant from both radiants by the same number of undulations, although in fact composed of two rays oppositely polarized, (as may be proved by stopping one of them, and examining the other singly,) yet it presents the phenomenon of a ray completely polarized in one plane, which is neither that of the one or the other of its component rays, but the original plane of polarization of the incident light. Suppose now, by a fine screw we shift gradually the place of the metal plate so as to bring the hole a little to one or the other side of its former place. The ray which illuminates it will appear to lose its polarized character as the motion of the plate proceeds, and at length will offer no trace of polarization; continuing the motion, and bringing in succession other points of the field of view under examination, the light which passes through the hole will again appear polarized, at first partially, and at length totally; not, however, as before, in the plane of primitive polarization, but in a plane making with it twice the angle included between it and the principal section of the first rhomboid, and so on alternately. Thus we are presented with the singular phenomenon of two rays polarized in planes at right angles, which produce by their concourse a ray either wholly polarized in one or the other of two planes, or not polarized at all, according to the difference of routes of the rays before their union.

997. Fresnel's general theory of double refraction.

In 1821, M. Fresnel presented to the Academy of Sciences of Paris a Memoir, containing the general application of the principle of transverse vibrations to the phenomena of double refraction and polarization as exhibited in biaxal crystals, which was read in November of that year. A brief report on the experimental parts of this Memoir by the Committee of the Academy appointed to examine it, about half a dozen pages, was published in the *Annales de Chimie*, vol. xx. p. 337, recommending it to be printed as speedily as possible in the collection of the *Mémoires des Savans Etrangers*. We are sorry to observe, that this recommendation has not yet been acted upon, and that this important Memoir, to the regret and disappointment of men of science throughout Europe, remains yet unpublished; though we trust (from the activity recently displayed by the Academy in the publication of their Memoirs in arrear) this will not long continue to be the case.* An abstract by the author himself, which appeared in the *Bulletin de la Société Philomatique* of 1822, and was subsequently reprinted in the *Annales de Chimie*, 1825, enables us, however, to present a sketch, though an imperfect one, of its contents, supplying to the best of our ability the demonstration of the fundamental propositions, and reaping a melancholy gratification from the inadequate tribute, which, in thus introducing for the first time to the English reader a knowledge of these profound and interesting researches, we are enabled to pay to departed merit. *His saltem accumulem donis—et fungar inani munere.* For even at the moment when we are recording his discoveries, their author has been snatched from science in the midst of his brilliant career by a premature death, like his hardly less illustrious contemporary, Fraunhofer, the early victim of a weakly constitution and emaciated frame, unfit receptacles for minds so powerful and active.

998. General expression for the elastic forces of a medium investigated.

M. Fresnel assumes, as a postulatum, that the displacement of a molecule of the vibrating medium in a crystallized body (whether that medium be the ether, or the crystal itself, or both together, in virtue of some mutual action exercised by them on each other,) is resisted by different elastic forces, according to the different directions in which the displacement takes place. Now it is easy to conceive, that in general the resultant of

* This delay has been productive of a singular consequence, which will suffice to show the small degree of publicity which labours, even the most important, can acquire by the circulation of such notices as those mentioned in the text. So lately as December 1826, the Imperial Academy of Sciences of Petersburg proposed as one of their prize questions for the two years 1827 and 1828, the following, "*To deliver the optical system of waves from all the objections which have (as it appears) with justice, been urged against it, and to apply it to the polarization and double refraction of light.*" In the programma announcing this prize, M. Fresnel's researches on the subject are not alluded to (though his *Memoir on Diffraction* is noticed,) and it is fair to conclude, were not then known to the Academy. Precisely one month before the publication of this programma, the Royal Society of London awarded their Rumford Medal to M. Fresnel, "for his application of the undulatory theory to the phenomena of polarized light, and for his important experimental researches and discoveries in physical optics." Our readers will be gratified to know, that the valuable mark of this high distinction reached him a few days before his death.

Light. Part IV.

all the molecular forces which act on a displaced molecule, is not necessarily parallel to the direction of its displacements when the partial forces are unsymmetrically related to this direction, but the proposition may be demonstrated *à priori*, as follows. Suppose three coordinates x, y, and z, to represent the partial displacements of any molecule M in their respective directions, and r $(= \sqrt{x^2 + y^2 + z^2})$ the total displacement, making angles α, β, γ, respectively with the axes of the x, y, z, so that $x = r \,.\, \cos \alpha$, $y = r \,.\, \cos \beta$, $z = r \,.\, \cos \gamma$. Now, since in this theory we assume that the displacements of the molecules are infinitely, or at least extremely small compared with the distances of the molecules *inter se*, it is evident that whatever be the law of molecular action, the force resulting from any displacement must (*cæteris paribus*) be proportional to the linear magnitude of that displacement, and can, therefore, be only of the form $r \,.\, \phi$, where ϕ is some unknown function of the angles α, β, γ, or their cosines. And, moreover, since such infinitely small displacements, in whatever direction made, neither alter the angular position, nor distance of the displaced molecule among the rest, by any sensible quantity, all their forces will act on it in its displaced position in the same manner as before. Hence the total force developed by the simultaneous displacements x, y, z, or by the single displacement r must be equivalent to (or the statical resultant of) the three which would be developed independently by the several partial displacements x, y, z. Now the force originating in the partial displacement x alone will result from $r\phi$ by making $r = x$ and ϕ equal to a, where a is the same function of 1, 0, 0, that ϕ is of $\cos \alpha$, $\cos \beta$, $\cos \gamma$. a therefore is a constant depending only on the position of the axes of the x, y, z with respect to the molecules of the crystal. And when this partial force $= a\,x$ is resolved into the directions of these several axes, since its direction (whatever it be) is determinate, the resolved portions can only be of the form $\mathrm{A}\,x$, $\mathrm{A}'\,x$, $\mathrm{A}''\,x$, where A, A′, A″ are in like manner dependent only on the position of the coordinates x, y, z with respect to the molecules, and not at all on α, β, γ, which are arbitrary, and where $\mathrm{A}^2 + \mathrm{A}'^2 + \mathrm{A}''^2 = a^2$. The same being true of the partial forces brought into play by the displacements y and z, it follows that the total force arising from the displacement r must be the resultant of the three forces

Principle of partial displacements.

$$f = \mathrm{A}\,x + \mathrm{B}\,y + \mathrm{C}\,z, \quad f' = \mathrm{A}'\,x + \mathrm{B}'\,y + \mathrm{C}'\,z, \quad f'' = \mathrm{A}''\,x + \mathrm{B}''\,y + \mathrm{C}''\,z,$$

respectively parallel to the axes of the x, y, z, where the coefficients are independent of α, β, γ, and where, in like manner, $\mathrm{B}^2 + \mathrm{B}'^2 + \mathrm{B}''^2 = b^2$, $\mathrm{C}^2 + \mathrm{C}'^2 + \mathrm{C}''^2 = c^2$. But we have $x = r \,.\, \cos \alpha$, $y = r \,.\, \cos \beta$, $z = r \,.\, \cos \gamma$, so that if we put

$$f = r \ \{ \mathrm{A} \,.\, \cos \alpha + \mathrm{B} \,.\, \cos \beta + \mathrm{C} \,.\, \cos \gamma \},$$
$$f' = r \ \{ \mathrm{A}' \,.\, \cos \alpha + \mathrm{B}' \,.\, \cos \beta + \mathrm{C}' \,.\, \cos \gamma \},$$
$$f'' = r \ \{ \mathrm{A}'' \,.\, \cos \alpha + \mathrm{B}'' \,.\, \cos \beta + \mathrm{C}'' \,.\, \cos \gamma \},$$

the resultant of f, f', f'' will be the force urging the displaced molecule.

999. Expression of the elasticity in any assigned direction. Acts obliquely to the direction of displacement.

Now these forces acting in the directions of the coordinates may each be decomposed into two, one in the direction of the displacement r, and the other at right angles to it in the planes respectively of r and x, r and y, r and z; the sum of the former will be

$$\mathrm{F} = f \,.\, \cos \alpha + f' \,.\, \cos \beta + f'' \,.\, \cos \gamma,$$

which is the whole force tending to urge the displaced molecule directly to its position of equilibrium. The latter will be respectively equal to $f \,.\, \sin \alpha$, $f' \,.\, \sin \beta$, and $f'' \,.\, \sin \gamma$; but as they act, although in one plane, yet not in the same direction, they will not destroy each other, unless they be to each other in the ratio of the sines of the angles they make with each other's direction. But it is evident, that since α, β, γ are arbitrary, this condition cannot hold good in general, because it furnishes two equations, which, taken in conjunction with the relation $\cos \alpha^2 + \cos \beta^2 + \cos \gamma^2 = 1$, suffice to determine α, β, γ. Hence it follows, that *the displaced molecule is, except in certain cases, urged by the elastic forces of the medium obliquely to the direction of its displacement.*

1000. Axes of elasticity defined and investigated

Mr. Fresnel next goes on to observe, that in general *every elastic medium has three rectangular axes, in the direction of which, if a molecule be displaced, the resultant of the molecular forces urging it will act in the direction of its displacement.* These are the excepted cases just alluded to, and to the axes possessing this property, (which he regards as the true fundamental axes of the crystal,) he gives the name of *Axes of Elasticity.*

To demonstrate this proposition we must observe, that, by mechanics, in order that the resultant of three rectangular forces f, f', f'' shall make angles α, β, γ with their three directions, and therefore be coincident in direction with r, they must be to each other in the ratio of the cosines of these angles, and therefore we must have the following equations expressive of this condition,

$$\frac{f}{f'} = \frac{\cos \alpha}{\cos \beta}; \quad \frac{f}{f''} = \frac{\cos \alpha}{\cos \gamma}; \quad \frac{f'}{f''} = \frac{\cos \beta}{\cos \gamma}.$$

These three equations are in general equivalent to two only, but when combined with the equation $\cos \alpha^2 + \cos \beta^2 + \cos \gamma^2 = 1$ resulting from the geometrical conditions of the case, they suffice to determine α, β, and γ; and if we put u, v, w for the cosines of these angles, furnish the following system of equations which every axis of elasticity must satisfy.

$$(\mathrm{A}\,u + \mathrm{B}\,v + \mathrm{C}\,w)\,v = (\mathrm{A}'\,u + \mathrm{B}'\,v + \mathrm{C}'\,w)\,u,$$
$$(\mathrm{A}\,u + \mathrm{B}\,v + \mathrm{C}\,w)\,w = (\mathrm{A}''\,u + \mathrm{B}''\,v + \mathrm{C}''\,w)\,u,$$
$$(\mathrm{A}'\,u + \mathrm{B}'\,v + \mathrm{C}'\,w)\,w = (\mathrm{A}''\,u + \mathrm{B}''\,v + \mathrm{C}''\,w)\,v,$$
$$u^2 + v^2 + w^2 = 1.$$

Light. Part IV.

Suppose by elimination we have derived from these equations the position of one axis of elasticity, then it will follow of necessity, that two others must exist, at right angles to it and to each other. To prove this, we must consider the connection between the partial forces developed by any displacement of the molecule M, and the molecular attractions and repulsions of the medium. Let ϕ be the action of any molecule dm on M, which we suppose to be exerted in the direction of their line of junction, and to be a function of their mutual distance ρ. Then, if we suppose M displaced by any arbitrary quantities δx, δy, δz (infinitely small in comparison with ρ) in the direction of the three coordinates, we have

1001. Three exist in any crystal at right angles to each other.

$$\delta\phi = \left(\frac{x}{\rho}\delta x + \frac{y}{\rho}\delta y + \frac{z}{\rho}\delta z\right) \cdot \frac{d\phi}{d\rho},$$

and putting $$\phi' = \frac{d\phi}{d\rho}, \quad \text{and } \frac{x}{\rho} = \cos\lambda, \quad \frac{y}{\rho} = \cos\mu, \quad \frac{z}{\rho} = \cos\nu,$$

we have $$\delta\phi = \phi' . \{ \delta x . \cos\lambda + \delta y . \cos\mu + \delta\rho . \cos\nu \}.$$

Consequently, since the force of the molecule dm, resolved into the directions of the coordinates, is respectively equal to

$$(\phi + \delta\phi)\, dm . \frac{x}{\rho}, \quad (\phi + \delta\phi)\, dm . \frac{y}{\rho}, \quad \text{and } (\phi + \delta\phi)\, dm . \frac{z}{\rho},$$

the sum of all these throughout the medium will be the total action on M; but since in the original position of the molecule M it is in equilibrio, we have

$$\int \phi\, dm . \frac{x}{\rho} = 0, \quad \int \phi\, dm . \frac{y}{\rho} = 0, \text{ and } \int \phi\, dm . \frac{z}{\rho} = 0,$$

so that the whole action of the medium on M in its displaced situation will be, in the three directions

General relation between the partial elasticities.

$$\int \frac{x}{\rho} dm . \delta\phi, \qquad \int \frac{y}{\rho} dm . \delta\phi, \qquad \int \frac{z}{\rho} dm . \delta\phi;$$

that is, in the direction of the x,

$$\int \phi' dm . \{ \cos\lambda^2 \delta x + \cos\mu^2 . \delta y + \cos\nu^2 . \delta z \};$$

δx, δy, δx, are the partial displacements of M in the directions of the coordinates, and are, therefore, the same we denoted in Art. 998 by x, y, z. Restoring these denominations, we see that, on this hypothesis, (the most natural which can be formed respecting the mode of molecular action) the coefficients A, B, C, can be no other than the following,

$$A = \int \phi' dm . \cos\lambda^2, \quad B = \int \phi' dm . \cos\lambda . \cos\mu, \quad C = \int \phi' dm . \cos\lambda . \cos\nu;$$

and by similar reasoning we find

$$A' = \int \phi' dm . \cos\lambda . \cos\mu, \quad B' = \int \phi' dm . \cos\mu^2, \quad C' = \int \phi' dm . \cos\mu . \cos\nu;$$
$$A'' = \int \phi' dm . \cos\lambda . \cos\nu, \quad B'' = \int \phi' dm . \cos\mu . \cos\nu, \quad C'' = \int \phi' dm . \cos\nu^2;$$

and, consequently, the following relations must necessarily subsist between these coefficients

$$B = A', \quad C = A'', \quad C' = B''.$$

This premised, suppose we have determined one axis of elasticity of the medium by the foregoing equations. Since the positions of the axes of the coordinates are arbitrary, we are at liberty to suppose that of the x coincident with the axis so determined, which renders $A' = A'' = 0$, and consequently $B = 0$ and $C = 0$, and $B'' = C'$, because the relations above demonstrated are general and independent of any particular situation of the axes. The equations of Art. 1000 then become

1002. One axis being given the situation of the other determined.

$$A u v = (B' v + C' w) u, \quad A u w = (B'' v + C'' w) u,$$
$$(B' v + C' w) w = (C' v + C'' w) v, \quad u^2 + v^2 + w^2 = 1.$$

Now if we put $u = 0$, or $a = 90°$, the two former of these are satisfied without any relation supposed between v and w, so that if we determine these from the two latter only, the whole system will be satisfied. These (making $u = 0$) give at once by elimination

$$w = \sqrt{\frac{1}{2}\left(1 \pm \frac{1}{\sqrt{4m^2+1}}\right)}; \quad v = -\sqrt{\frac{1}{2}\left(1 \mp \frac{1}{\sqrt{4m^2+1}}\right)},$$

where $m = \frac{C'}{B' - C''}$. Now since m^2 is necessarily positive, $4m^2 + 1$ is so, and is > 1; therefore $\frac{1}{\sqrt{4m^2+1}}$ is real and < 1, consequently w^2 and v^2 are both positive, and therefore v and w both real, and less than unity. Hence it follows, that there are necessarily two axes at right angles to the x which satisfy the conditions of axes of elasticity, and the opposite signs of v and w show that they are at right angles to each other.

For simplicity, therefore, we will in future suppose the directions of the coordinates to be coincident with those of the axes of elasticity, so as to make 1003.

$$A = a,\ A' = A'' = 0;\quad B' = b,\ B = B'' = 0;\quad C'' = c,\ C = C' = 0;$$

then we have by Art. 998 for the partial forces,

$$f = a\,x = a\,r\,.\cos\alpha,\quad f' = b\,y = b\,r\,.\cos\beta,\quad f'' = c\,z = c\,r\,.\cos\gamma,$$

and by 999,

$$F = r\,\{\,a\,.\cos\alpha^2 + b\,.\cos\beta^2 + c\,.\cos\gamma^2\,\}$$

for the whole force urging the molecule M in the direction of the r, generally assumed, in which it will be observed that

$$a = \int \phi'\,.\cos\lambda^2\,dm,\quad b = \int \phi'\,.\cos\mu^2\,dm,\quad c = \int \phi'\,.\cos\nu^2\,dm.$$

1004. The *surface of elasticity* defined and investigated

M. Fresnel next conceives a surface, which he terms the "Surface of Elasticity," constructed according to the following law:—on each of the *axes of elasticity*, and on every radius r drawn in all directions, take a length proportional to the square root of the elasticity exerted on the displaced molecule by the medium in the direction of the radius, or to $\sqrt{F}$. Then if we call R this length, or the radius vector of the surface of elasticity, we shall have

$$R^2 = \{\,a\,r\,.\cos\alpha^2 + b\,r\,.\cos\beta^2 + c\,r\,.\cos\gamma^2\,\} \times \text{const.}$$

Its radius vector expressed.

The values of R parallel to the axes are then had by the equation

$$R^2 = \text{const}\ a\,r,\quad R^2 = \text{const} \times b\,r,\quad R^2 = \text{const} \times c\,r$$

which (for brevity, as we shall have no further occasion to recur to our former denominations) we shall express simply by a^2, b^2, c^2, so that the equation of the surface of elasticity will be of the form

$$R^2 = a^2\,.\cos X^2 + b^2\,.\cos Y^2 + c^2\,.\cos Z^2,$$

where X, Y, Z, now stand for α, β, γ, the angles made by R with the axes of the coordinates.

1005. Velocity and plane of polarization of an interior wave determined.

Let us now imagine a molecule displaced and allowed to vibrate in the direction of the radius R, and retained in that line, or at least let us neglect all that part of its motion which takes place at right angles to the radius vector. Then the force of elasticity by which its vibrations are governed will be proportional to R^2, and the velocity of the luminous wave propagated by means of them, in a direction transverse to them (or at right angles to R) will be proportional to R, so that the surface of elasticity being known, the velocity of a wave transmitted through the medium in a given direction, and with a given plane of polarization will be had at once as follows. Parallel to the surface of the wave, and at right angles to its plane of polarization draw a straight line. This will be the direction of the vibrations by which the wave is propagated. Parallel to this line draw a radius vector to the surface of elasticity, and it will represent the wave's velocity.

1006. Equation of the surface of elasticity.

The equation of the surface of elasticity, if we put for R, cos X, cos Y, cos Z, their values in terms of three coordinates will become

$$(x^2 + y^2 + z^2)^2 = a^2\,x^2 + b^2\,y^2 + c^2\,z^2.$$

It is, therefore, in general a surface of the fourth order. If we suppose it cut by a plane passing through its centre, whose equation must therefore in general be of the form $m\,x + n\,y + p\,z = 0$, the curve of intersection will be a species of oval whose diameters are not necessarily all equal.

1007. Resolution of an incident wave into two. Polarized in opposite planes.

Suppose now any molecule set in vibration in this plane, then at any period of its motion it will not be urged directly to its point of rest but obliquely, so that it will not describe a straight line, but will circulate in a curve more or less complicated; its motion in this, however, will always be resolvable into two vibratory rectilinear ones at right angles to each other, one parallel to the greatest, and the other to the least diameter of the section. Each of these vibratory motions will, by the laws of motion, be performed independently of the other, and therefore the motion propagated through the crystal will affect every molecule of it in the same way as if two separate and independent rectilinear vibrations (at right angles as above) were propagated through it, with different velocities. Consequently every system of waves propagated from without into the crystal, will necessarily on entering it be resolved into two propagated with different velocities, and polarized in planes at right angles to each other, *viz.* those parallel respectively to the greatest and least diameter of a section of the surface of elasticity parallel to the plane of either wave. And as every difference in the velocities of two waves propagated parallel to each other through a medium, gives rise to a corresponding difference in their planes at their emergence from it into another, where they assume a common velocity; these waves will at their egress no longer be parallel, and the rays which are perpendicular to them will be inclined to each other, thus producing the phenomena of double refraction; and it is evident that the waves at their egress must retain the planes of polarization they received in the crystal, because any molecule of the exterior medium at the junction of the media will begin to move only in the plane in which it was displaced by the contiguous molecule in the medium.

1008. This theory then accounts perfectly both for the bifurcation of the emergent ray, and the opposite polarizations of the two portions into which it is divided. These portions will coincide in direction, and there will be no double refraction when the section of the surface of elasticity above mentioned is (if such can ever be the case) a circle, because all its radii being then equal, the elasticity is the same in all directions, and all vibrations performed in it will have equal periods, so that in this case the resolution of the incident wave into two no longer takes place, nor is its plane of polarization changed. Now the section in question becomes a circle, when $x^2 + y^2 + z^2 = \text{const} = r^2$, or when $a^2\,x^2 + b^2\,y^2 + c^2\,z^2 = r^4$. Combining these with $m\,x + n\,y + p\,z = 0$, we get

$$r^4 = r^2 (x^2 + y^2 + z^2),$$

$$p^2 r^4 = r^2 (p^2 x^2 + p^2 y^2 + (m x + n y)^2),$$

and

$$p^2 r^4 = p^2 a^2 x^2 + p^2 b^2 y^2 + c^2 (m x + n y)^2,$$

and equating these, and considering that the equation thence resulting ought to be verified independently of any particular values of x, y, we get

Investigation of the optic axes.

$$r^2 (m^2 + p^2) = a^2 p^2 + m^2 c^2,$$

$$m n r^2 = m n c^2,$$

$$r^2 (p^2 + n^2) = b^2 p^2 + n^2 c^2.$$

These equations cannot be satisfied except by supposing either m, n, or p to vanish, or the section in question to pass through one or other of the axes. If we suppose $m = 0$, we have $r = a$, $\left(\frac{n}{p}\right)^2 = \frac{a^2 - b^2}{c^2 - a^2}$, which shows that $\left(\frac{n}{p}\right)^2$ cannot be positive, and of course $\frac{n}{p}$ not real, unless a, the semiaxis of the surface through which the section passes, be that intermediate in length between b and c, the other two semiaxes.

It appears then, that the surface of elasticity admits of two circular sections and no more, formed by diametral planes passing through the mean axis of the surface, and (since $\frac{n}{p}$ has two values equal but of opposite signs) that these sections are both equally inclined to each of the other two axes. The normals to these sections are the directions of no double refraction, or the *optic axes* of the crystal. Of these, then, there will be two and two only, in all crystals which possess three unequal axes of elasticity, and rays propagated along them will suffer neither double refraction, nor change of polarization. 1009

1010. Dispersion of the axes of different colours explained.

The position of these axes depends wholly on the values of a, b, c, the semiaxes of the surface of elasticity. We have, however, no other measure of the elasticity of the medium than the velocity with which the rays are propagated through it; and if, as the phenomena of ordinary dispersion indicate, the rays of different colours be propagated in one and the same medium with velocities somewhat different, (an effect which might result from certain suppositions relative to the extent of the sphere of action of its molecules compared with the lengths of an undulation,) the semiaxes a, b, c, which must be taken proportional to the velocities of propagation, must be supposed to vary a little for waves of different lengths. Now this variation may not be in the same ratio for all the three semiaxes, and thus a variation in the values of $\frac{n}{p}$ will arise. But $\frac{n}{p}$ is the tangent of the inclination of the plane of section to the plane of the $x\,y$, or of half the angle the two circular sections make with each other, *i. e.* the cotangent of half the angle between the optic axes, which will thus vary, and give rise to that separation of axes of different colours, and their distribution over a certain angle, *in the plane containing any two* of the same colour, which observation shows to exist, (Art. 921 and 922.)

1011. Application of the Huygenian construction to the general case.

The general laws of double refraction flow with great facility from these principles. We have only to resume the construction and reasoning of Art. 806 and 807, *et seq.*, substituting for the ellipsoid of revolution, which the Huygenian theory assumes as the figure of a wave originating in any molecule of the crystal, the surface, whatever it be, which, in the general case, terminates a wave so propagated, and investigating the point of contact I (fig. 170) of this surface with a plane I K T passing through the line K T drawn as there described. There is this difference, however, in the two cases, or, at least, in the method of treating them, that in the theory there stated the form of the wave is made a matter of arbitrary assumption, in the present case it is to be determined *à priori*. This will render it necessary to depart in some respects from the course before adopted. If we know, *à priori*, the form of the wave, the position of the tangent plane is given; *vice versâ*, if we can determine the position of this plane in all cases, *à priori*, the figure of the wave, which must be such as to touch all such planes, under the conditions of the case, becomes known.

1012 Direction and velocity of a plane wave.

Now, in Art. 807, it is shown that the tangent plane is in all cases coincident with the position assumed within the crystal, by the surface of a plane indefinite wave propagated from an infinitely distant luminary, perpendicular to the line of incidence R C. It follows, moreover, from Art. 811, that if we know the velocity with which such a plane wave advances within the crystal in a direction perpendicular to its surface, we may calculate its inclination to the surface of incidence by the law of ordinary refraction, assuming an index of refraction which is to that of the ambient medium as the velocity of the wave before incidence is to its velocity within the medium perpendicular to its own surface. The reader will here keep in view the distinction noticed in Art. 813 between the velocity of the *wave* and that of the *ray* conveyed by it, whose direction, generally speaking, is oblique to its surface. Now the velocity of a wave within the medium in any direction is given by the equation of the surface of elasticity, whose radius vector expresses it in all cases. But it has been shown, that every vibration impressed on the molecules of the crystal is resolved into two rectilinear ones propagated with velocities proportional to the greatest and least diameters of that section of the surface of elasticity which is parallel to the plane in which they are performed. Now it is the same thing, (as far as the law of double refraction is concerned,) whether we regard the bifurcation to take place by the separation of a single exterior ray into two interior ones, or a single interior into two exterior. We will take the latter case, and suppose the

Light. Velocities of an ordinary and extraordinary plane wave investigated.

ordinary and extraordinary plane waves to be parallel *within* the medium. Their velocities may then be investigated as follows: the equation of the surface of elasticity being Part IV.

$$R^4 = a^2 x^2 + b^2 y^2 + c^2 z^2,$$

if we take, for the equation of the ~~second~~ secant plane,

$$z = m x + n y,$$

and put V for the maximum or minimum radius vector of the surface in the section in question, V will be the value of R, which makes $d R = 0$, and therefore will be given by elimination from the following system of equations

$$V^2 = x^2 + y^2 + z^2,$$
$$V^4 = a^2 x^2 + b^2 y^2 + c^2 z^2,$$
$$z = m x + n y,$$

and their differentials, regarding V as constant. This elimination, which is complicated enough, must be conducted as follows: first, if among the differential equations we eliminate $d x$, $d y$, $d z$; and for z in the whole system substitute its value, we shall get, putting $p = a^2 - b^2$; $q = a^2 - c^2$; $r = b^2 - c^2$;

$$V^4 = (a^2 + m^2 c^2) x^2 + (b^2 + n^2 c^2) y^2 + 2 m n c^2 x y,$$
$$V^2 = (1 + m^2) x^2 + (1 + n^2) y^2 + 2 m n x y,$$
$$0 = m n q x^2 - m n r y^2 + k x y,$$

where

$$k = p + n^2 q - m^2 r = (1 + n^2) q - (1 + m^2) r.$$

These, by elimination, give the following, in which

$$M = k^2 + 4 m^2 n^2 q r;$$
$$M x^2 = V^2 (V^2 - c^2) \{ (1 + n^2) k + 2 m^2 n^2 r \} - r k V^2,$$
$$M y^2 = - V^2 (V^2 - c^2) \{ (1 + m^2) k - 2 m^2 n^2 q \} + r q V^2,$$
$$M x y = - m n \{ (1 + n^2) q + (1 + m^2) r \} V^2 (V^2 - c^2) + 2 m n q r V^2;$$

and by equating the square of the last of these to the product of the two first, we find, after all reductions, the following equation for determining V:

$$(V^2 - a^2) (V^2 - b^2) + m^2 (V^2 - b^2) (V^2 - c^2) + n^2 (V^2 - a^2) (V^2 - c^2) = 0.$$

1013. General equation of a wave propagated from a point in the medium.

The roots of this equation determine the maximum and minimum values of the radius vector in the plane of section, and therefore the velocities of ordinary and extraordinary plane waves moving parallel to each other within the crystal; and these found, the figure of the wave becomes known, from the condition that its surface must always be a tangent to a plane distant by the quantity V from the secant plane whose equation is $z = m x + n y$; and that, whatever be the values of m and n. Its investigation is therefore reduced to a purely geometrical problem. Required the equation of a curve surface, which shall touch every plane parallel to a plane whose equation is $z = m x + n y$; and distant from it by a quantity V, a function of m and n given by the above equation, which, being resolved, will be found to lead to the following equation

$$\left.\begin{aligned}(a^2 x^2 + b^2 y^2 + c^2 z^2) (x^2 + y^2 + z^2) - a^2 (b^2 + c^2) x^2 - b^2 (a^2 + c^2) y^2 \\ - c^2 (a^2 + b^2) z^2 + a^2 b^2 c^2\end{aligned}\right\} = 0.$$

1014. Nonexistence of the Cartesian law of refraction in biaxal crystals.

The surface represented by this equation is, generally speaking, of the fourth order, and consists of two distinct surfaces, or sheets, (*nappes.*) One of these, by its contact with the plane in question, determines the direction of the ordinary, and the other of the extraordinary ray. Now, it is important to remark, that this equation, so long as particular values are not assigned to a, b, c, is not decomposable into quadratic factors, so that neither of the sheets of which it consists is spherical, or ellipsoidal; and, consequently, neither the ordinary nor the extraordinary ray follows either the Cartesian or Huygenian law of refraction. This is a consequence too remarkable not to have been put to the test of experiment. Two methods have been put in practice by M. Fresnel for this purpose. The first consisted in measuring directly the velocities of the two rays in plates of topaz cut in different directions with respect to their axes by the method explained under the head of interferences, (Art. 738 and 739.) Since a difference of velocity of the interfering rays displaces the diffracted fringes as a difference of thickness would do, it is manifest that if, in two plates differently cut, but of precisely the same thickness, the fringes formed by the ordinary rays are differently displaced when the plates are combined successively with one and the same equivalent plate of glass, or any other standard medium, their velocity cannot be the same in both plates; and if such difference be observed to take place, both in the fringes formed by the interference of the ordinary and of the extraordinary rays severally, with a compensated pencil, it is clear that neither can have a constant velocity. Now the condition of equal thickness is secured by cementing the two plates edge to edge, and grinding and polishing them together, and carefully examining the surfaces after the operation, to be satisfied of their precise continuity, which may be done by the reflected image of a distant object, and yet more delicately by pressing slightly on them a convex lens of long focus, over their line of junction. If the coloured rings formed between the surfaces be uninterrupted, we are sure that this condition

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is rigorously satisfied. The experiment so made, M. Fresnel found to confirm the conclusion to which the above theory leads. But in corroboration of this important result, the following method was also used.

1015. Another experiment to prove the same.

In topaz the extraordinary refraction is stronger than the ordinary; so that the ordinary ray, when the two are separated by a prism of that medium, may be at once recognised, by being the least deviated. M. Fresnel procured two prisms to be cut from one topaz, in both of which the base was parallel to the cleavage planes, and therefore perpendicular to a line bisecting the angle between the optic axes and to the principal section of the crystal, *i. e.* to the mean axis of elasticity; but in one the plane of the refracting angle was coincident with, and in the other perpendicular to, that section, these being the planes in which the difference between the velocities of the ordinary ray is the greatest, as is easily seen from what has above been said. These prisms were cemented side by side, so as to have their bases in one plane and their refracting edges in one straight line; and were then very carefully ground and polished to plane surfaces, so that the refracting angles in both could not be otherwise than precisely equal. In this situation the compound prism A B C, fig. 199, 1, (which is seen in perspective in fig. 199, 2,) whose refracting angle A B C was about 92°, was achromatised by two prisms E B A and D C A of crown glass, in which circumstances a slight, uncompensated refraction remained in favour of the topaz prism. Looking now through the side E B, the whole combination was turned round the refracting edge as an axis, till the image of a distant object, a black line on a white ground, appeared stationary; so that the refracted rays, both ordinary and extraordinary, must have traversed the prisms very nearly parallel to the base, or at right angles to the mean axis, but in the different planes above mentioned in each. Now it was observed, that the least refracted image of the black line so seen, that is the ordinary one, was broken at the junction of the two prisms, being more deviated by one than by the other, while the most refracted or extraordinary image formed a continuous line in both. This latter fact (which, at first sight, would lead us to suspect that the extraordinary image had been mistaken for the ordinary one) is a consequence of the theory above explained, and is an additional confirmation of it.

1016. Case of uniaxal crystals.

When two of the axes of elasticity (as b and c, for instance) are equal, the general equation of the surface of the wave becomes decomposable into two factors, and may be put under the form

$$(x^2 + y^2 + z^2 - b^2)\{a^2 x^2 + b^2 . (y^2 + z^2) - a^2 b^2\} = 0,$$

which is the product of the equation of a sphere with that of an ellipsoid of revolution. In this case the two circular sections coincide with the plane of the $y z$, and the two optic axes with the axis of the x. We have here then the case of uniaxal crystals, and are thus furnished with an *à priori* demonstration, both of the Huygenian law of elliptic undulations, in the case of the extraordinary wave in such crystals, and of the constancy of the index of refraction in that of the ordinary. The manner in which this results as a corollary from the general case is at once elegant and satisfactory.

1017. Construction of the wave by an ellipsoid.

M. Fresnel gives the following simple construction for the curve surface bounding the wave in the case of unequal axes, which establishes an immediate relation between the length and direction of its radii. Conceive an ellipsoid having the same semiaxes a, b, c; and having cut it by any diametral plane, draw perpendicular to this plane from the centre two lines, one equal to the greatest, and the other to the least, radius vector of the section. The loci of the extremities of these perpendiculars will be the surfaces of the ordinary and extraordinary waves; or, in other words, their lengths will be the lengths of the radii of the waves in those directions, and will therefore measure the velocity of the two *rays* propagated in those directions, in the same way as the radii of the Huygenian ellipsoid are proportional to the velocities of the extraordinary ray in their direction.

1018. Origin of the rule of the product of the two sines.

Finally, if we divide unity by the squares of the two semiaxes of a diametral section of the ellipsoid, the difference of these quotients will be found to be proportional to the product of the sines of the angles which the perpendicular to this section makes with the two normals to the planes of the circular sections of the ellipsoid. Now, in all the crystals hitherto known, these sections differ very little from the circular sections of the surface of elasticity, and may, without sensible error, be supposed to coincide with them; consequently, the two normals in question may be taken for this purpose as the optic axes of the crystal. We have thus the origin of that law, deduced from the phenomena of the coloured lemniscates, which makes the difference of the squares of the reciprocal velocities proportional to the product of the sines made by the ray with the optic axes; and thus the phenomena of the polarized rings are all made to depend on the same general principles.

1019.

Such is the beautiful theory of Fresnel and Young, (for we must not in our regard for one great name forget the justice due to the other, and to separate them and assign to each his share would be as impracticable as invidious, so intimately are they blended throughout every part of the system; early, acute, and pregnant suggestion characterising the one,—and maturity of thought, fulness of systematic developement, and decisive experimental illustration, equally distinguishing the other. If the deduction in succession of phenomena of the greatest variety and complication from a distinctly stated hypothesis, by strict geometrical reasoning, through a series of intermediate steps, in which the powers of analysis alone are relied on, and whose length and complexity is such as to prevent all possibility of foreseeing the conclusions from the premises, be a characteristic of the truth of the hypothesis,—it cannot be denied that it possesses that character in no ordinary degree; but, however that may be, as a generalization the reader will now be enabled to judge whether the encomium we passed on it in a former Article be merited. We can only regret that the necessary limits of this Essay, which is already extended greatly beyond our original design, forbid our entering farther into its details.

1020. Dr. Brewster's theory of polarizing axes.

The *axes of elasticity* are those which M. Fresnel regards as the fundamental axes of a doubly refractive medium. The *optic axes* can in no view of the subject be regarded as such, for several obvious reasons. First, they are seldom symmetrically situated relative to fundamental lines in the crystalline form; secondly, because they vary in position according to the colour of the incident light; thirdly, because it is found that for one and the same coloured illumination, and in the same crystal, their situation varies by a variation of temperature.

Light. This important fact has been lately ascertained by M. Mitscherlich, and we shall presently have occasion to speak further of it. From all these reasons it follows, that we can regard them only as resultant lines, to which no *à priori* properties can be supposed to belong, but which simply satisfy the condition $v - v' = 0$, according to the laws which regulate the constitutions of the functions v, v', the velocities of the two rays, in terms of those quantities which we may regard as fundamental data, and the situation of the ray within the medium. The axes of elasticity themselves may, perhaps, be regarded as mere resultants from the equations of Art. 1000, and determined from other remoter data dependent on the fundamental lines in the crystalline form, and the intensity and distribution of the molecular forces within it. Accordingly, Dr. Brewster considers the optic axes as the resultants of others which he terms *polarizing axes*, and from which he conceives to *emanate* polarizing *forces* producing the phenomena of the rings and of the double refraction and polarization observed. We shall not here stop to examine into the propriety of these terms. The reader who may have doubts on the subject will, in what follows, mentally substitute other and more general phrases in their place expressive of relation and causality, while we proceed to state the assumptions with which he sets out, and the conclusions he very ingeniously deduces from them. Part IV.

1021. A single polarizing axis. *Postulate* 1. A polarizing axis, when single, has the characters of an axis of no double refraction, and is coincident with the axis of the Huygenian spheroid in such crystals as have but one. A positive axis acts as the axis in quartz, &c. may be supposed to do, and a negative, as that of carbonate of lime, &c.

1022. Polarizing force. *Post.* 2. The polarizing force of a single axis in any medium is proportional to, and measured by, the tint developed in the ordinary and extraordinary pencils into which a doubly refracting prism analyzes a polarized ray, which has traversed a given thickness of the medium.

1023. *Corol.* 1. The polarizing force of a single axis in the same medium is as the square of the sine of the angle made by the ray traversing it internally, with the axis.

1024. *Corol.* 2. The same force is also inversely as the thickness necessary to be traversed at a given angle to develope the same or equal tints. This may be regarded as the *intrinsic* polarizing force or *intensity* of the axis.

1025. Composition of tints in the case of two axes. *Post.* 3. When two axes exist in one medium and operate together, they polarize a tint whose measure (see Art. 906) is the diagonal of a parallelogram whose sides measure, on the same scale, the tints which would be polarized by either, separately, and include between them an angle *double* of the mutual inclination of two planes passing through the ray and either axis respectively.

1026. Formula for the compound tint. *Corol.* 1. If t and t' be the numerical measures of the tints polarized by either of two axes separately, T that polarized by their joint action, and C the angle between the planes just described, the tint T will be given by the equation

$$T^2 = t^2 + 2\,t\,t' \,.\, \cos 2\,C + t'^2.$$

1027. *Corol.* 2. If a and b represent the intensities of the axes, and α and β the angles which the ray makes with each respectively, we have $t = a \,.\, \sin \alpha^2$; $t' = b \,.\, \sin \beta^2$, and

$$T^2 = (a \,.\, \sin \alpha^2)^2 + (b \,.\, \sin \beta^2)^2 + 2\,a\,b \,.\, \sin \alpha^2 \,.\, \sin \beta^2 \,.\, (1 - 2 \,.\, \sin C^2),$$
$$= \{ a \,.\, \sin \alpha^2 + b \,.\, \sin \beta^2 \}^2 - 4\,a\,b \,.\, \sin \alpha^2 \,.\, \sin \beta^2 \,.\, \sin C^2,$$

or else

$$T^2 = \{ a \,.\, \sin \alpha^2 - b \,.\, \sin \beta^2 \}^2 + 4\,a\,b \,.\, \sin \alpha^2 \,.\, \sin \beta^2 \,.\, \cos C^2.$$

1028. If γ be the angle contained between the polarizing axes, since α, β, γ are the sides of a spherical triangle, and C the angle included between the sides α and β, or opposite to γ, we have

$$\cos C = \frac{\cos \alpha \,.\, \cos \beta - \cos \gamma}{\sin \alpha \,.\, \sin \beta},$$

and if this be written for cos C in the latter of the expressions above given for T^2, we find on reduction

$$T^2 = \{ a \,.\, \sin \alpha^2 + b \,.\, \sin \beta^2 \}^2 - 4\,a\,b \{ 1 - \cos \alpha^2 - \cos \beta^2 - \cos \gamma^2 + 2 \,.\, \cos \alpha \,.\, \cos \beta \,.\, \cos \gamma \}.$$

1029. *Corol.* If the polarizing axes be at right angles to each other, $\gamma = 90°$ and $\cos \gamma = 0$, and the expression for the compound tint becomes $T^2 = \{ a \,.\, \sin \alpha^2 + b \,.\, \sin \beta^2 \}^2 - 4\,a\,b\,(\sin \alpha^2 - \cos \beta^2)$.

1030. *Proposition. Two rectangular polarizing axes, either both positive or both negative, being given, two other axes, or fixed lines, may be found, such that calling θ and θ' the angles made with them respectively by a ray traversing a spherical portion of the medium, the tint polarized shall be proportional to* $\sin \theta \,.\, \sin \theta'$.*

Resultant axes arising from the joint action of two rectangular polarizing axes. Fig. 199. Let A C and B C (fig. 199) be the two polarizing axes including a right angle, of which let B C be the more powerful. Let O C be a ray penetrating the crystal in that direction; and in a plane P C Q perpendicular to A C B, draw any two lines P C, Q C, making equal angles with B C, either of which we will represent by x. Then if a sphere about C as a centre be conceived, it will intersect the planes A C B, P C Q, O C A, O C B, O C P, O C Q in lines of great circles B A, P B Q, O A, O B, O P, O Q, and we shall have P B = Q B = x, O A = α, O B = β, O P = θ, O Q = θ'; and by Spherical Trigonometry, from the triangle O B P, we have

$$\cos O B P \left(= \sin O B A = \sin A O B \,.\, \frac{\sin O A}{\sin A B} = \sin \alpha \,.\, \sin C; \text{ since } A B = 90° \right)$$
$$= \frac{\cos \beta \,.\, \cos x - \cos \theta}{\sin \beta \,.\, \sin x},$$

* M. Biot appears to have first noticed the fact announced in this proposition, *viz.* that Dr. Brewster's hypothesis of polarizing axes leads to a result *mathematically* identical with his own elegant law of the product of the sines. He has, however, suppressed his demonstration. Dr. Brewster's verification of this coincidence of results seems to have been founded on a numerical comparison of Biot's experiments on sulphate of lime with his own theory.

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and therefore $$-\cos\theta = \sin\alpha \,.\, \sin\beta \,.\, \sin x \,.\, \sin C - \cos\beta \,.\, \cos x,$$

and similarly from the triangle O B Q, since O B Q $= 90° +$ O B A, we obtain a second relation

$$+\cos\theta' = \sin\alpha \,.\, \sin\beta \,.\, \sin x \,.\, \sin C + \cos\beta \,.\, \cos x;$$

and, adding and subtracting, (putting, for brevity's sake, $\cos\theta' = p$, $\cos\theta = q$,)

$$p + q = 2 \,.\, \cos\beta \,.\, \cos x; \qquad p - q = 2 \,.\, \sin\alpha \,.\, \sin\beta \,.\, \sin x \,.\, \sin C.$$

These equations express the geometrical relations subsisting between the lines P C, Q C, and the axes A C, B C; and, if combined with the equations of Art. 1028 and 1029, suffice to eliminate α, β, and C, and to express T in terms of x, θ, and θ' alone. To execute this, we have by the equations just demonstrated

$$\left(\frac{p+q}{2 \,.\, \cos x}\right)^2 = \cos\beta^2; \qquad \left(\frac{p-q}{2 \,.\, \sin x}\right)^2 = \sin\alpha^2 \,.\, \sin\beta^2 \,.\, \sin C^2;$$

and in the latter, putting $1 - \cos C^2$ for $\sin C^2$, and for $\cos C^2$ its value given by Art. 1028, which, since $\gamma = 90°$, becomes simply

$$\sin\alpha^2 \,.\, \sin\beta^2 \,.\, \cos C^2 = \cos\alpha^2 \,.\, \cos\beta^2,$$

we have

$$\left(\frac{p-q}{2 \,.\, \sin x}\right)^2 = \sin\alpha^2 \,.\, \sin\beta^2 - \cos\alpha^2 \,.\, \cos\beta^2,$$

$$= \sin\alpha^2 - \cos\beta^2.$$

Hence we get, for the values of $\sin\alpha^2$ and $\sin\beta^2$,

$$\sin\alpha^2 = \left(\frac{p+q}{2 \,.\, \cos x}\right)^2 + \left(\frac{p-q}{2 \,.\, \sin x}\right)^2,$$

$$\sin\beta^2 = 1 - \left(\frac{p+q}{2 \,.\, \cos x}\right)^2;$$

and, substituting these in the equation of Art. 1029,

$$T^2 = \left\{ b + \frac{a-b}{4 \,.\, \cos x^2}(p+q)^2 + \frac{a}{4 \,.\, \sin x^2}(p-q)^2 \right\}^2 - \frac{a\,b}{\sin x^2}(p-q)^2.$$

Such is the general form of the expression for the tint, when referred to arbitrary axes in the manner here supposed, and it is complicated enough; but if we fix the position of the new axes so as to make $\sin x^2 = \frac{a}{b}$ the complication disappears; we have then $\frac{a}{4 \,.\, \sin x^2} = \frac{b}{4}$, and $\frac{a-b}{4 \,.\, \cos x^2} = \frac{b}{4}$, so that the value of T^2 reduces itself to

$$T^2 = b^2 \left\{ \left(1 - \left(\frac{p+q}{2}\right)^2 + \left(\frac{p-q}{2}\right)^2\right)^2 - (p-q)^2 \right\},$$

$$= b^2 \{ (1 - pq)^2 - (p-q)^2 \} = b^2 \{ 1 - p^2 - q^2 + p^2 q^2 \},$$

$$= b^2 (1 - p^2)(1 - q^2) = b^2 \,.\, \sin\theta^2 \,.\, \sin\theta'^2,$$

restoring the values of p and q, or $\cos\theta'$ and $\cos\theta$, consequently

$$T = -b \,.\, \sin\theta \,.\, \sin\theta'.$$

The negative sign is prefixed for the reason stated further on in Art. 1034.

1031. Thus we see that the combined action of the two axes in the manner here supposed, on Dr. Brewster's principles, will give rise to a series of isochromatic lines arranged in the form of sphero-lemniscates about two poles P, Q, determined by the condition

$$\sin B\,P = \sin B\,Q = \sqrt{\frac{\text{intensity of the feebler axis}}{\text{intensity of the stronger}}};$$

and the lines C P, C Q so determined have therefore the character of the optic axes in biaxal crystals, and may be designated with Dr. Brewster by the name of resultant axes. We must be careful, however, not to confound a *resultant* with a *polarizing* axis in this theory.

1032. Combination of a positive with a negative axis.

If the polarizing axes be not of the same denomination, as if one be positive and the other negative, the value of sin B P becomes imaginary, and the tints cannot be so arranged. But if we suppose the new axes to lie in this case in the *same* plane with the polarizing ones, as in fig. 200, all other things remaining, we have here

$$\cos O\,B\,A = +\cos O\,B\,Q, \qquad \text{and } \cos O\,B\,A = -\cos O\,B\,P,$$

but

$$\cos O\,B\,A = -\frac{\cos\alpha}{\sin\beta}, \qquad \text{and } \cos O\,B\,Q = \frac{\cos\beta \,.\, \cos x - \cos\theta'}{\sin\beta \,.\, \sin x},$$

so that we find

$$\cos\theta' = p = \cos\beta \,.\, \cos x + \cos\alpha \,.\, \sin x;$$

Light. and similarly $\cos\theta = q = \cos\beta \,.\, \cos x - \cos\alpha \,.\, \sin x$, Part IV.

whence, by adding and subtracting, we get at once

$$\cos\alpha = \frac{p-q}{2\,.\,\sin x}; \qquad \cos\beta = \frac{p+q}{2\,.\,\cos x},$$

which, substituted in the value of T^2, give

$$T^2 = \left\{(a+b) - \left(\frac{a}{\sin x^2} + \frac{b}{\cos x^2}\right)\frac{p^2+q^2}{4} + \left(\frac{a}{\sin x^2} - \frac{b}{\cos x^2}\right)\frac{pq}{2}\right\}^2$$
$$-4ab + \frac{ab}{\sin x^2\,.\,\cos x^2}(p^2+q^2) + \frac{2ab(\sin x^2 - \cos x^2)}{\sin x^2\,.\,\cos x^2}pq.$$

Now, if in this we suppose $\frac{a}{\sin x^2} + \frac{b}{\cos x^2} = 0$, or $\tan x^2 = -\frac{a}{b}$, it will, on substitution and reduction, take the form

$$T^2 = \frac{(1-p^2)(1-q^2)}{\cos x^4}\,.\,b^2 = \frac{b^2\,.\,\sin\theta^2\,.\,\sin\theta'^2}{\cos x^4}$$

and $$T = \frac{-b}{\cos x^2}\,.\,\sin\theta\,.\,\sin\theta';$$

that is, restoring the value of x, $\left(\text{since } \tan x^2 = -\frac{a}{b}, \text{ and therefore } \cos x^2 = \frac{b}{b-a}\right)$, finally,

$$T = -(b-a)\,.\,\sin\theta\,.\,\sin\theta'.$$

1033. Position of resultant axes in this case. Thus, in this case also, the isochromatic lines are sphero-lemniscates, and the only difference is that their poles lie now in the plane of the polarizing axes, instead of at right angles to it; and that whereas in the former case the semi-angle between them (x) was given by the equation $\sin x = \sqrt{\frac{a}{b}}$, that is, $\cos x = \sqrt{\frac{b-a}{b}}$, in this it is given by the equation $\cos x = \sqrt{\frac{b}{b-a}}$.

1034. Cases of the resolution of a single axis into two. *Corol.* 1. In the case when $a = b$, or when the two polarizing axes are of the same denomination and of equal intensity, we have $\sin x = 1$, or $x = 90°$, so that the angle between the resultant axes being 180°, they form one straight line, the lemniscates become circles, and the single *resultant* axis has now the characters of a *polarizing* axis. Hence, *vice versâ*, a single polarizing axis, in any direction, may be resolved into two others equal in intensity, at right angles to it and to each other, and of an opposite denomination to the resolved axis. This follows from the negative sign of T, which is prefixed in extracting the square root in Art. 1030 and 1032; because in the case supposed, when the arc A B is 90° the angle C or A O B is necessarily greater than 90°, and 2 C the angle of the parallelogram of tints > 180°; so that the diagonal will be to be measured backwards through the angle, or must be a negative quantity.

1035. Composition of three equal rectangular axes. *Corol.* 2. Since a single axis is equivalent to two equally intense axes of an opposite character at right angles to it and to each other, if we superadd to both another equal axis also of the opposite kind, and in the direction of the first, this will destroy the effect of the first, and therefore the combination of three equal and similar axes arising on the other side at right angles to each other, will be equivalent to none at all. Thus, three equal rectangular axes of the same character destroy each other's effects. This is Dr. Brewster's account of the want of polarization and double refraction in crystals whose primitive form is the cube, regular octohedron, &c., and whose secondary forms indicate a perfect symmetry in their molecules with respect to three rectangular axes.

1036. There is no necessity to pursue further the general subjects of this species of composition of axes and of tints. Indeed, it appears to us that the rule for the parallelogram of tints, as laid down by Dr. Brewster, becomes inapplicable when a third axis is introduced; for this obvious reason, that when we would combine the compound tint arising from two of the axes (A, B) with that arising from the action of the third (C,) although the *sides* of the new parallelogram which must be constructed are given, (*viz.* the compound tint T, and the simple tint t'',) yet the wording of the rule leaves us completely at a loss what to consider as its *angle*, inasmuch as it assigns no single line which can be combined with the axis C in the manner there required, or which *quoad hoc* is to be taken as a resultant of the axes A, B. For further information therefore on this subject we shall content ourselves with referring the reader to his original Paper in the *Transactions of the Royal Society*, 1818.

§ X. *Of Circular Polarization.*

1037. The first phenomena referable to the class of facts to whose consideration this section will be devoted, were noticed by M. Arago in his Memoir published among those of the Institute for 1811 on the colours of crystallized plates. He observed that when a polarized ray was made to traverse at right angles a plate of rock crystal

(quartz) cut perpendicularly to the axis of double refraction, on analyzing the emergent ray by a doubly refracting prism, the two images had complementary colours, and that these colours changed when the doubly refracting prism was made to revolve; so that in the course of a half revolution, the extraordinary image (for example) which at first was red, became in succession orange, yellow, yellow-green, and violet, after which the same series of tints would of course recur. It is evident that this is just what would take place, supposing the several coloured rays at their emergence from the rock crystal to be polarized in different planes; and to this conclusion M. Arago came in a second Paper, subsequently read to the Institute. The subject was resumed by M. Biot, in a Paper published in the *Mém. de l'Inst.*, 1812; and his labours were completed in a second extremely interesting Paper read to that body in September, 1818.

1038. Rotatory phenomena of quartz.

When a polarized ray is made to traverse the axis of Iceland spar, beril, and other uniaxal crystals, we have seen that it undergoes no change or modification; and that when analyzed at its egress by a doubly refracting prism, having its principal section in the plane of primitive polarization, the ordinary image will contain the whole ray, or the complementary tints will be white and black. Quartz, however, is an exception to this rule. A polarized ray transmitted, however precisely, along its axis, is still coloured and subdivided, and that the more evidently, the *thicker* is the plate. If we place on a proper apparatus, such as that described in Art. 929 and figured in fig. 189, a very thin plate of this body, and turn round the analyzing prism M in its cell, till the extraordinary image is at its minimum of brightness, it will in this position have a sombre violet, or purple tinge, because the yellow or most luminous rays, which are complementary to purple, are now completely extinguished. Let the angle of rotation of the prism in its cell, measured on the divided circle R, and which in this case will be small, be noted; and then let the rock crystal plate be detached, and another cut from the same crystal, but of twice the thickness, be substituted. The tint of the extraordinary image will no longer be violet; but if the prism be made to revolve through an additional equal arc in the same direction, the violet or purple tint will be restored, and the minimum of brightness attained; and, in general, if the thickness of the plate (always supposed cut from the same crystal) be greater or less in any ratio, the angle of rotation through which the prism must be moved in the same direction, to produce a minimum of intensity and a purple tint in the extraordinary image, is increased or diminished in the same ratio. In consequence, if the plate be sufficiently thick, one or more circumferences will be required to be traversed; and as only the excesses over whole circumferences can be read off, this may produce some confusion or doubt, unless we take care to use a succession of thicknesses so gradually increasing as not to allow of a *saltus* of a whole, or a half circumference.

1039. Rotation of the plane of polarization.

From this experiment we collect, that the plane of polarization of a mean yellow ray which has traversed the axis of a quartz plate, has been turned aside from its original position, through an angle proportional to the thickness of the plate; and, therefore, assumes at its egress a position the same as it would have, had it revolved uniformly in one direction, during every instant of the ray's progress through the plate. The same holds good for all the other homogeneous rays; but to prove it, we must abandon the use of white light, and operate with pure rays of the particular colour we would examine. If we use pure red light, for instance, or defend the eye with a pure red glass, the same will be observed, only that instead of a violet tint and a minimum of light, we shall have a total obliteration of the extraordinary pencil when the prism attains its proper position, thus proving, what in the former mode of observation might have been doubtful, that the polarization of the emergent ray is *complete*.

1040. Law of rotation of the simple coloured rays.

In examining in this way the quantity by which one and the same plate of quartz turns aside the planes of polarization of the different homogeneous rays, M. Biot ascertained that the *more refrangible rays are more energetically acted on than the less*, and have their planes of polarization deviated through a greater arc. According to this eminent philosopher, the constant coefficient, or index, which represents the velocity with which the plane of polarization may be conceived to revolve, is proportional to the square of the length of an undulation of the homogeneous ray under consideration; so that if we call λ the length of an undulation, and t the thickness of the plate, the deviation produced will be equal to $k \cdot \lambda^2 t$, k being a certain constant. The value of this constant he assigns at $\frac{18°.414}{(6.18614)^2}$, when t is reckoned in millimètres; and the following is stated by him as the numerical amount of the deviations in degrees (sexagesimal) produced by one millimètre of thickness of rock crystal on the several rays:

Designation of the homogeneous ray.	Arc of rotation corresponding to one millimètre.
Extreme red	17°.4964
Limit of red and orange.....	20°.4798
Limit of orange and yellow ..	22°.3138
Limit of yellow and green....	25°.6752
Limit of green and blue	30°.0460
Limit of blue and indigo	34°.5717
Limit of indigo and violet....	37°.6829
Extreme violet	44°.0827

Light. Part IV.

In the course of these researches M. Biot was led to the very singular discovery of a constant difference subsisting in different specimens of rock crystal, in the *direction* in which this rotation or angular shifting of the plane of polarization of a ray traversing them takes place. In some specimens it is observed to be from right to left, in others from left to right. To conceive this distinction, let the reader take a common cork-screw, and, holding it *with the head towards him*, let him turn it in the usual manner, as if to penetrate a cork. The head will then turn the same way with the plane of polarization of a ray in its progress *from* the spectator through a *right-handed* crystal may be conceived to do. If the thread of the cork-screw were reversed, or what is termed a *left-handed thread*, then the motion of the head as the instrument advanced would represent that of the plane of polarization in a left-handed specimen of rock crystal. It will be observed, that we do not here mean to say that the plane of polarization *does* so revolve in the interior of a crystal, but that the ray at its egress presents the same phenomena as to polarization *as if* it had done so. This is necessary, for we shall see presently that a very different view of the subject may be taken.

1041. Right and left handed quartz.

1042. Phenomena of plagiedral crystals

In crystals which present this remarkable difference, when cut and polished, and when the external indications of crystalline form are obliterated, no other difference can be detected. Their hardness, transparency, refractive and double refractive powers are the same; and, with the exception of the direction in which it takes place, their effects in deviating the planes of polarization of the rays which traverse them are alike. Experiments subsequent to M. Biot's researches have, however, established, as a result of extensive induction, a very curious connection between this direction and the crystalline forms affected by individual specimens. In the variety of crystallized quartz, termed by Hauy, *Plagiedral*, there occur faces which (unlike those in all the more common varieties) are unsymmetrically related to the axes and apices of the primitive form, whether regarded as the rhomboid or bipyramidal dodecahedron. Fig. 201 represents such a crystal, in which when the apex A is set upwards, the faces C, C, C, are observed to lean all in one direction, *viz.* to the right, with respect to the axis, as if distorted from a symmetrical position by some cause acting from left to right all round the crystal. When the vertex B is set upwards, the same distortion, and in the same direction, is observed in the plagiedral faces D, D, D, and crystals of quartz are excessively rare, if they exist at all, in which two plagiedral faces leaning opposite ways occur. Now it has been ascertained, that in crystals where one or more of these faces, however minute and even of microscopic dimensions, can be seen, we may thence predict with certainty the direction of rotation in a plate cut from it, which is always that in which the plagiedral face appears to lean with respect to an observer regarding it as the reader does the figure, which represents a *right-handed* crystal. *Hence we are entitled to conclude, that whatever be the cause which determines the direction of rotation, the same has acted in determining the direction of the plagiedral faces.* Other crystallized minerals, as apatite, &c. also present plagiedral and unsymmetrical faces; but, independent of their extreme rarity, they are not possessed of the property of rotation; so that at present we are unable to say whether this curious law be general, or to conjecture to what principles it will hereafter prove to be referable.

1043. Superposition of plates of rock crystal.

When two plates of rock crystal are superposed, if they be both right-handed or both left, their joint rotatory effect will be the sum of their respective ones, *i. e* each ray's plane of polarization will be shifted through an angle equal to the sum of those through which it would have been shifted by their separate actions. If their characters be opposite, it will be their difference, *i. e.* the *index of rotation* in a right-handed crystal being regarded as positive, it will be negative in a left-handed one.

1044. Amethyst.

The amethyst (and, possibly, also the agate in some cases) presents the very remarkable and curious phenomenon of these two species of quartz crystallized together in alternate layers of very minute thickness. Accordingly, when a crystal of amethyst is cut at right angles to the axis, and examined by polarized light transmitted exactly along the axis, and analyzed as usual, it offers a striped or fringed appearance, as represented in fig. 202, variegated with different colours, according to the different planes of polarization assumed by the rays emergent at its several points, and presenting, according to the distribution of its elements, the most beautiful combinations and contrasts of coloured fasciæ and spaces. For a particular account of these phenomena, the reader is referred to a Paper by Dr. Brewster, (*Edinburgh Transactions*, vol. xi.) who first observed and publicly described them, though we have reason to believe them to have been known to others by independent observation previous to the publication of his very curious and interesting Memoir. The layers may be distinctly seen *cropping out* to the surface in a fresh fracture of the mineral, and imparting that peculiar undulated fracture which is the chief mineralogical character of this substance by which it is known from ordinary quartz.

1045. Rotatory phenomena in liquids.

But the phenomena of rotation as above described are not confined to quartz. Many liquids, and even vapours exhibit it, a circumstance which would seem very unexpected, when we consider that in liquids and gases the molecules must be supposed unrelated to each other by any crystalline arrangement, and independent of each other; so that to produce any such phenomena, each individual molecule must be conceived as unsymmetrically constituted, *i. e.* as having a right and a left side. M. Biot and Dr. Seebeck appear about the same time to have made this singular and interesting discovery; but the former has analyzed the phenomena with particular care, and it is from his Memoir above cited that we extract the following statements. The liquids in which he observed a right-handed rotatory property, according to our sense of the word above explained, in which the observer is supposed to look in the direction of the ray's motion, are oil of turpentine, oil of laurel, vapour of turpentine oil, and an alcoholic solution of artificial camphor produced by the action of muriatic acid on oil of turpentine. The left-handed rotation was observed by him in oil of lemons, syrup of cane sugar, and alcoholic solution of natural camphor. In all these, the intensity of the action, or the velocity of rotation, was much inferior to quartz. The following are their *indices of rotation*, or the arcs of rotation produced by one millimètre of thickness in the plane of polarization of a certain homogeneous red ray chosen by M. Biot for a standard, as calculated from his data.

Right-handed.	Index of rotation.	Left-handed.	Index of rotation.
Rock crystal	+ 18°.414	Rock crystal	− 18°.414
Oil of turpentine	+ 0°.271	Oil of lemon	− 0°.436
Ditto, another specimen	+ 0°.251	Concentrated syrup of sugar	− 0°.554
Ditto, purified by repeated distillations	+ 0°.286		
Oil of laurel			
Solution of 1753 parts of artificial camphor in 17359 of alcohol	+ 0°.018		

1046. Law of rotation in mixtures.

It follows further from M. Biot's researches, that when any two or more liquids are mixed together, or combined with plates of rock crystal, the rotation produced by the compound medium will be always the sum of the rotations produced by the several simple ones, in thicknesses equal to their actual thicknesses present in the combination, the thicknesses in mixed liquids being assumed in the ratio of the volumes of each respectively mixed; so that calling T the compound thickness, and R the resulting index of rotation, we shall always have

$$R \,.\, T = r \,.\, t + r' \,.\, t' + r'' \,.\, t'' + \&c.$$

where r, r', &c. are the indices (with their signs) of the elementary ingredients, and t, t', &c. their thicknesses. Thus, when 66 parts by measure of oil of turpentine, having the index + 0.258 are made to act against 38 of oil of lemon, we have

$$+ 66 \times 0.251 - 38 \times 0.436 = 0.002,$$

so that these thicknesses ought almost exactly to compensate each other; and such was, in fact, the result of M. Biot's experiment, the whole pencil transmitted being found to retain its primitive polarization without the least trace of an extraordinary image. Again, when into two tubes of the same bore, but of very unequal lengths, equal quantities of oil of turpentine were poured, and the rest of their lengths filled with sulphuric ether, which has no rotatory property, or in which $r = 0$, the two compound thicknesses thus differently constituted gave identically the same tints in all positions of the analyzing prism. Thus we see that dilution or mixture which only separate, without decomposing the molecules, do not alter their rotatory power. Nay, even when reduced to vapour, M. Biot found, that oil of turpentine still preserved its property and peculiar character; and, had not the explosion of his apparatus prevented accurate measures, would probably enough have been found to retain the same index of rotation allowing for the change of density. From these circumstances he concludes that the rotatory power is essentially inherent in the molecules of bodies, and carried with them into all their combinations. But this is too rapid a generalization; for ~~neither~~ sugar ~~nor camphor~~ in the solid state possess this property, though examined for it in the same circumstances as quartz is, by transmitting the polarized ray along their optic axes; and, on the other hand, quartz held in solution by potash, or (as Dr. Brewster has found) melted by heat, and thus deprived of its crystalline arrangement, manifests no such property. This obscure part of chemical optics well deserves additional attention.

do not

1047. Fresnel's theory of circular polarization.

M. Fresnel's researches have been directed to the rotatory phenomena with the same brilliant success which has distinguished his other inquiries into the nature of light; and he has shown that they may be explained by conceiving the molecules of the ether, which propagate rays along the axis of quartz, or rotatory fluids, instead of vibrating in straight lines, to revolve uniformly in circles, in the manner explained in Art. 627, (where we have shown (Corol.) that such a mode of vibration may subsist, and must arise from the interference of two rectangular vibrations of equal amplitude, but differing in phase by a quarter undulation,) and by admitting that, in virtue of some peculiar mechanism in the molecules of the media in question, such circular vibrations, when performed from right to left, bring into play an elasticity slightly different from that which propagates them forward when performed in the contrary direction. The colours produced by such media he conceives to originate in the interference of two pencils thus *circularly polarized*, and lagging the one behind the other by an interval of retardation proportioned to their difference of velocities.

1048. Peculiar double refraction produced by circularly polarizing media.

But to make this last hypothesis admissible, it is incumbent on us to show that the phenomenon which necessarily accompanies a difference of velocities, *viz.* a bifurcation of the pencil in the act of refraction at oblique surfaces, really takes place. This has accordingly been shown by M. Fresnel, by an experiment which, though of great delicacy, is decisive and satisfactory. From a crystal of quartz he procured to be cut a prism having its refracting angle 150°, and its faces equally inclined to the axis; so that a ray traversing it internally parallel to its axis should be incident at equal angles, *viz.* of 75° on either face. As this is too great to allow of the ray's egress, he cemented on the surfaces the two halves of another precisely similar prism cut from another rock crystal of an opposite rotatory character. Thus in fig. 203, A C B is the first prism, and the side C B of the second prism C B E being cemented on to C B, this prism is bisected by the plane B D, and the half of it D B E transferred to the other side, and cemented with its side B C in contact with A C, thus producing the achromatic parallelepiped F A B D; so that if a ray be incident on Q in the direction P Q parallel to the base A B, *i. e.* to the axis of the two crystals, it will traverse all three in the direction of the axes of their spheroids of double refraction; and, therefore, so far as the Huygenian law of double refraction is concerned, ought to undergo no division. Now it is evident, that if the ray P Q *be* at its entry into A F C divided into two circularly polarized in opposite directions, the one (R) moving quicker than the other (L,) then, at quitting the surface A C, a bifurcation must take place, the ray R being *least*, and L *most* refracted. In this state they are incident on the medium A C B, and now the portions R and L, by reason of the opposite nature of the media, exchange velocities; so that R, which at its emergence from the face A C of F A C was *least* refracted *upwards*, will now

be *most* refracted *downwards;* and thus the separation of the images will be doubled, and the same will take place at the common face C B. Thus this combination, both from the doubling of the separation, and the greatness of the angles of incidence, is peculiarly well adapted to render sensible any bifurcation, or difference of velocities, however small, which may exist along the axis. Accordingly, with the compound prism, so constructed, a double refraction is produced; and the two rays are really observed to emerge, making a sensible angle with each other.

1049. Characters of circular polarization. But it is, moreover, observed, that though thus separated by a real double refraction, the two pencils have not acquired the characters which double refraction usually impresses on the ordinary and extraordinary rays, at their emergence, but very different ones. In common cases of double refraction the two emergent pencils are each wholly polarized in opposite planes, and either of them when examined with a doubly refracting prism gives two unequal images, one alternately more and less bright than the other, as the prism revolves through successive quadrants. This is not the case with the two pencils produced in the case before us, for

First, Either of them, when examined with a doubly refracting prism, gives constantly two images of equal intensity, in whatever plane the principal section of the latter be placed. In this respect, then, they present the characters of unpolarized light, and may be regarded as each consisting of two rays polarized at right angles to each other. But

1050. Other characters of circularly polarized rays. Secondly, They differ from ordinary, or unpolarized light, in a very remarkable property, which was first discovered by Fresnel, and is a chief distinctive character of this kind of polarization. Suppose either of them to be incident at right angles on the surface A B of a parallelepiped of crown glass of the refractive index 1.51, having its angles A B C and A D C each $54\frac{1}{2}°$, it will then be totally reflected at the internal surface B C; and (if the parallelepiped be long enough) again in the same plane at the opposite surface A D, and will emerge at length perpendicularly through the surface B C. But the emergent ray, instead of comporting itself as ordinary light, will now be found to be *completely polarized* in a plane 45° inclined to that in which the reflections were made, whatever may have been the position of that plane. If both the pencils be treated in this manner, it will be found that the one, after its two total reflexions will assume a plane of polarization 45° in azimuth to the *right*, and the other 45° to the *left* of the plane of the reflexions.

Thus we see that the effect of double refraction along the axis of quartz is to impress on either of the emergent pencils opposite polarizations, or modifications, of a nature totally distinct from that given to a ray by ordinary reflexion, or by double refraction through Iceland spar, &c.; and, as in the last described experiment, so long as the ray enters perpendicularly into the first surface of the glass parallelepiped, it is indifferent in what plane the two reflexions are operated, and since when presented to a doubly refracting prism in any plane indifferently it always divides itself into two equal pencils, it is evident that the ray thus modified has *no sides*, *i. e.* no particular relations to certain regions of space; and therefore that the epithet *circular polarization*, apart from all theoretical considerations, may be naturally applied to this peculiar modification. But the characters above described are not the only ones belonging to a ray thus modified, for

Thirdly, Such a ray being transmitted through a thin crystallized lamina, and parallel to its axis, is divided by subsequent double refraction into two rays of complementary colours, thus marking a decided difference between it and a ray of common light; while, on the other hand, these colours are not the same with those which would arise from a ray of light polarized in the usual way and similarly analyzed, but differ from them by an exact quarter of a tint, either in excess or defect, as the case may be.

1051. Fourthly, A ray so modified by this peculiar double refraction, when transmitted again along the axis of rock crystal, or through columns of oil of turpentine, of lemons, &c., and then analyzed by a double refracting prism, gives rise to no phenomena of colour, differing in this from polarized, and agreeing with common light.

1052. Another mode of producing circular polarization. Another independent mode of impressing on a ray all this assemblage of characters has been discovered by M. Fresnel. It consists in inverting the process described in Art. 1049. Thus, into the side C D of the glass parallelepiped there mentioned, let a common polarized ray be introduced at a perpendicular incidence, the parallelepiped being so placed that the plane of internal reflexion at the side A D shall be 45° inclined to that of its primitive polarization. Then, after undergoing two total internal reflexions at G and F, it will emerge at E deprived of its characters of ordinary polarization and endowed with those of circular, and being no way distinguishable from one of the pencils produced by double refraction along the axis of rock crystal.

1053. It remains to show, however, that the characters here described, as impressed on a ray by transmission along the axis of rock crystal, are really those which ought to belong to a ray propagated by circular vibrations. And, first, it follows from Art. 627, that this latter ray is the resultant of two rays polarized at right angles, and differing in their phases by a quarter undulation. It must, therefore, of necessity possess the first character, *viz.* that of division into two equal pencils by double refraction in any plane, for the same reason that unpolarized light is so divided, the difference of phases having nothing to do with this character.

1054. In the next place, a ray propagated by circular vibrations when incident on rock crystal in the direction of the axis, will (by hypothesis) be propagated along it by that elasticity which is due to the direction of its rotation, the wave then will enter the crystal without further subdivision, and there will be no difference of paths, or interfering rays at its emergence; and, of course, no colours produced on analyzing by double refraction, which is another of the characters in question.

1055. Tints produced by circularly polarized rays. When a ray propagated by circular vibrations is incident on a crystallized lamina it may be regarded as composed of two, one polarized in the plane of the principal section, the other at right angles to it, of equal intensity, and differing in phase by a quarter undulation. Each of these will be transmitted unaltered, and therefore at their emergence and subsequent analysis will comport themselves in respect of their interferences, just as would do the two portions of a ray primitively polarized in azimuth 45°, and divided into two by the

Light. double refraction of the lamina, provided that a quarter undulation be added to the phase of one of these latter rays. Now such rays will, as we have shown at length in Art. 969, produce by the interference of their doubly refracted portions, the ordinary and extraordinary tints due to the interval of retardation within the crystallized lamina. Hence, in the present case, the tints produced will be those due to that interval, *plus* or *minus* the quarter of an undulation added to, or subtracted from, the phase of one of the portions; and, consequently, will differ one-fourth of a tint, or order, from that which would arise from the use of a beam of ordinary polarized light incident in azimuth 45° on the lamina. Part IV.

1056. Modifications impressed on light by total reflexion.

There remains but one more character of the rays transmitted along the axis of quartz, which we must show to belong to a ray propagated by circular vibrations, *viz.* that described in Art. 1049. But in order to this it will be necessary to state the result of M. Fresnel's researches on the modifications which light undergoes by total reflexion in the interior of transparent bodies.

When a ray polarized in any azimuth is incident on a reflecting surface which reflects the whole of the incident light, if we decompose it into two, the one having its vibrations performed parallel, and the other perpendicular to the surface, and regard each of these as independent of the other; it is evident that the reflexion of these portions will be performed under very different circumstances, the ethereal molecules having in the former case to glide as it were on the surface, and therefore parallel to the strata in which their density is constant, while in the latter each molecule in the act of vibration will pass into strata of variable density. The reflexions therefore will be performed at different depths in the two cases; and from this cause will arise a difference of route, and a consequent difference of phase in the reflected portions, so that the total reflected ray will no longer be capable of being regarded as one having a single origin, but as two of unequal intensities, oppositely polarized, and differing in phase by a quantity depending on the angle of incidence and the refractive power of the medium. From peculiar considerations, of a delicate nature, and depending on a discussion of the imaginary forms assumed by the general expressions for the intensity of a ray reflected at any angle (Art. 852) when applied to the case of total reflexion, M. Fresnel has been led to the following expression for the difference of phases (δ) of the two portions in question.

$$\cos \delta = \frac{2\,\mu^2 . (\sin i)^4 - (\mu^2 + 1) . (\sin i)^2 + 1}{(\mu^2 + 1)\,(\sin i)^2 - 1},$$

where μ is the index of refraction, and i the angle of internal incidence. This formula, it is to be observed, is given by him, not as strictly demonstrated, but merely as highly probable, as an interpretation of the analytical meaning of the imaginary formula alluded to. The mode of its deduction being, however, independent of experiment, and entirely *a priori*, it is clear that if found verified by careful experiment in circumstances properly varied, it may be received as a physical law, like any other result of the same kind. Now we have already seen, that in the case of crown glass, where $\mu = 1.51$ and $i = 54\frac{1}{2}°$, a polarized ray, having its azimuth 45°, reckoned from the plane of total reflexion, has its polarization destroyed, and becomes resolved into a ray having the other characters of a resultant from two differing 45° in phase, by two total reflexions at this angle, (Art. 1056.) But if in the above formula we make $\mu = 1.51$, and $i = 54°\ 37'$, we shall find $\delta = 45°$, and $2\,\delta = 90°$, so that the above equation is verified in this case. M. Fresnel also found that the same effect was produced by three reflexions when the angle of incidence was 69° 12′, and by four when 74° 42′, both agreeing with the formula which gives in the former case $\delta = \frac{1}{3}\,90°$, and in the latter $\delta = \frac{1}{4}\,90°$, for the difference of phase gained or lost by one portion on the other at each reflexion. Similar verifications were obtained by performing two reflexions at the internal surface of glass, and two at the confines of glass and water at angles of 68° 27′.

1057. Explanation of the rotatory phenomena.

It appears, then, that when a ray polarized in azimuth 45° undergoes two total reflexions at the angles, and in the manner described, it becomes circularly polarized; and if *vice versâ*, the two elements of a ray so circularly polarized be made to retrace their course, they will reunite into a ray polarized completely in one plane. Thus we see, that all the characters of the rays transmitted along the axis of rock crystal agree with those of a ray so compounded, and possessing *circular polarization*. In order, then, to explain the phenomena presented by a polarized ray when incident on a plate of this substance cut at right angles to its axis, we must first regard the ray as resolved into two others (which we will call A and B) of equal intensity; the one A polarized in a plane 45° inclined to the *right*, the other 45° inclined to the *left* of the vertical, (which, to fix our ideas, we shall take for the plane of primitive polarization.) Now, since by Art. 615 a ray polarized in any plane may be regarded as equivalent to two rays each of half its intensity, differing in their phases by a quarter undulation, let us conceive the ray A as resolved into two, A a polarized in the plane + 45°, and having its phase advanced + $\frac{1}{8}$ undulation, and another A b also polarized at + 45°, but having its phase retarded, or − $\frac{1}{8}$ undulation, so that A a and A b differ $\frac{1}{4}$ undulation in their phases. Similarly, let B be regarded as decomposed into B a polarized at − 45°, and having its phase + $\frac{1}{8}$ undulation, and B b polarized also at − 45°, but having its phase − $\frac{1}{8}$ undulation different from B. Thus will the original ray be resolved into the four A a, A b, B a, B b. Now, let us combine these two and two in a cross order, then A a combined with B b will be equal rays, polarized in opposite planes, and differing $\frac{1}{4}$ undulation in their phases, and will therefore compose one circularly polarized ray, in which the rotation is from *right* to *left*. Similarly, the pair A b, B a will compound another equally intense circularly polarized ray having its rotation the contrary way. Now these will (*ex hypothesi*) be transmitted through the quartz with unequal velocities, and thus an interval of retardation will arise, and if the surface of egress or ingress be *oblique* to the axis, a double refraction will take place; and two circularly polarized rays will emerge in different directions, as experiments show they do. If *perpendicular* they will emerge superposed, and will compound one ray. Let us now examine what will be the character and state of polarization of this compound ray. To this end conceive a molecule of ether C to be at once agitated by two circular motions in opposite directions; one in a circle equal and similar to A P in

Light. Part IV.

Fig. 205.

the direction A P, the other in a circle equal and similar to B Q, and in the direction B Q, fig. 205. Let A, B be two molecules setting out at once from A, B in these circles with equal velocities, then will the motion of C at any instant be equal to that compounded of the motions of A and B at that instant. When A comes to P let B come to Q, then arc A P = B Q, and the motions at P and Q will be each resolved into two, those of which parallel to C D (a perpendicular to P Q) conspire, while those in the directions P D and Q D parallel to P Q oppose, and being equal destroy each other; thus C will move only in virtue of the sum of the two former, and its vibrations will therefore be rectilinear, and in the plane C D perpendicular to P D Q. If the thickness of the plate of quartz were nothing, or such that the interval of retardation were an exact number of undulations, A, B would lie at opposite extremities of a diameter, and C D the new plane of polarization would be perpendicular to A M that diameter, or coincident with the plane of primitive polarization. But if not, the quicker motion will have gained on the other a part of a circumference M B, which is to a whole circumference as the thickness of the plate is to that which would produce a difference of a whole undulation; and at the emergence of the two waves into air, after which they circulate with equal velocity, if we suppose the one molecule to be setting out from A, the other will be setting out, not from M the opposite extremity of the diameter, but from B, and therefore C D the new plane of polarization (which from what has just been shown must always bisect the angle A C B) will no longer be coincident with C N the primitive plane of polarization, at right angles to A M, but will make an angle D C N with it equal to half B C M, and therefore proportional to M B, or to the interval of retardation, *i. e.* to the thickness of the plate. Thus the system of rays emerging from the rock crystal plate will compound one ray polarized in one plane, and in the position the original plane would have had, had it revolved uniformly round the ray as an axis during its passage through the plate. Thus we have a complete and satisfactory explanation of the apparent rotation of the plane of polarization, as observed by Biot in the case of a homogeneous ray.

1058. It is observed, that the spectra formed by the double refraction of rock crystal along its axis are very highly and unequally coloured. The violet rays are most *separated*, and therefore the difference of velocities of the two rotating pencils is much greater for violet than for red rays. Consequently, the apparent velocity of rotation of the plane of polarization will also be greater for the violet rays in the same proportion, and thus arise all the phenomena of coloration observed and described by M. Biot. It is scarcely possible to imagine an analysis of a natural phenomenon more complete, satisfactory, and elegant. With regard to the physical reason of the difference of velocity in the two circular polarized pencils within the quartz, it is true we remain in the dark; but the fact of such difference existing is now shown to be no hypothesis, but a fact demonstrated by their observed difference of refraction, and by the observed characters of the two emergent rays.

§ XI. *Of the Absorption of Light by Crystallized Media.*

1059. Absorption of polarized light by double refracting crystals.

Crystallized media, endowed with the property of double refraction, are found to absorb the differently coloured rays differently, according to their planes of polarization, and the manner in which these planes are presented to the axis of the crystal, and also to exert very different absolute absorbing energies on rays of one colour polarized in different planes. A remarkable instance of this has been already often referred to in the case of the brown tourmaline, a plate of which, cut parallel to the axis, absorbs almost entirely *all* rays polarized in the plane of the principal section, and lets pass only such among oppositely polarized rays as go to constitute a brown colour.

1060. Property of the tourmaline

Explained.

When such a plate, then, is exposed to natural light, since at the entrance of each ray into its substance it is resolved into two, one polarized in the plane of the principal section, and one perpendicular to it, the former is absorbed in its progress by the action of the crystal, while the brown portion of the latter escaping absorption, but retaining at its egress the polarization impressed on it, after traversing the plate, appears with its proper colour, and *wholly polarized in a plane at right angles to the axis.* Thus the curious phenomenon of the polarization of light by transmission through a plate of tourmaline, or other coloured crystal, is explained, or at least resolved into the more general fact of an absorbing energy varying with the internal position of the plane of polarization. The crystal, in virtue of its double refractive property, divides the ray into two, and polarizes them oppositely; and the unequal absorption of these two portions *subsequently* causes the total suppression of one, and the partial of the other of the portions so separated. Thus we see that the polarized beam obtained by transmission through a tourmaline must always be of much less than half the intensity of the incident light.

1061. Gradual destruction of the ordinary ray.

The destruction of the pencil polarized in the principal section is not, however, sudden; for if the plate of tourmaline be very thin, the emerging pencil will only be partially polarized, indicating the existence in it of rays belonging to the other pencil. This is best shown by cutting a tourmaline into a prism having its refracting edge parallel to the axis, and its angle small, so as to produce a wedge whose thickness increases not too rapidly. If we look through this at a distant candle, we shall see only one image, *viz.* the extraordinary through the back of the wedge, (if thick enough;) but as the eye approaches the edge, the ordinary image appears at first very faint, but increasing in intensity till, at the very edge, it becomes equal to the other. At the same time the colour of the latter, which at first was intense, becomes diluted; and the images approximate not only to equality of light, but to similarity of tint. We see by this, too, that in strictness the ordinary pencil is never *completely* absorbed by any thickness, however great; but as it diminishes in geometrical progression as the thickness increases in arithmetical, the absorption may for all practical purposes be regarded as total at moderate thicknesses.

Light. Part IV.

The indefatigable scrutiny of Dr. Brewster, to whom we owe nearly all our knowledge on this subject, has shown that the same property is possessed in greater or less perfection by the greater number of coloured doubly refracting media; and the expression of the property may be rendered general by considering all doubly refractive media as possessing two distinct absorbing powers or two separate *scales of absorption* for the two pencils, or (adopting the language of § III. part 2) as having two distinct *types*, or curves expressing the law of absorption throughout the spectrum. If these types be both straight lines parallel to the abscissa, the crystal will be colourless. Such are limpid carbonate of lime, quartz, nitre, &c. If they be similar and equal curves, the medium, although coloured, will present the same colour, and the same intensity of tint, in common as in polarized light. If dissimilar, or if, although similar, their ordinates are in a ratio of inequality, the character, in the former case, and the intensity in the latter, will vary on a variation of the plane of polarization of the incident beam, so that if a plate cut from such a crystal be exposed to a beam of polarized white light, and turned round in its own plane, or otherwise inclined to the beam, its colour will change either in hue or depth or both. Dr. Brewster has remarked such change of colour and the phenomena connected with it in a great variety of crystals both with one and two axes, of which he has given a list in a most interesting Paper on the the subject in the *Philosophical Transactions*, 1819, p. 1, which we strongly recommend to the reader's perusal. It may be familiarly seen in a prism of smoked quartz of a pretty deep tinge, which held with its axis in the plane of polarization appears of a purple or amethyst colour, while if held in a direction at right angles to this position, its colour is a yellow brown.

1062. Media possess two distinct absorbing powers.

But in order to analyze the phenomena more exactly, we must examine the two pencils separately. To this end Dr. Brewster took a rhomboid of yellow carbonate of lime of sufficient thickness to give two distinct images of a small circular aperture placed close before it, and illuminated with white light, when he observed that the image seen by extraordinary refraction appeared of a deeper colour and less luminous than the other, being an orange yellow, while the ordinary image was a yellowish white. He found, moreover, that the difference of colour was greater as the paths of the refracted rays within the crystal were more inclined to the axis, being 0 when the rays passed along the axis, and a maximum when at right angles to it. If we denote by Y_o and Y_e the ordinates of the curves, expressing the law of absorption as in Art. 490, for the ordinary and extraordinary pencil respectively, these will both therefore decrease as we proceed from the red to the violet end of the spectrum, corresponding to types of the character of that represented in fig. 114; but Y_e being smaller, and decreasing more rapidly than Y_o. Moreover, since $Y_o = Y_e$ in the axis, and since as we recede from the axis Y_o increases (because the colour of the ordinary pencil becomes whiter and more luminous) while Y_e diminishes by the same degrees, (the extraordinary becoming deeper and less bright,) we shall represent both these changes satisfactorily by putting

1063. Absorption of the rays in the two pencils examined in crystals with one axis.

Formulæ for the light transmitted in each.

$$Y_o = Y\,(1 + k\,.\,\sin\theta^2)\,; \qquad Y_e = Y\,(1 - k\,.\,\sin\theta^2).$$

These give $Y_o + Y_e = 2\,Y =$ constant, or independent of θ, which agrees with an observation of Dr. Brewster, that in every situation the combined tints of the two images are exactly the same with the natural colour of the mineral, (which, in this instance, appears to have been alike in all directions.)

1064. Cases of two distinct colours.

In this case, then, the colour of a plate of the crystal of given thickness exposed to natural light will be the same, whether the plate be cut parallel or perpendicular to the axis. But Dr. Brewster has observed, that this is not always the case, but that great differences occasionally exist in this respect. Thus he found, that in some specimens of *sapphire* the colour when viewed *along* the axis was deep blue, and when *across* it yellowish green. In Idocrase an orange-yellow tint is seen along the axis, and a yellowish green across it. Specimens of tourmaline also are not uncommon in which the tint across the axis is green, while along the axis it is deep red; and, in general, this mineral is always much more opaque in the direction of the axis than in any other; so much so, indeed, that plates of a very moderate thickness cut across the axis are nearly impermeable to light. One of the most remarkable instances of this kind we have met with is a variety of sub-oxysulphate of iron, which crystallizes in regular hexagonal prisms, and which viewed through two opposite sides of the prism is light green, but along the axis, a deep blood red, so intense that a thickness of $\frac{1}{20}$ inch allows scarcely any light to pass. It is obvious, that to such cases the formulæ of the last article do not extend. But a slight modification will enable us to embrace the phenomena in an analytical expression. For if we take

Investigation of formulæ for these cases.

$$y_o = X_o + Y_o\,.\,\sin\theta^2\,; \qquad y_e = X_e + Y_e\,.\,\sin\theta^2\,;$$

where X_o, Y_o, &c. as well as y_o, y_e represent functions of λ (the length of an undulation) being the ordinates of so many curves, or types of tints, whose relations are to be determined, we have

$$y_o + y_e = (X_o + X_e) + (Y_o + Y_e)\ \sin\theta^2.$$

Now this is the tint which a sphere of the medium of a diameter $= 1$ will exhibit when viewed by natural light along a diameter inclined $\theta°$ to the axis. If we represent by A and B the ordinates of the types of the tints it is observed to exhibit in the directions of the axis, and perpendicular to it, we have, when $\theta = 0$,

$$y_o + y_e = A = X_o + X_e\,;$$

and when $\theta = 90°$,

$$y_o + y_e = B = (X_o + X_e) + (Y_o + Y_e),$$

whence we have

$$Y_o + Y_e = B - A\,;$$

and the tint exhibited by ordinary light at the inclination θ to the axis, will be represented by

Expression of the colour transmitted in common light.

$$y_o + y_e = A + (B - A)\,.\,\sin\theta^2,$$
$$= A\,.\,\cos\theta^2 + B\,.\,\sin\theta^2.$$

Light. Thus in the case of our sub-oxysulphate of iron, A is the ordinate of the type of a deep blood-red tint, and B in like manner represents a bright pale green, so that we shall have at any intermediate inclination θ Part IV.

$$\text{tint} = (\text{deep red}) \times \cos\theta^2 + (\text{light green}) \times \sin\theta^2,$$

which represents faithfully enough the gradual passage of one hue into the other as the inclination changes.

1065. When illuminated by polarized light. Suppose now the incident beam polarized in any plane, and let the plane in which the ray and the axis of the sphere lie make an angle $= a$ with that plane. Then would $\cos a^2$ and $\sin a^2$ represent the intensities of the ordinary and extraordinary pencils which superposed make up the emergent beam, were the crystal limpid; but in virtue of its absorbent powers, they will be reduced respectively to

$$y_o = \cos a^2 (X_o + Y_o \sin\theta^2), \quad \text{and } y_e = \sin a^2 (X_e + Y_e \cdot \sin\theta^2),$$

so that at their emergence they will no longer make up white light, but a variable tint whose type has for its ordinate

$$(X_o \cdot \cos a^2 + X_e \cdot \sin a^2) + (Y_o \cdot \cos a^2 + Y_e \cdot \sin a^2) \cdot \sin\theta^2,$$

in which it will be recollected that $X_o + X_e = A$, and $Y_o + Y_e = B - A$.

To determine the individual values of X_o, &c. however, we must have two more conditions, and these will be found by considering, first, that in the direction of the axis the tint must be independent of a, which gives $X_o \cdot \cos a^2 + X_e \cdot \sin a^2$ independent of a, and therefore $X_o = X_e$, and either of them $= A$. To get another condition, let the tints be noticed which the sphere or crystal exhibits when its axis is perpendicular to the visual ray; and, first, coincident with, next, perpendicular to, the plane of polarization, *i. e.* when $a = 0$, and $a = 90°$.

These are respectively $X_o + Y_o$, and $X_e + Y_e$;

and calling these a and b, we have

$$Y_o = a - X_o = a - A, \quad Y_e = b - A.$$

Hence the final expression for the tint seen in polarized light will be

$$A + \{ (a - A) \cdot \cos a^2 + (b - A) \sin a^2 \} \cdot \sin\theta^2,$$

that is,

$$A \cdot \cos\theta^2 + \{ a \cdot \cos a^2 + b \cdot \sin a^2 \} \cdot \sin\theta^2,$$

in which it will be observed that a and b are complements of each other to the tint B, because

$$a + b = X_o + Y_o + X_e + Y_e = B, \text{ by Art. 1064.}$$

1066. Dichroism. Such is the expression for the apparent hue of crystals with one axis, which exhibit a variable colour in common or polarized light, according to their position with respect to the incident light. The phenomenon in question may be generally termed *dichroism*, though the word has usually been applied only to that particular case where a marked change in the character of the tint takes place, as from red to green, &c.

1067. Dichroism in biaxal crystals. Fig. 206. The dichroism of biaxal crystals differs in many of its phenomena from those having only one optic axis. If we look through a plate, or into a crystal of any biaxal mineral, having the property in question, illuminated by natural light in such a direction that the visual ray within the crystal shall pass along, and in the immediate neighbourhood of, one of the axes, we shall perceive a phenomenon like that represented in fig. 206, consisting of two similar and equal sombre spaces A B one on either side of the pole P, and of the principal section P P′, and if we look along the other axis P′ a similar pair of spaces will be seen in its neighbourhood. In the mineral called *dichroite* by Hauy, (on account of the striking difference of its colours in different positions,) or *iolite* (from its violet hue) by others,* of which the phenomena have been described by Dr. Brewster in the Paper already cited, these spaces are of a full blue colour, while the intermediate region towards O, along the line O P C, and the space beyond P towards C are yellowish white. In epidote the sombre spaces are *brown*, and the region around O and in the principal section green, of a greater or less degree of dilution. In this latter mineral (at least in some of its more ordinary varieties of crystalline form, *viz.* in long striated prisms much flattened, and terminated by dihedral summits placed obliquely, so as to truncate two of the angles of the prism) the phenomena are seen without any artificial section, merely by looking in obliquely, across the axis of the prism; and the same is true of many other minerals, as, for instance, the axinite, in which the transition of colour is extremely remarkable and beautiful.

Colours of iolite.

Phenomena of epidote.

1068. Connection of the phenomena with the polarized rings and optic axes Fig. 207. The phenomena of dichroism in biaxal, as well as in uniaxal crystals, are evidently related to the optic axes, and depend on the planes of polarization assumed by the intromitted light, during its transit through the crystal to whose absorptive power it is subjected. Now, if we consider the form and situation of the sombre spaces where the greatest absorptive energy is exerted, we are at once struck by their analogy with those occupied by the more vividly coloured parts of the rays about the axes in the situation of fig. 179. That figure represents (Art. 900) the extraordinary set of rings as seen in a crystal whose principal section is in the plane of primitive polarization. Fig. 207 represents the ordinary or complementary set as seen around either of the axes, the pole P, and the principal section being here occupied with white light, and very bright, in consequence of its containing the whole incident light, while the lateral or coloured portions occupied by the rings are less illuminated, the colours originating in an abstraction of certain rays.

Conceive now a number of such sets of coloured rings not all of exactly the same dimensions, nor having

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* Mohs, with his usual contemptuous disregard of, or rather hostility to, all ordinary convenience and received usage, chooses to call this mineral "*prismatic quartz*." Such a nomenclature *must* ere long work out its own destruction, but while it subsists the nuisance is intolerable. We cannot but lament, that such a cause should exist to raise up prejudice against a system in many respects so useful and valuable.

Light. precisely the same pole, but very nearly so, to be superposed on one another, then would the *colours* be obliterated and blended into white light by their overlapping, but still the general intensity of the light in the lateral regions would remain much feebler than in the principal section, and the effect would be precisely that of fig. 206, *viz.* two sombre, cloudy, fan-shaped spaces traversed by a narrow ray of vivid light, opening out from P towards C and O. Such would be the case with a limpid crystal, supposing such a slight degree of confusion of structure as to produce the non-coincidence of the axes from all its molecules. In this case, however, neither of the spaces in question would appear coloured, nor would the phenomena be seen at all without the use of polarized light and its subsequent analysis. But if we conceive the crystal, instead of limpidity, to possess the property of *double absorption*, the suppressed and transmitted portions will be, not white light, but light of the colour of one or other of the pencils into which it is resolved by double refraction, according to its plane of polarization and the thickness of the medium it has traversed; and the analysis of the emergent ray may be regarded as performed, at least imperfectly by the difference of absorptive powers acting differently on the two pencils. In support of this it may be noticed, that when we examine the system of rings in the usual way, by polarized light, in crystals presenting the above phenomenon, they are usually found to be very irregular, several sets evidently overlapping and interfering with one another, and rendering the non-coincidence of all the axes a matter of ocular demonstration.

Part IV. Analogy in respect of intensity of illumination

1069. DIGRESSION. Theory of the polarized rings resumed.

In Art. 931 we investigated the law of intensity of the illumination of the polarized rings in different parts of their periphery for uniaxal crystals. As what is there said does not apply, however, to biaxal ones, and as the present subject has led us to the consideration of the more general case, it will not be irrelevant, if we digress at this point, in order to show, what modifications the statement there made must receive to embrace the phenomena of biaxal crystals.

1070. Biot's general law for the planes of polarization in biaxal crystals.

M. Biot has stated the general law of polarization in biaxal crystals, from his elaborate researches on that subject (*Mém. sur les Lois Générales de la Double Réfraction et Polarisation, &c. Mém. Acad. Sci.* 1819) to be as follows:

If two planes be drawn through the course of a ray within a crystal and through the two optic axes, and a third plane bisecting the angle included between the two former, this will be the plane of polarization if the ray be an ordinary one—but one perpendicular to it if extraordinary. Thus in fig. 209, C P and C P′ being the optic axes, and A C a ray penetrating the crystal, if P A, P′ A be joined by arcs of circles on the sphere H O K A having C for its centre, and the angle P A P′ be bisected by the arc A N, the plane A C N bisecting the dihedral angle between the planes P C A and P′ C A is the plane of ordinary polarization, and a plane perpendicular to it that of extraordinary. This is the law of fixed polarization, and expresses generally the planes of polarization assumed by the two rays at their emergence from doubly refracting crystals. It is a consequence of Fresnel's general theory, (though deducible from it by a train of analytical reasoning far too intricate and refined to allow of its insertion in a treatise like the present,) and, having been experimentally established long before that theory was devised, must be looked on as a strong additional proof of its conformity to nature.

1071. Doctrine of movable polarization applied to biaxal crystals. Fig. 208. Law of intensity of the rings in different points of their periphery.

The doctrine of movable polarization, however, which, so far as respects the phenomena of the colours and intensity of the rings, has been shown by M. Biot in the same excellent paper, to represent with fidelity their various affections, whether in uniaxal or biaxal crystals, requires the resulting ray to assume at its emergence a plane of polarization alternately coincident with, and making with the primitive plane of polarization *twice* the angle which the plane of fixed polarization *so determined* would make; so that if we draw A M (fig. 208) bisecting the angle P A P′, the emergent ray will be affected by subsequent analysis, as if polarized either in the plane of primitive polarization, or making with it an angle equal to twice C M A, and from this it is easy to derive the law of intensity in question; for the ray by which the point A of the rings is formed consists of two portions, of which (A) is affected by subsequent analysis by a prism of Iceland spar, as *if* it were polarized in a plane making an angle $2\,C\,M\,A = \psi$ with the plane of primitive polarization, (in which we suppose the principal section of the analyzing prism to be placed,) and the other, complementary to this (1 − A) retains its primitive polarization. The portion A then will be divided between the ordinary and extraordinary image in the proportion $(\cos 2\psi)^2 : (\sin 2\psi)^2$, and (considering only the latter,) A being its intensity at its emergence from the crystal, $A \cdot (\sin 2\psi)^2$ will be its intensity in the extraordinary image, or in the primary set of rings, while the whole of the portion 1 − A will pass into the ordinary or complementary set, as in Art. 932; so that we have only to express this in terms of the azimuth of the crystallized plate itself, and the direction of the ray within the crystal. For this purpose, put a = angle C O P = azimuth of the principal section of the plate reckoned from the plane of primitive polarization; $\theta = A\,P$, $\theta' = A\,P'$, and let us (for simplicity) consider only at present the case when P and P′ are near, as in nitre, so that arcs of circles may be regarded as straight lines, and spherical as plane triangles, (see Art. 907.) Now if in fig. 208 we put ϕ for the angle P N A, or the angle made by the plane of ordinary polarization with the principal section, we shall have $\psi = C\,M\,A = C\,O\,P + M\,N\,O = C\,O\,P + P\,N\,A = a + \phi$. To find ϕ we have only to consider that $\sin\phi^2 = \left(\frac{P\,A}{P\,N}\right)^2 \times \sin(P\,A\,N = \frac{1}{2}\,P\,A\,P')^2$; but since N A bisects the angle of the triangle P A P′ and cuts the base, $P\,N = P\,P' \times \frac{P\,A}{P\,A + A\,P'} = \frac{2\,a\,\theta}{\theta + \theta'}$, and

Analytically expressed.

$$(\sin\tfrac{1}{2}\,P\,A\,P')^2 = \tfrac{1}{2}(1 - \cos P\,A\,P') = \frac{4\,a^2 - (\theta - \theta')^2}{4\,\theta\,\theta'},$$

so that

$$\sin\phi^2 = \frac{(\theta + \theta')^2\,\{4\,a^2 - (\theta - \theta')^2\}}{16\,a^2 \cdot \theta\,\theta'}$$

Light. A more symmetrical value of ϕ will, however, be had by expressing the value of $\sin 2\phi$, which being equal to $4 . \sin \phi^2 (1 - \sin \phi^2)$ is immediately given by substitution of the foregoing. If we execute the reductions we shall find that, putting S for $\frac{\theta + \theta' + 2a}{2}$ = half the sum of the sides of the triangle P A P′ Part IV.

$$\sin 2\phi = \frac{2(\theta + \theta')(\theta - \theta')}{(2a)^2} \cdot \frac{\sqrt{S(S-\theta)(S-\theta')(S-2a)}}{\theta\theta'}.$$

Now $\frac{2\sqrt{S(S-\theta)(S-\theta')(S-2a)}}{\theta\theta'}$ is the well known expression for the sine of the angle P A P′ included between the sides θ, θ', and therefore calling this angle P, we have

$$\sin 2\phi = \frac{(\theta + \theta')(\theta - \theta')}{(2a)^2} . \sin P.$$

The nature of this expression renders the transition from plane to spherical triangles easy, and we may conclude consequently, that, in crystals where the axes make any angle $2a$, that if we take

$$\sin 2\phi = \frac{\sin(\theta + \theta') . \sin(\theta - \theta')}{(\sin 2a)^2} . \sin P,$$

and $\psi = a + \phi$, we shall still have the intensity of the extraordinary rings represented by A $(\sin 2\psi)^2$, and that of the ordinary by $1 - A + A . (\cos 2\psi)^2$, that is, $1 - A (\sin 2\psi)^2$, their sum being, as it ought, unity.

1072. Form of the black cross in biaxal crystals investigated. The black cross which divides the system of the primary rings, is too remarkable a feature not to require express notice. Its form, it is evident, must be determined by the condition that the line M A shall be everywhere perpendicular to C O D; in which circumstances the locus of A will be a curve marking out its central or blackest portion. The problem then is reduced to a purely geometrical one, Required a curve P A such that a line drawn from A bisecting the angle between lines A P, A P′ drawn to two given points P, P′, shall always be perpendicular to a given line C O D. To resolve this, retaining the former notation, and putting O M $= x$, M A $= y$, O A $= r$, we have

$$\cos AOP = \cos(AOM - a) = \frac{x . \cos a + y . \sin a}{r} = \frac{N}{r},$$

$$\sin AOP = \frac{y . \cos a - x . \sin a}{r} = \frac{M}{r},$$

putting N and M for the respective functions in the numerators of the fractions.

Now since P A M is half the angle P A P′, it is easy to see that we must have $2 \times$ angle O′A M $=$ P A O $-$ P′A O.

But,
$$\cos PAO = \frac{\theta^2 + r^2 - a^2}{2r\theta}; \quad \cos P'AO = \frac{\theta'^2 + r^2 - a^2}{2r\theta'};$$

and
$$\sin PAO = \sin AOP \times \frac{PO}{PA} = \frac{aM}{r\theta}; \quad \sin P'AO = \frac{aM}{r\theta'};$$

consequentlv we have, first,

$$\sin 2OAM, \text{ or } \frac{2xy}{r^2} = \frac{aM}{2r^2}\left\{\frac{\theta'^2 + r^2 - a^2}{\theta\theta'} - \frac{\theta^2 + r^2 - a^2}{\theta\theta'}\right\} = \frac{aM}{2r^2\theta\theta'}(\theta'^2 - \theta^2), \qquad (a)$$

and, secondly,

$$\cos 2OAM = \frac{y^2 - x^2}{r^2} = \frac{1}{r^2}\left\{\frac{(\theta^2 + r^2 - a^2)(\theta'^2 + r^2 - a^2)}{4} + a^2M^2\right\}.$$

Now we have further,

$$\left.\begin{aligned}\theta^2 &= a^2 + r^2 - 2ar . \cos AOP = a^2 + r^2 - 2aN\\ \theta'^2 &= \qquad\qquad\qquad\quad = a^2 + r^2 + 2aN\end{aligned}\right\}; \qquad \theta'^2 - \theta^2 = 4aN,$$

which substituted in the values of sin 2 O A M and cos 2 O A M above, give the equations

$$xy . \theta\theta' = a^2 . MN,$$
$$(y^2 - x^2) . \theta\theta' = r^4 + a^2(M^2 - N^2),$$

and, eliminating $\theta\theta'$ from these, we obtain

$$a^2(y^2 - x^2) . MN = xy\{r^4 + a^2(M^2 - N^2)\}.$$

In this it only remains to substitute for M and N their values $y . \cos a - x . \sin a$, and $y . \sin a + x . \cos a$, which done, the whole will be found divisible by r^4, and will reduce itself to the very simple equation

Light. Part IV.

$$x\,y = a^2 \,.\, \sin a \,.\, \cos a = \frac{a^2}{2} \,.\, \sin 2\,a.$$

When the axes are near it is an hyperbola.

The black cross then is an hyperbola, passing through the poles P, P′, and having the planes of primitive polarization, and one perpendicular to it (C D and $c\,d$) for its asymptotes, and which as a approaches to 0, or 90°, approaches nearer and nearer to its asymptotes, with which it at last coincides in the limiting case; all which particulars are exactly conformable to fact, and may easily be verified by turning a plate of nitre round between crossed tourmalines. When the inclination of the axes is so considerable, that the rings about both poles cannot be seen at once, there will arise modifications from the substitutions of the sines, &c. of arcs for the arcs themselves, which it is not worth while to enter into.

1073. Empirical formula expressing the phenomena of dichroism

To return now to the phenomena of dichroism. That portion of the light transmitted by a biaxal coloured medium which has relation to the optic axes, and which forms the sombre brushes of colour (in fig. 206,) and the bright spaces which divide them, have evidently for their analytical expression a function of the form

$$\mathrm{Y} \,.\, (\cos 2\,\phi)^2 + \mathrm{B} \,.\, (\sin 2\,\phi)^2; \quad (a)$$

where Y and B are functions of λ, and represent the ordinates of the types of two fundamental tints, ϕ representing as before the angle P N A, fig. 208, or the angle made by the plane of ordinary polarization with the principal section. But besides this, the phenomena described by Dr. Brewster, as exhibited by the iolite, require us to admit two other portions, which may be more naturally referred, not to either of the optic axes but to the line C O (fig. 209) bisecting them, and having for its expression a function of the form $y \,.\, \cos \mathrm{O\,A}^2 + b \,.\, \sin \mathrm{O\,A}^2$. In this mineral, when exposed to common light (or to polarized, provided we place its principal section at right angles to that of polarization,) the lateral brushes A, B, fig. 206, are blue, and the bright rays which divide them, passing through the poles P, P′ are white, or yellowish white, and so far the phenomena agree with the expression (a) if we suppose Y to represent a bright yellowish white, and B a blue. But according to that expression alone, the blue spaces should be continued down to the equator C a b D, fig. 206, and there ought to be two directions C D and $a\,b$ in which the mineral viewed transversely to the axis of the prism (which is perpendicular to the plane C a b D) should appear yellow, and two others, m n and p q, in which it should transmit a blue colour, while in the direction of the axis O it should appear yellow. Now, on the contrary, the equatorial colour is nearly uniform and pale yellow, while that along the axis O is blue; and in proceeding from the equator toward the axis O of the prism, the yellow diminishes, and the blue gains strength, whether we set out from C and D, or from a and b, precisely as would be indicated by the other formula

$$y \,.\, (\sin \mathrm{O\,A})^2 + b \,.\, (\cos \mathrm{O\,A})^2,$$

y representing a yellow white and b a blue tint. If, therefore, we put O A $= \nu$, the joint expression

$$\mathrm{T} = (\mathrm{Y} \,.\, \cos 2\,\phi^2 + \mathrm{B} \,.\, \sin 2\,\phi^2) + (y \,.\, \sin \nu^2 + b \,.\, \cos \nu^2); \quad (b)$$

will be found to represent pretty correctly the variations of colour as far as they can be judged of by the eye. Thus, at O where $\nu = o$, and $\phi = 90°$, we have $\mathrm{T} = \mathrm{Y} + b$, which may indicate either a yellow, a white, or a blue, according as we suppose Y or b to be predominant. The fact being, that the tint at O is blue, we must suppose the latter to express the more decided colour. As we proceed from O along the sections O C, O D, or O a, O b, in both of which $\sin 2\,\phi = o$, we have

$$\mathrm{T} = (\mathrm{Y} + y \,.\, \sin \nu^2) + b \,.\, \cos \nu^2 = (\mathrm{Y} + b) + (y - b) \,.\, \sin \nu^2$$

Now y expressing a yellow white and b a strong blue, $y - b$ will express a proportionally vivid yellow, and therefore the blue tint Y + b seen along the axis will be diluted with more and more yellow as we approach the equator; at P P′, then, (by a proper assumption of numerical values) it will be rendered nearly neutral, after which the yellow will predominate, and, at the equator, will remain alone sensible, the expression for T then becoming $\mathrm{T} = \mathrm{Y} + y$, at the points C, a, b, D. Let us next consider the case when $\cos 2\,\phi = o$, or $\phi = 45°$, that is to say, along the axes or most intense lines of the lateral brushes. In this case we have

$$\mathrm{T} = \mathrm{B} + (y \,.\, \sin \nu^2 + b \,.\, \cos \nu^2) = (\mathrm{B} + b \,.\, \cos \nu^2) + y \,.\, \sin \nu^2.$$

Now if we suppose B and b to represent blue tints, since (in the case of iolite) the angle between the axes or P P′ $= 62°\,50'$ and O P $= 31°\,25'$, we have in the immediate vicinity of the poles, $(\sin \nu)^2 = \frac{1}{4}$ nearly, and $\cos \nu^2 = \frac{3}{4}$, so that in the immediate neighbourhood of P the tint of the most intense part of the brushes will be $\mathrm{B} + \frac{3}{4}\,b + \frac{1}{4}\,y$, which, on very reasonable suppositions of the numerical values of B, b and y will denote a full and rich blue. But as we approach the equator at m, n, p, q, $\cos \nu^2$ diminishing and $\sin \nu^2$ increasing, the sombre tint B is continually more feebly reinforced by the tint $b \,.\, \cos \nu^2$ and more strongly counteracted by $y \,.\, \sin \nu^2$, till at length it will be overpowered, and the colour in these points, as in C, a, D, b, will be yellow only somewhat less decided than in the latter, its tint being represented by $\mathrm{T} = y + \mathrm{B}$ instead of $y + \mathrm{D}$.

1074. Determination of the coefficients from the observed tints.

In general, if we put A for the tint transmitted along the axis O of the prism, P for that seen along the poles, L for that of the lateral branches at their origin close to the poles, and E for the mean equatorial tint, we shall have for determining Y, y, B, b, the equations

$$\mathrm{A} = \mathrm{Y} + b, \quad 2\,\mathrm{E} = 2\,y + \mathrm{B} + \mathrm{Y},$$
$$\mathrm{P} = \mathrm{Y} + y \,.\, \sin a^2 + b \,.\, \cos a^2; \quad \mathrm{L} = \mathrm{B} + y \,.\, \sin a^2 + b \,.\, \cos a^2,$$

on elimination from these, it will appear that there is an equation of condition to be satisfied, *viz.*

$$2\,(\mathrm{A} - \mathrm{P}) = (2\,\mathrm{A} - 2\,\mathrm{E} \quad \mathrm{P} + \mathrm{L}) \,.\, \sin a^2; \quad (c)$$

Light. and that supposing it satisfied, one of the tints, as y, will (so far as these conditions are concerned) remain arbitrary, and the others will be given by the equation

$$\left.\begin{aligned} 2\,Y &= 2\,E + P - L - 2\,y \\ 2\,B &= 2\,E - P + L - 2\,y \\ 2\,b &= 2\,A - 2\,E - P + L + 2\,y \end{aligned}\right\};\quad (d)$$

in which y must, however, be such as to render Y, B, b real tints, *i. e.* expressed by positive numbers.

1075. Application to the iolite. To apply this, for example's sake, to the case of the iolite, let us regard every white ray as consisting of two complementary rays of bright yellow and bright blue of equal efficacy; and suppose that by observation we have ascertained its equatorial tint E to be a very pale but strongly luminous yellow white, consisting of 110 such yellow rays, and 99 such blue ones, producing a joint intensity = 209. Moreover, let the tint seen along the axis of the prism (A) be a blue, of a good colour, but considerably less intensity, represented by 10 such yellow + 20 such blue rays = 30. That seen along the optic axes (P) to be a white represented by 36 yellow + 36 blue = 72, and that of the most intensely coloured portions of the lateral brushes = L to be a stronger blue than that seen in the axis of the prism, such as may be represented by 28 yellow + 66 blue = 94. These numbers are chosen so as to satisfy the equation of condition, taking $a = 30°$, and if we substitute them we shall find

$y + y$ = 114 yellow + 84 blue; B + y = 106 yellow + 114 blue; $y - b$ = 104 yellow + 64 blue,

y remaining indeterminate; if we suppose its composition to be m yellow + n blue, we may determine m and n by the two conditions that b shall (as we have before supposed) represent, a pure blue without any mixture of yellow, and Y a very pale yellow, such as would result from a mixture of yellow and blue in the ratio of 10 to 9. These conditions are satisfied by taking m = 104 and n = 75; so that we have, finally,

Y = 10 yellow + 9 blue; B = 2 yellow + 39 blue;

y = 104 yellow + 75 blue; b = 0 yellow + 11 blue;

and these being taken for the values of the coefficients in the expression (*b*) Art. 1073, it will be found on trial to reproduce the tints actually observed. In fact, the extreme equatorial tints being y + Y and y + B, will be respectively represented by 114 yellow + 84 blue, and 106 yellow + 114 blue; the former is a very pale yellow, but highly luminous, being equivalent to 30 rays of yellow diluted with 168 of white; while the latter is a blue so pale as to be undistinguishable from white, and also highly luminous, being equivalent to 8 rays of blue diluted with 212 of white.

1076. Phenomena exhibited by various crystals. The reader will perceive that the formula in question is merely empirical, and that more numerous experiments than we possess will be required to establish or disprove it. It is unfortunately, however, difficult to meet with biaxal crystals sufficiently dichromatic for the purposes of decisive experiment, and at the same time large and transparent enough to admit of being cut into the forms and examined in the directions required, through a thickness sufficient for a full developement of their colours. Such are indeed hardly less rare than the most precious gems; and this circumstance is a great obstacle to the advancement of our knowledge in one of the most interesting branches of optical inquiry, which that of dichroism certainly deserves to be considered. Among artificial crystals, however, there is room to suppose that subjects fit for such experiments may be met with. One remarkable instance of dichroism among these has been mentioned in the sub-oxysulphate of iron. To this we may add the potash-muriate of palladium, which exhibits along the axis of the four-sided prism in which it crystallizes a deep red, and in a transverse direction a vivid green. (Wollaston, *Phil. Trans.* 1804. On a new metal in Crude Platina.) The curious property of the purpurates of ammonia, potash, &c. described by Dr. Prout, (*Phil. Trans.* 1808,) which by transmitted light exhibit an intense red, and by reflected, on one surface, a dull reddish brown, and on another a splendid green, appears referable, not so much to the principles of dichroism properly so called, as to some peculiar conformation of the green surfaces, producing what may be best termed a *superficial colour*, or one analogous to the colour of thin plates, and striated or dotted surfaces. A remarkable example of such superficial colour, differing from the transmitted tints, is met with in the green fluor of Alston-moor, which on its surfaces, whether natural or artificial, exhibits, in certain lights, a deep blue tint, not to be removed by any polishing.

1077. Unequal effects of heat on the colours of the two pencils. Dr. Brewster has shown that the action of heat often modifies in a very remarkable manner the colour of doubly refracting crystals, producing a permanent change in the scale of absorption of the crystals as affecting one of the pencils and not the other. Thus, having selected several crystals of Brazilian topaz which displayed no change of colour by exposure to polarized light, (and in which, of course, the types of both absorptions must have been alike,) and bringing them to a red heat, or even boiling them in olive oil, or mercury, they experienced a permanent change, and had acquired the property of absorbing polarized light unequally. He then took a topaz in which one of the pencils was yellow and the other pink; and by exposing it to a red heat, he found the extraordinary pencils more powerfully acted on than the ordinary, the yellow colour being discharged entirely from the one, while only a slight change was produced in the pink tint of the other. This change of colour in the topaz by heat (though not its intimate nature) is well known to jewellers, who are in the habit of thus developing in this gem a colour more highly prized. It is remarkable, that while hot the topaz is perfectly colourless, and acquires the pink colour gradually in cooling. By the repeated action of very intense heat Dr. Brewster was never able to modify or remove this permanent pink tint. How far violent compression, slow application, and abstraction of the heat, or other modifying circumstances, might prevent its developement, it

would be interesting to examine; since we cannot help being otherwise struck by the force of the argument geologists may draw, from the existence in rocks of a mineral which mere elevation of temperature unaccompanied with change of composition, thus irrevocably alters.

1078. General character of dichroite crystals. Effects of uncrystallized inter rupting films.

One general character of all dichroite bodies is, that when natural light is transmitted through a plate of sufficient thickness, in any direction not coincident with one of the optic axes, the emergent beam is wholly or partially polarized by reason of the unequal action of the medium on the two pencils, and the consequent suppression of one of them. And, in general, whatever cause tends to interfere unequally with their free transmission through a medium, will produce a similar effect. Thus, for example, if the continuity of a doubly refracting medium be interrupted by a film of any uncrystallized substance, since the two pencils by reason of their angular separation are incident on this film at different angles; and since, moreover, their relative refractive indices, with respect to the medium composing the film, differ, they will undergo partial reflexion at the film in different proportions, and thus an inequality will arise in the parts transmitted. If the refractive index of the film be precisely equal to the ordinary refractive index of the crystal (supposed, for simplicity, to be uniaxal) the ordinary ray, it is evident, will undergo no disturbance or diminution, while the extraordinary will be changed in direction and diminished in intensity by partial reflexion at its ingress and egress, at every such film which may exist in the medium. If the films be extremely numerous, and if, moreover, they be not disposed in planes, but in undulatory or irregular surfaces through the medium, this will make no difference, so far as the ordinary ray is concerned, which will still pass undisturbed through the system, (except so far as any opacity in the matter of the films may extinguish a portion of it;) but the extraordinary ray will be rendered confused, and dispersed, its egress from the films not being performed (by reason of their curvature) at the same angles as its ingress, and that irregularly, according to their varying inclination. Hence will arise a phenomenon precisely such as is presented by the agate, and other irregularly laminated bodies, through plates of which, if a luminary be viewed, it is seen distinctly, but as if projected on a curtain of nebulous light; and if examined with a tourmaline, or doubly refracting prism, the distinct image, and the nebulous light, are found to be oppositely polarized. If we examine a piece of agate with a magnifier, the laminated structure and unequal refraction of the laminæ are very apparent; it appears wholly composed of a set of exceedingly close layers, not arranged in planes, but in undulating or crinkled lines like a number of figures of 333333 placed close together. The planes of polarization of the nebulous and distinct image are parallel and perpendicular to the general direction of the layers, which through any very small portion of the substance is generally pretty uniform.

Phenomena of agate.

1079. Action of a crystallized interrupting film. Fig. 210.

But the film interposed may, itself, be crystallized, and inserted between adjacent portions of a regular crystal, according to the crystallographic laws which regulate the juxtaposition of the molecules at the common surfaces of macled or hemitrope crystals. Let A D E F (fig. 210) be such a plate interrupted by a crystallized lamina B C E F, bounded by parallel planes, and let us consider what will happen to a ray S *a* incident at *a*. It is evident, that were the crystallized lamina away, or were its molecules homologously situated with those of the portions on either side of it; in the latter case, we should have an uninterrupted crystal; in the former, two prisms disposed with their principal sections parallel, and acting in opposition to each other; in either case, the emergent ordinary and extraordinary pencils separated by double refraction at the first surface will emerge parallel to the incident ray, and therefore to each other. But the principal section of the crystallized film being non-coincident with those of the two prisms A B E, C F G, it will alter the polarization of the portions *a b*, *a c*; and in place of their being, as in the former case, each refracted singly by the second prism C F G, they will now each be refracted doubly, so that in place of two emergent rays there will now be four. The subdivision of the rays within the interposed lamina may evidently be disregarded, for they will be refracted in passing from the film into the second prism in the same *direction*, where contiguous, as they would were an infinitely thin plate of air interposed. Now, in that case, they would emerge from the film in pairs respectively parallel to the incident rays *a b*, *a c*, and therefore to each other. Hence the refraction at the second prism will be precisely the same as if the lamina were suppressed, and in its place the rays *a b*, *a c* had received at *a* the polarizations they acquire by its action. Now, these being in opposite planes, it is evident that each of the rays *a b*, *a c* would undergo both an ordinary and an extraordinary refraction. Let us denote these four emergent pencils so arising by O O, O E, E O, E E, and suppose *a b* to be the direction taken by the ordinary refracted portion of S *a*, and *a c* that of the extraordinary. Then, since O O has been refracted ordinarily by the prism C F G, and was incident on it in the direction of the ordinary ray *a b*, its direction on emerging will be parallel to S *a*. Similarly, E E is refracted extraordinarily, and being incident in the direction *b c* of the extraordinary portion of S *a*, it also will emerge parallel to S *a*, and thus the two rays O O, E E will emerge parallel, and their systems of waves will be superposed. But the portions O E and E O, the one being incident in the ordinary direction, but refracted extraordinarily, the other incident in the extraordinary direction and refracted ordinarily, will neither emerge parallel to the original ray S *a*, nor to each other; and this will give rise to two lateral images, one on each side of the central or direct image, which will have, moreover, an intensity equal (except in extreme cases) to the sum of those of the lateral images.

1080. Phenomena of interrupted Iceland spar.

If the film E B C F be very thin, or if either of its optic axes be nearly coincident with the direction in which the light traverses it, the difference of paths and velocities within it will give rise to an interference of the pairs of rays going to form either pencil emergent from the film, and thus will arise the colours of the rings in each image. Those on either side the central one will be consequently tinged with the respective colours of the primary and complementary set of rings; while the central image, being formed by the precise superposition of two similar complementary pencils will appear white.

All these phenomena actually occur, and have been described by Dr. Brewster, and explained by him on the principles here laid down, in certain not uncommon specimens of Iceland spar, which are interrupted by such

hemitrope films, passing through the longer diagonals of opposite faces of the primitive rhomb. If we look at a candle through such an interrupted rhomb, it will be seen accompanied by a pair of lateral images such as here described, and exhibiting frequently the complementary tints with great splendour.

1081. Phenomena of idiocyclophanous crystals. If the luminary from which the ray S *a* issues be small, the lateral images will be separated by a dark interval from each other and from the central one, but if large they will overlap. If infinite (as where the uniform light of the sky is viewed) all the images will be superposed. But the field of view will not necessarily be uniform and white. The central image will form an intense white screen, or ground, on which will be projected the lateral ones. Now, if the film be so constituted as to have within the visible field of view of one only of the lateral images the pole of one of its sets of rings, (which will be the case whenever one of its optic axes is not very remote from perpendicularity to the surface of the plate A D, so as to admit of one of the rays O E or E O traversing the film in the direction of its axis,) that set of rings will not be seen projected centrally on the corresponding set complementary to it of the other lateral image, by reason of the angular separation of these two images. Of course its colours will not be neutralized, and it will be visible *per se*, though very faint, being diluted by the whole white light of the central image (O O, E E) and by the whole visible and nearly uniform portion of the other lateral one (O E.).

1082. Fig. 211. This is not the only way in which a crystal perfectly colourless may exhibit its sets of rings by exposure to common daylight without previous polarization, or without subsequent analysis of the transmitted pencil. The general mass of the crystallized plate may have one of its optic axes in the direction of the visual ray, as in fig. 211, and the portion of it C D *d c* included between two films B C *c b* and D *d e* E will then form precisely such a combination as that above described, and will exhibit a set of rings feeble in proportion to the rarity and minuteness of the films, and the consequently small area of their outeropping surfaces B C, D E. These are not hypothetical cases. Dr. Brewster states himself to have met with specimens of nitre exhibiting their rings *per se*. Such are rare. But in the bicarbonate of potash it is an accident of continual occurrence; and, indeed, almost universal. The films in both cases are easily recognised, and their position and that of the system of rings seen leave no doubt of the correctness of the explanation here given. Such crystals, of which more will no doubt be hereafter recognised, may be termed *idiocyclophanous* till a better term can be thought of.

§ XII. *On the effects of Heat and Mechanical Violence in modifying the action of Media on Light, and on the application of the Undulatory Theory to their explanation.*

1083. General account of the phenomena. It was ascertained independently, and about the same time by Dr. Seebeck and Dr. Brewster, that when glass, which in its ordinary state offers none of the phenomena of double refracting media, is heated or cooled unequally, it loses this character of indifference, and presents phenomena of coloration, &c. analogous, in many respects, to those exhibited by doubly refracting crystals. If the heat communicated be below the temperature at which glass softens, the effect is transient, and vanishes when the glass attains a uniform temperature throughout its substance, whether by the equable distribution of the caloric throughout its mass, or by its abstraction in cooling. But if the temperature communicated be so high as to allow the molecules of the glass to yield to the mechanical forces of dilatation and contraction produced in the act of cooling and take a new arrangement, the effect is permanent, and glass plates so prepared have many points of resemblance with crystallized bodies. Dr. Brewster afterwards ascertained, that mechanical compression or dilatation applied to glass, jellies, gums, and singly refractive crystals (such as fluor spar, &c.) is capable of imparting to them the same characters. If the medium to which the pressure is applied be perfectly elastic, like glass, the effect, like that of heat, is transient. But if during the continuance of the compression or dilatation, the particles of the medium are allowed to take their own arrangement and state of equilibrium, then when the external force is withdrawn a permanent polarizing character will be found to exist.

1084. Accompanied by double refraction. As periodical colours are not produced in phenomena of this class without a resolution of the incident light into two pencils moving with different velocities, and as a difference of velocities is invariably accompanied with a difference of refraction at inclined surfaces, it might be expected that media thus under the influence of heat or pressure should become doubly refractive. This has been verified by direct experiment by M. Fresnel, who has shown that a peculiar species of double refraction is thus produced.

1085. Effect of heat analogous to that of pressure. As the unusual heating or cooling of glass and other substances, is well known to produce in the parts heated or cooled a corresponding inequality of bulk, and thus to bring the parts adjacent into a state of strain in all respects analogous to that arising from mechanical violence, and as, in fact, the effects of heat in communicating double refraction to glass, whether transient or permanent, are all, as we shall see, (with one very obscure and doubtful exception) commensurate with the amount of the strain thus transiently or permanently induced, we have little hesitation in regarding the inequality of temperature as merely the remote, and the mechanical tension or condensation of the medium as the proximate cause of the phenomena in question, and are very little disposed to call in the agency of a peculiar crystallizing fluid, endowed with properties analogous to those of magnetism, electricity, &c., to account for the phenomena, still less to regard media under the influence of heat or pressure as in any way thereby rendered more crystalline than in their natural state of equilibrium.

1086. In gasiform, or fluid media, no such phenomena are observed to be developed by either heat or pressure; the reason is obvious, the pressure is equally distributed in all directions, and the elasticity of the ether (on the undulatory hypothesis) preserves its uniformity.

But in solids the case is different. The molecules cannot shift their places one among the other, and the

Light. Part IV.

effect of a compression in any direction is, *first*, to urge contiguous particles nearer together in that direction, and thereby to call into action their repulsive forces, more than in the natural state, to maintain the equilibrium; secondly, but much more slightly to urge contiguous particles in a direction perpendicular to that of the pressure laterally *asunder*, by reason of the increase of the oblique repulsive force developed by the approach of the molecules in the line of pressure to those which lie obliquely to that line. But this action, which in fluids would cause a motion of the lateral particles out of the way, in solids is ultimately equilibrated by an increase of the attractive forces of the adjacent molecules in a line perpendicular to the line of pressure; and thus we see that every external force applied to a solid is accompanied with a condensation of its particles in the direction of the force and a dilatation in a perpendicular direction. It is probable, however, that this latter is extremely minute, on account of the rapid diminution of the molecular forces by increase of distance, rendering the diagonal action insensible. But the former may easily be conceived to produce in the ether, in virtue of its connection (whatever it be) with the molecules of refracting media, a difference of elasticity in the two directions in question, accompanied with all the necessary concomitants of interfering pencils, periodical colours, and double refraction. The effect of dilatation will be the converse of that of compression, the direction of maximum elasticity in the one case being that of minimum in the other.

Mode of action of pressure on the molecules of solids.

1087. Effects of compression described.

These views are in perfect accordance with the experiments described by Brewster and Fresnel on compressed and dilated glass. According to the former (*Phil. Trans.* 1816. vol. 106) the effect of pressure on the opposite edges of a parallelepiped of glass is to develope in it "neutral" and "depolarizing axes," the former parallel and perpendicular to the direction of the pressure, the latter 45° inclined to them; in other words, a parallelepiped of glass so compressed, will when exposed to a ray polarized in the plane parallel or perpendicular to the sides to which the pressure is applied, produce no change in its polarization and develope no periodical colours, while if polarized in 45° of azimuth with respect to those sides, it will develope a tint, descending in the scale of the coloured rings as the pressure increases.

1088. Explanation on the undulatory doctrine.

In this case, if the pressure be uniformly applied over the whole length of each opposite side, the elasticity of the ether in every point of the plate will be uniform in either direction at every point of the plate, being a maximum in one, and a minimum in that at right angles to it. The incident light therefore if polarized in azimuth a^0 will resolve itself into two pencils of unequal intensity (*viz.* $\cos a^2$ and $\sin a^2$) polarized in these two planes, and differing at their egress by an interval of retardation proportional to $t \times (v' - v)$, where t is the thickness traversed, and $v' - v$ the difference of velocities of the pencils, which when received on a double refracting prism will (as in the case of a crystallized plate (Art. 969) give rise to complementary periodical tints in the two images, the extraordinary image vanishing when $a = 0$, or 90, and the contrast being a maximum at 45°. It is, of course, extremely difficult to give such a perfect equality of pressure, so that we must not be surprised if a perfect uniformity of tint over the whole surface of the glass should not take place. In the experiment, however, described by Dr. Brewster (Prop. I. of the Memoir cited) this seems to have been the case.

1089. Opposite effects of compression and dilatation.

If we suppose the elasticity of the ether in compressed glass *less* in the direction of the force applied (and where consequently the medium is densest, according to the general law) than in the perpendicular, the contrary will be the case in dilated. Hence, supposing the forces equal, in two similar plates, the extraordinary waves, or those whose vibrations are performed in the direction of the pressure, and which are therefore polarized at right angles to that direction, will advance most rapidly in the former case, the ordinary in the latter. Consequently, if we regard the interval of retardation or the tint, $t\,(v' - v)$ as negative in the former case, it will be positive in the latter; and the tints in the two cases will present the opposite characters of those exhibited by doubly refracting crystals of the two classes described in Art. 940, *et seq.* see also Art. 803, as negative and positive, or repulsive and attractive. Two such plates, therefore, placed homologously, or with the directions of the forces coincident, ought to neutralize each other, and if crossed at right angles should reinforce each other; and in general, if t be the thickness and f the compressing force applied to any plate (supposing the difference of velocities to be proportional to the force, and regarding dilating forces as negative) we shall have for homologously situated plates

$$\mathrm{T} = \text{tint polarized by any number of plates}$$
$$= (f.t + f'.t' + f''.t'' + \&c.)$$

Law of superposition.

In the case of crossed plates the thicknesses of those placed transversely are to be regarded as negative, just as in the case of the superposition of crystallized plates. All these results are conformable to the experiments of Dr. Brewster.

1090. Tints produced by bending a glass plate.

The phenomena of contracted and dilated glass may most easily and conveniently be produced by bending a long parallel plate of glass having its longer edges polished, and passing the light through them across its breadth. In this case, as in all cases of flexure, the convex surface is in a state of dilatation, and the concave of compression, while there exists a certain intermediate line or boundary between these oppositely affected regions in which the substance is in its natural state of equilibrium, and on both sides of which neutral line the degree of strain increases as we *recede* from it towards either surface. Fig. 212 is a section of such a bent plate, much exaggerated, through which light, polarized in a plane 45° inclined to its length, has been passed and analyzed as usual. The neutral line is marked by a divided black stripe, and the tints on either side of it descend in Newton's scale, being arranged in stripes disposed according to the lines 11, 22, 33, 44, &c. The tints, however, on opposite sides of the neutral line have opposite colours, being positive on the side of the dilatation, or towards the convexity, and negative on the compressed or concave side. In a plate of glass 1.5 inch broad, 0.28 thick and six inches long, Dr. Brewster developed seven orders of colours before the glass broke with the bending force applied. This experiment affords an exceedingly beautiful illustration of the action of compressing and bending forces on solids, and furnishes ocular evidence of the state of strain into which their several parts

Fig. 212.

State of strain ascertained by the tints.

are brought by external violence. The ingenuity of Dr. Brewster has not overlooked its application to the useful and important object of ascertaining the state of strain and pressure on the different parts of architectural structures, as stone bridges, timber framings, &c., by the use of glass models actually put together as the buildings themselves. We must recollect always, however, that the information thus afforded will only be distinct when the load intended to be sustained is many times the weight of the materials.

1091. Effects of several co-existing strains. If a plate of glass be subjected to several distinct compressions and dilatations in different directions, Dr. Brewster finds, that its action will be the same as the combined action of several plates each subjected to one of the forces employed. Thus a square of glass compressed equally on all its four edges exerts no polarizing action.

1092. Pressure applied at a point. If a pressure be applied at a single point of a mass of glass, or rather at two opposite points, it will diverge from these points in all directions into the mass, and the lines of equal pressure, which are in fact the isochromatic lines, must have their form determined in some measure by the figure of the compressing screw or tool at its point of contact with the glass, for this figure regulates the form and curvature of the indentation immediately under it. Dr. Brewster has figured several of the curves produced by the application of such pressure to different parts of the same parallelepiped of glass, for which the reader is referred to his Paper, as well as for a variety of beautiful figures produced by crossing plates differently strained.

1093. Effects of vibration. M. Biot has observed, that in some instances glass maintained in a state of vibration by the action of a bow or otherwise, depolarizes light, *i. e.* restores the vanished pencil. This is a necessary consequence of the alternate compressions and dilatations which follow each other in rapid succession in all the vibrating molecules. Nodal lines (see ACOUSTICS) being exempt from such variations of density ought to be marked by black bands, and may thus, perhaps, be rendered evident to the eye.

1094. Polarization by compressed jellies. When masses of jelly (especially of isinglass) are pressed between plates they acquire a polarizing action. If dilated by proper management, and in that state allowed to dry and harden, the character so impressed, according to Dr. Brewster, is permanent when the dilating force is removed; to explain which, we must consider that the exterior coats indurate more rapidly than the interior, and when they have acquired the consistency of a solid, they will be capable of resisting the subsequent contraction of the interior portions and keeping them in a dilated state, even when the original dilating force is removed. That force only served to determine the figure and dimensions of the exterior crust, and when once that crust is fully formed and indurated, it becomes capable of maintaining them without the further aid of the cause which gave them rise. The polarizing power of isinglass thus developed is very great, and even exceeds that of some doubly refractive crystals, such as beryl; a plate of isinglass whose thickness is 624 polarizing the tint which would be reflected by a plate of air whose thickness is unity, while a plate of beryl parallel to the axis, to polarize the same tint, will require a thickness = 720. Glass compressed, or dilated, by an equal force, would require a thickness (according to Dr. Brewster) = 12580 to produce the same tint.

1095. Transient effects of heat below the softening point. We come now to consider the transient effects of unequal temperature below the softening point of glass. The immediate effect of an increase or diminution of temperature in one point of a piece of glass, is to produce a mechanical strain on all the surrounding part, which if the difference of temperature is considerable, is of the utmost violence, and capable of breaking asunder the thickest pieces of glass; an effect with which every one is familiar. Now, as we know that strain alone developes a polarizing action, the rule of philosophy, "*non plures causas admitti debere*," *&c.* which forbids the admission of a second cause when one adequate to the effect is known to be in action, will hardly justify us in attributing a peculiar action to the caloric, independent of its power of altering the dimensions of matter.

1096. Case of a rectangular plate of glass heated at one edge. When a heated iron bar is applied along the edge of a parallelepiped of glass held in a polarized beam, analyzed as usual, the vanished image is restored in various degrees of intensity in different parts of the glass. The *neutral axes* are parallel and perpendicular to the heated edge, and the axes in whose azimuth the tint polarized is the strongest, at 45° of inclination. If held in that azimuth, the first effect of the heat is to produce a line, or, as it were, a wave of white light at the heated edge, which advances gradually upon the glass, driving before it a dark and undefined wave. Nearly at the same instant, and *long before the slightest increase of temperature can have reached the further extremity of the glass plate*, a similar but fainter white wave advances from the edge opposite to the heated one, driving before it a similar undefined dark wave; and at no perceptible interval of time another white fringe appears in a very diluted state about the centre of the plate, advancing equally towards the heated edge on one side and that most remote on the other, and thus condensing the two undefined dark waves into two black fringes. The white tints are succeeded by tints of a lower order in the scale of colour, yellow, red, purple, blue, &c., till at length the whole scale of the colours of thin plates is seen arranged in four sets of fringes parallel to the heated edge, and having for their origins the black fringes above mentioned. At the same time, other lateral fringes are produced along the edge perpendicular to the heated one. Thus in all six sets are seen; two *exterior*, *viz.* those parallel to the heated edge, and *outside* of the black fringes; two *interior*, in the same direction, but between the black fringes; and two *terminal*, along the lateral edges. The whole phenomena is as represented in fig. 213. The fringes along the heated edge A B are most distinct and numerous, those along the opposite, C D, less so, and the interior and terminal fringes least of all.

Fig. 213.

1097. Action of heat in straining the glass. As glass is an extremely bad conductor of heat, and as culinary heat is propagated through glass *entirely* by conduction, it follows, that the sudden application of an elevated temperature to the edge A B must produce a dilatation in it, not participated in by the rest of the glass. If, therefore, the stratum of molecules A B were detached from the rest of the glass, it would elongate itself so as to project at its two ends beyond the edges A C, D B. When the heat of this stratum communicated itself to the next, that also would elongate itself, but in a less degree; and thus after a very long time, during which the heat had penetrated to the farther extremity of the glass, its outline would assume the form *a* C D *b*, the lines *a* C, *b* D being certain curves depending on the law of propagation and the time elapsed. This would be the state of things were the glass plate composed

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of *discrete* strata, each of which could dilate independently of all the rest. And since in each of these (regarded as infinitely thin) the temperature and strain would be uniform, there would arise no polarizing action. But, in reality, the case is quite different; every stratum is indissolubly connected along its whole extent with the strata adjacent, and can neither expand nor contract without forcing them to participate in its change of dimension. In so far, then, as two adjacent strata participate in the change of temperature they expand together; but when one is hotter than the other, the former is found to expand *less*, and the other *more* than if they were independent. Now the strain thus induced on any stratum is not, like the caloric which causes it, confined by the conducting power of the medium, but propagates itself *instantly* (with diminished energy) to the strata beyond, by reason of the mutual action of the molecules.

1098. State of the various regions of the plate as to strain determined. Fig. 214.

The general problem, then, to investigate the actual state of strain of any molecule at any moment is one of some complexity, inasmuch as it depends at once on the laws of the slow propagation of heat, and the instantaneous but variable participation of change of figure necessary to establish among the particles a momentary equilibrium under the circumstances of temperature at the time; but, without attempting minutely to analyze the effects, if we content ourselves with acquiring a general idea how they arise, we shall find little difficulty. For in fig. 214, if we conceive the stratum A B *b a* adjacent to the border A B to be dilated by the heat, the rest of the glass retaining its original temperature; if this stratum could expand separately, its edges A *a*, B *b* would project out beyond the general edges C *a*, D β; and if we regard two terminal strata C A E G, D B F H, as detached from the interior portion C D β *a*, and free to move by the force applied at their extremities A, B, they would be raised by the dilatation of the portion A B *b a* into the situation represented in the figure, turning round C, D as fulerums, and leaving triangular intervals C *a a*, D *b* β vacant, and in these circumstances there would be no strain on any part of the system. But the cohesion of the glass prevents the formation of these vacancies, and the bars or levers C A E G, D B F H cannot move into this situation without dragging with them, and therefore distending the strata of C D β *a*. Let P Q be any such stratum, and let it be distended to *p q*. Then by its elasticity it will tend to draw the bars C A E G and B D H F together; and its action will therefore tend, first, to produce a pressure on the fulcrums C, D, urging the points C D together, and therefore bringing the stratum C D into a state of compression. Secondly, to produce also a pressure on A *a*, B *b*, or a resistance to the dilatation of A B *b a*, which its increased temperature would naturally produce. It will therefore tend to *compress back* the strata of A B *b a* into a smaller length than what would be natural to them in their heated state, *i. e.* to bring *them* also into a relatively compressed state. Thirdly, the tension of *p q* being sustained at C, D and A, B, will tend to bend inwards the levers A C G E, B D H F, rendering them concave at the edges G E, H F, and convex at C A, D B, and thus distending the lines C A, D B, and compressing the strata adjacent to E G, H F.

1099. Production of fringes of opposite characters. Fig. 215.

From this reasoning it is clear, that the glass, in consequence of these various strains, will assume a figure concave on all its edges, but chiefly so at the lateral ones A C, D B, as in fig. 215; and that the state of strain of its various parts will be as there expressed, all the edges being compressed, but principally A B and C D, and the interior distended. The limit between the distended and compressed portions parallel to A B must necessarily be marked by neutral lines *a b*, *c d* on either side of which the strain will increase, being a maximum in the middle and on or near the edges. Consequently, it ought to polarize four sets of fringes, having *a b*, *c d* for their origins, and of which the two external (or those between these lines to the edge) ought to have a character opposite to those of the internal, the portion of the intromitted pencil polarized parallel to A B being propagated faster than that parallel to A C in the one case, and slower in the other. This opposition of characters is conformable to Dr. Brewster's observations, who states (*Phil. Trans.* 1816) that the parts of the glass which exhibit the two exterior sets of fringes (adjacent to the edges A B, C D) have "the structure of" attractive crystals, while the parts which exhibit the interior and terminal sets have that of repulsive ones; meaning, of course, in the language of the undulatory doctrine, that the order of velocities of the doubly refracted pencils is reversed in passing from one region of the glass to the other, for of its actual structure we can know nothing. That the terminal fringes ought (as observed) to have the same character as the interior is a necessary consequence of the above reasoning, for the terminal regions D B, A C are compressed in directions *parallel* to their edges, and therefore *perpendicular* to the direction in which the central portion is distended; and we have already seen that compression in one direction is equivalent (so far as the character of the tints produced is concerned) to distension in that perpendicular to it.

The terminal fringes.

1100. Neutral lines separating adjacent sets of fringes.

Lastly, the black lines separating the terminal fringes from the interior ones, arise from the combined action of the tension of the interior region parallel to A B (fig. 214) exerting itself on any point as *q* on the inner border of the terminal portion D B F H, (which we have regarded as an elastic bar, or lever,) and the distension of the line D B also exerting itself at *q*, and arising from the convexity given to this line. In virtue of these two forces, every point *q* in a certain line at a proper distance from the extreme edge H F, will be equally distended in opposite directions, and will therefore be in a neutral state, as to polarization, and, of course, appear black. The terminal fringes are less developed than the rest, because they arise simply from the flexure of the edges H F, G E, which is an indirect effect of the principal force, and is very small, (owing to the small dilatability of glass by heat, and consequent minuteness of the versed sine of the curve into which they are distorted,) and the line of indifference separating them from the others lies near the edges; for the same reason, the tension of the convex line D B being small, and therefore putting itself in equilibrium with that of the distended column *p q* at a point *q* near its extremity, where it is evident that the strain parallel to *p q* must be much diminished; the greater portion of the whole tension of *p q* being resisted by the spring of laminæ situated still further from the edge than D B.

1101.

If a lamina of glass, uniformly heated, be suddenly cooled at one of its edges, the reverse of all these effects will arise; the outer column A B *a b* (fig. 214) will suddenly contract and compress violently the columns

beyond $\alpha\beta$, from which no heat has yet been abstracted, and drag inwards the ends of the terminal levers E A G C, B F H D, which will thus be violently pressed on the parts β Q and α P as fulcra; and their action being thus transmitted to the opposite edge C D will tend to lengthen it, and thus bring it, as well as the edge A B, into a distended state. The terminal edges will also be sprung outwards. The strain on every point will be exactly the reverse of what is expressed in fig. 215, and a corresponding inversion of the characters of the tints will take place; all which is agreeable to Dr. Brewster's observation, (Prop. 14 of the *Memoir* cited.)

Phenomena of a heated glass rectangle cooled at one edge.

1102. When a crack takes place in a piece of unequally heated glass, the directions and intensities of the straining forces in every part, which depend wholly on the cohesion of its molecules, and the continuity of the levers, springs, &c. into which it may be mentally conceived to be divided, is suddenly altered; and the fringes are accordingly observed to take instantly a new arrangement, and assume forms related to the figure of that part of the glass which preserves its continuity. To analyze the modifications arising from variations of external figure and different applications of the heat, would be to involve ourselves unnecessarily in a wilderness of complexity. One simple case may, however, be noticed, in which the centre of a circular piece of glass is heated. Each exterior annulus of this will be placed in a state of distension parallel to its circumference, and will compress all within it by a force parallel to the radius. The central point will be neutral, being equally confined in all directions, and the annuli adjacent to the centre will in like manner be compressed both radially and circumferentially. The radial strain continues as we recede from the centre, but the circumferential diminishes, and at length, as already said, changes to a state of distension, and of course passes through a neutral state, thus giving rise to a black circle and concentric fringes of opposite characters, the whole of which will be intersected by the arms of a black cross parallel and perpendicular to the plane of primitive polarization, and which of course remains fixed while the plate is turned round in its own plane.

Effect of a crack.

Case of a circular plate heated in the centre.

1103. There is only one experiment of Dr. Brewster which seems hostile to the theory here stated. He made a partial crack with a red-hot iron in a very thick piece of glass, and allowed it to close by long standing, which it did, so as to disappear entirely. In this state, the glass, when unequally heated, exhibited the same fringes, as if no crack had existed; but the moment the crack was opened by a slight heat applied near it, they suddenly changed their figure, and assumed that due to the portion having the crack for a part of its outline. It seems, however, that a very great adhesive force takes place between the surfaces of glass when thus in optical contact; and to those who are aware how the free expansion and contraction of dissimilar metallic bars may be commanded, and the bars in consequence made to ply on change of temperature by mere forcible juxtaposition, without soldering, till the difference of expansion has reached a certain point, when they give way with a snap and regain their state of equilibrium, the anomaly will not appear in the light of a radical objection. (*We think it not improbable that the musical sounds said to issue at sunrise from certain statues, may originate in some pyrometrical action of the kind here alluded to. We have often been amused by a similar effect produced in the bars of the grate of a fire-place.*)

Singular effect of a crack allowed to close.

1104. Such are, in general, the transient effects of a heat below the softening point of glass, unequally distributed through its substance. But if a mass of glass be heated up to, or beyond that point, so as to allow its molecules to glide with more or less freedom on one another, and adapt themselves to any form impressed on the mass, and then suddenly cooled, either by plunging into water, or by exposure to cold air, the heat is abstracted from its external strata with so much greater rapidity than it can be supplied by conduction from within, that they become rigid, while the inner portions are still soft and yielding. At this instant, there is therefore no strain in any part; but, the abstraction of the heat still going on, the internal parts at length become solid, and tend, of course, to contract in their dimensions. In this, however, they are prevented by the external crust already formed, which acts as an arch or vault, and keeps them distended, at the same time that these latter portions themselves are to a certain extent forced to obey the inward tension, and are strained inwards from their figure of equilibrium. Glass in this state is said to be unannealed. If the cooling has been sudden, and the mass considerable, it either splits in the act of cooling, or flies to pieces, when cold, spontaneously, or on the slightest scratch which destroys the continuity of its surface; and the pieces when put together again (which, however, is seldom practicable, as it usually flies into innumerable fragments, or even to powder, as is familiarly shown in the glass tears called Rupert's drops, which exhibit a very high polarizing energy from their intense strain, and which burst with a violence amounting to explosion, on the rupture of their long slender tails) are found not to fit, but to leave a slight vacancy; thus satisfactorily proving the state of unnatural and violent distension in which its internal parts have been held. The case is precisely analogous to that of a gelatinous substance allowed to indurate under the influence of dilating forces. (See Art. 1094.)

Phenomena of unannealed glass

Rupert's drops.

1105. If the cooling be less sudden, and carefully managed, the glass, though much more brittle than ordinary annealed glass, is yet susceptible (with great caution) of being cut and polished; and in this state, if polarized light be passed through it, it exhibits coloured phenomena of astonishing variety and splendour, forming fringes, irises, and patterns of exquisite regularity and richness, according to the form and size of the mass, and the degree of strain to which it is subjected. In all these cases if the external form be varied, the pattern varies correspondingly, as it is easy to perceive it ought; for if any part of the exterior crust be removed, that part of the strain which it sustained will fall on the remainder, and on the new surface produced. Figures 216, 217, and 218, represent the patterns exhibited by a circular, a square, and a rectangular plate of about $\frac{1}{3}$ inch thick, the two latter being placed so as to have one side parallel to the plane of primitive polarization. Figure 219 and 220 represent the patterns shown by the two latter in azimuth 45°, and fig. 221 that arising from the crossing of two plates equal and similar to fig. 220, each being in azimuth 45°. In all these cases the laws of superposition of Art. 1089 are observed, when similar points of similar plates are laid together. If symmetrically, the tints polarized is the same as would be polarized by one plate whose thickness is their sum; if crosswise, their difference.

Patterns exhibited by circular, square, and rectangular unannealed plates. Fig. 216—221.

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If a square or rectangular plate be turned about in its own plane, from azimuth 0°, the arms of the black cross dividing it into four quarters become curved, as in fig. 222, and pass in succession over every part of the disc; thus showing that the positions of the axes of elasticity of the molecules vary for every different point of the plate, and in different parts of it have every possible situation. We shall not here attempt to analyze the mechanical state of the molecules in any case, as it would lead us too far; but merely mention an experiment of Dr. Brewster, which is sufficient to show the conformity of our theory of these figures with fact. According to this excellent observer, the fringes parallel to the edge A B of the rectangle (fig. 220) are similar in their character to those produced by setting the corresponding edge of a similar plate of annealed glass on a hot iron. Now, in the latter case, the exterior fringes adjacent to A B, C D arise from a compressed state of the columns parallel to A B; and the interior, from a distended. And, in the unannealed plate the distribution of the forces is almost exactly similar to that described in Art. 1098 and 1099. In fact, such a plate may be likened, in some respects, to a frame of wood over which an elastic surface is stretched like a drum. The four sides will all be curved inwards by its tension, and they will all be compressed in the direction of their length by the direct tension, independent of the secondary effect produced by their curvature. The terminal fringes in the articles referred to arise solely from the secondary forces thus developed; but the analogy between the cases would be complete, if, instead of supposing the annealed plate heated at one edge only, the heat were applied at all the four simultaneously, by surrounding it with a frame of hot iron. For a farther account of the beautiful and interesting phenomena produced by unannealed glass, we must refer the reader to Dr. Brewster's curious Paper already cited.

1106. Effect of turning round an unannealed plate in its own plane. Fig. 222. Relation of these phenomena to those of transiently heated annealed plates.

M. Fresnel has succeeded in rendering sensible the bifurcation of the pencils produced by glass subjected to pressure, by an ingenious combination of prisms having their refracting angles turned opposite ways, and of which the alternate ones are compressed in planes at right angles to each other, thus (as in the case of the double refraction along the axis of quartz) doubling the effect produced.

1107.

The effects produced by unequal heat and pressure on crystallized bodies, in altering their relations to light transmitted through them, are less sensibly marked than in uncrystallized, being masked by the more powerful effects produced by the usual doubly refractive powers. In crystals, however, where these powers are feeble, or in which they do not exist in any sensible degree, (as in fluor spar, muriate of soda, and other crystals which belong to the tessular system, Dr. Brewster has shown that a polarizing and doubly-refractive action is developed by these causes just as in uncrystallized ones; and M. Biot, by applying violent pressure to crystallized substances while viewing through them their systems of rings in the immediate vicinity of their axes where the polarizing action is very weak, has succeeded in producing an evident distortion of the rings from the regularity of their form, thus rendering it manifest, that it is only the extreme feebleness of the polarizing action so induced in comparison with the ordinary action of the crystal, which prevents its becoming sensible in all directions.

1108. Effects of unequal heat and pressure on crystallized bodies.

In applying what is here said to heat, however, we consider only its indirect action, or that arising from its unequal distribution, inducing a strain, and thus resolving itself into pressure, as above shown. But Professor Mitscherlich in a most interesting series of researches (which we hope, ere long to see embodied in a regular form, but of which at present only the most meagre and imperfect details have reached us) has shown that the action of heat on crystallized bodies, even when uniformly distributed, so that the whole mass shall be at one and the same temperature, is totally different from what obtains in uncrystallized. In the latter (as well as in crystals of the tessular system) an elevation of temperature, common to the whole mass, produces an equal dilatation in all directions, the mass merely increases in dimensions, without change of figure. In crystals, however, not belonging to the tessular system, *i. e.* whose forms are not symmetrical relative to three rectangular axes, the dilatation caused by increase of temperature is so far from being the same in all directions, that in some cases a dilatation in one direction is accompanied with an actual contraction in another.

1109. Mitscherlich's researches on the dilatation of crystals by heat.

Of this important fact, (the most important, doubtless, that has yet appeared in pyrometry,) M. Mitscherlich has adduced a remarkable and striking instance in the ordinary Iceland spar, (carbonate of lime.) This substance when heated, *dilates* in the direction of the axis of the obtuse rhomboid which is the primitive form of its crystals, and *contracts* in every direction at right angles to that axis, so that there must exist an intermediate direction, in which this substance is neither lengthened nor contracted by change of temperature. A necessary consequence of such inequality of pyrometric action is, that the angles of the primitive form will undergo a variation, the rhomboid becoming less obtuse as the temperature increases, and this has been ascertained to be the case by direct measurement; M. Mitscherlich having found, that an elevation of temperature from the freezing to the boiling point of water produced a diminution of 8′ 30″ in the dihedral angle at the extremities of the axis of the rhomboid, (*Bulletin des Sciences publié par la Société Philomatique de Paris*, 1824, p. 40.)

1110. Pyrometrical properties of Iceland spar.

M. Mitscherlich assured himself of the fact in question by direct measurement of a plate of Iceland spar parallel to the axis, at different temperatures, by the aid of the "Spherometer," a delicate species of *calibre* contrived by M. Biot for measuring the thickness of any laminar solid by the revolution of a screw whose point is just brought into light contact with the surface, and by which the 10,000th of an inch is readily appreciated and measured. The experiment is necessarily one of great delicacy, but our readers may assure themselves at least of *the general fact* of unequal change of dimension by change of temperature, by a very simple experiment requiring almost no apparatus. Let a small quantity of the sulphate of potash and copper, (an anhydrous salt easily formed by crystallizing together the sulphates of potash and of copper,) be melted in a spoon over a spirit lamp. The fusion takes place at a heat just below redness, and produces a liquid of a dark green colour. The heat being withdrawn, it fixes into a solid of a brilliant emerald green colour, and remains solid and coherent till the temperature sinks nearly to that of boiling water, when all at once its cohesion is destroyed; a commotion takes place throughout the whole mass, beginning from the surface, each molecule, as if animated,

1111. Mode of ascertaining them. Pyrometrical property of sulphate of potash and copper.

starting up and separating itself from the rest, till, in a few moments, the whole is resolved into a heap of incoherent powder, a result which could evidently not take place, had all the minute and interlaced crystals of which the congealed salt consisted contracted equally in all directions by the cooling process, as in that case their juxtaposition would not be disturbed. Phenomena somewhat similar, and referable to the same principles, have (if we remember right) been encountered by M. Achard in the fusion of various frits for glasses, &c.

1112. Double refraction of crystals variable with temperature. Remarkable property of sulphate of lime.

The relation of the optical and crystallographical characters of bodies is so intimate, that no change can be supposed to take place in the latter without a corresponding alteration in the former. As the rhomboid of Iceland spar becomes less obtuse by heat, and therefore approximates nearer to the cube, in which the double refraction is nothing, it might be expected that the power of double refraction should diminish, and this result has been verified by M. Mitscherlich by direct measurement. More recently, the same distinguished chemist and philosopher has ascertained the still more remarkable and striking fact, that the ordinary sulphate of lime or gypsum, which, at common temperatures, has two optic axes *in* the plane of its laminæ, inclined at 60° to each other, undergoes a much greater change by elevation of temperature; the axes gradually approaching each other, collapsing into one, and (when yet further heated) actually opening out again in a plane *at right angles* to the laminæ, thus affording a beautiful exemplification of Fresnel's theory of the optic axes as above explained.

1113 This singular result we cite from memory, having in vain searched for the original source of our information; but it might have been expected, from the low temperature at which the chemical constitution of this crystal is subverted, by the disengagement of its water, that the changes in its optical relations by heat would be much more striking than in more indestructible bodies. We have not, at this moment, an opportunity of fully verifying the fact; but we observe, that the tints developed by a plate of sulphate of lime now before us, exposed as usual to polarized light, rise rapidly in the scale when the plate is moderately warmed by the heat of a candle held at some distance below it, and sink again when the heat is withdrawn, which, so far as it goes, is in conformity with the result above stated. Mica, on the contrary, similarly treated, undergoes no apparent change in the position of its axes or the size of its rings, though heated nearly to ignition. The subject is in the highest degree interesting and important, and lays open a new and most extensive field for optical investigation. It is in excellent hands, and we doubt not will, ere long, form a conspicuous feature in the splendid series of crystallographical discovery which has already so preeminently distinguished its author.

§ XIII. *Of the Use of Properties of Light in affording Characters for determining and identifying Chemical and Mineral Species, and for investigating the intimate Constitution and Structure of Natural Bodies.*

1114. Relation between the refractive powers and chemical composition of bodies.

Newton, who "looked all nature through," was the first to observe a connection between the refractive powers of transparent media and their chemical properties. His well known conjecture of the inflammable nature of the diamond, from its high refractive power, so remarkably verified by the subsequent discovery of its one and only chemical constituent, (*carbon,*) was, perhaps, less remarkable for its boldness, at a period when Chemistry consisted in a mere jargon, in which *salt*, *sulphur*, *earth*, *oil*, and *mercury* might be almost indifferently substituted for one another, than it would have been fifty years later. His divination of the inflammable nature of one of the constituents of water is at least equally striking as an instance of sagacity, and even more remarkable, for the important influence which its verification has exercised over the whole science of Chemistry. These instances suffice to show the value of the refractive index, either taken in conjunction with the specific gravity of a medium, or separately as a physical character. The refractive indices of a vast variety of bodies have been ascertained by the labours of Newton and later experimenters, among whom Dr. Brewster and Dr. Wollaston have been the largest contributors to our knowledge. They may be grouped together in a general way, in order of magnitude, as follows:

1115. Classification of bodies according to their refractive densities.

Class 1. Gases and vapours. Refractive index from 1.000 to 1.002, under ordinary circumstances of pressure and temperature.

Class 2. $\mu = 1.05 \ldots . \mu = 1.45$. Comprising the condensed gases; ethereal, spirituous, and aqueous liquids; acid, alkaline, and saline solutions, (not metallic.)

Class 3. Comprising, first, almost all unctuous, fatty, waxy, gummy, and resinous bodies; camphors, balsams, vegetable and animal inflammables, and all the varieties of hydro-carbon. Secondly, stones and vitreous compounds, in which the alkalis and lighter alkaline earths in combination with silica, alumina, &c. are the predominant ingredients. Thirdly, saline bodies not having the heavy metals, or the metallic acids predominant ingredients. $\mu = 1.40 \ldots . . 1.60$.

Class 4. Pastes, (glasses with much lead,) and, in general, compounds in which lead, silver, mercury, and the heavy metals, or their oxides abound. Precious stones, simple combustibles in the solid state, including the metals themselves.

$$\mu = 1.60 \text{ and upwards.}$$

These classes, however, admit of so many exceptions and anomalies, and are themselves so vague and indefinite, that we shall not attempt to distribute the observed indices under any of them, but rather prefer, for convenience of reference, presenting the whole list in the form of a Table, arranged in order of magnitude, in which all these classes are mingled indiscriminately—a form, in some measure, consecrated by usage.

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Table of Refractive Indices, or Values of μ for Rays of Mean Refrangibility, (unless expressed to the contrary.) Dr. Wollaston's results, however, are all (according to Dr. Young, Philosophical Transactions, vol. xcii. p. 370,) *to be regarded as belonging to the Extreme Red Rays.* 1116.

N. B. In this Table the authorities are referred to as follows:

Br. Brewster, *Encyclop. Ed.*, and Treatise on *New Philosophical Instruments.* Bos. Boscovich.
B. Y. Dr. Young's Calculations of Dr. Brewster's Unreduced Observations. *Quarterly Journal*, vol. xxii.
Bi. Biot. F. Faraday. Du. Dulong. M. Malus. N. Newton. Fr. Fraunhofer.
W. Wollaston, *Phil. Trans.* He. From our own observation. Eul. Euler the younger.
C. and H., authorities cited by Dr. Young in his *Lectures.*

Vacuum	1.000000	
GASES,		
at the freezing temperature and pressure = 29in.922 = 0m.76.		
Hydrogen	1.000138	Du.
Oxygen	1.000272	Du.
Atmospheric air	1.000294	Bi.
Azote	1.009300	Du.
Nitrous gas	1.000303	Du.
Carbonic oxide	1.000340	Du.
Ammonia	1.000385	Du.
Carburetted hydrogen	1.000443	Du.
Carbonic acid	1.000449	Du.
Muriatic acid	1.000449	Du.
Hydrocyanic acid	1.000451	Du.
Nitrous oxide	1.000503	Du.
Sulphuretted hydrogen	1.000644	Du.
Sulphurous acid	1.000665	Du.
Olefiant gas	1.000678	Du.
Chlorine	1.000772	Du.
Protophosphuretted hydrogen	1.000789	Du.
Cyanogen	1.000834	Du.
Muriatic ether	1.001095	Du.
Phosgen	1.001159	Du.
Vapour of sulphuret of carbon	1.001500	Du.
Vapour of sulphuric ether (boiling point at 35° centig.)	1.001530	Du.

LIQUIDS AND SOLIDS.

	μ =	
Ether expanded by heat to three times its volume	1.0570	Br.
Tabasheer from Vellore, a yellowish transparent variety	1.1111	Br.
First new fluid discovered by Dr. Brewster in cavities in topaz	1.1311	Br.
Tabasheer, transparent, from Nagpore	1.1454	Br.
Ditto ditto ditto another specimen	1.1503	Br.
Ditto, whitest variety, from Nagpore	1.1825	Br.
New fluid discovered by Dr. Brewster in amethyst, at 83½° Fahr.	1.2106	Br.
Second new fluid discovered by Dr. Brewster in topaz, at 83° Fahr.	1.2946	Br.
Nitrous oxide liquefied by pressure	much less than water	F.
Muriatic acid gas ditto ditto Carbonic acid gas ditto ditto .. nearly equal..	both much less than water	F.
Ice	1.307	Br.
	1.3085	Br.
	1.3100	W.
Chlorine liquefied by pressure	rather less than water	F.
Cyanogen liquefied by pressure	perhaps less than water	F.
Ditto ditto	1.316	Br
Sulphurous acid liquefied by pressure	equal to water	F.
Water	1.336	N. W. Br.
Sulphuretted hydrogen liquefied by pressure	rather greater than water	F.
Ammonia liquefied by pressure	greater than water, and greater than all the other liquefied gases	F.
Aqueous humour of the eye	1.3366	Br.
Ditto of the haddock	1.341	B.Y.
Vitreous ditto	1.336	W.
Vitreous humour of the haddock	1.3394	Br.
Ditto	1.340	B.Y.
Ditto of the lamb	1.345	B.Y.
Ditto of the pigeon	1.353	B.Y.
Saliva	1.339	B.Y.
Expectorated mucus	1.339	B.Y.
Salt water (? sea water)	1.343	Br.
Cryolite	1.344 1.349	Br.
Vinegar (distilled)	1.344	Eul.
Ditto	1.372	H.
Vinegar	1.347	B.Y
Acetic acid (? strength)	1.396	Br.
Jelly fish (Medusa Æquora)	1.345	Br.
White of egg	1.351	Eul.
Port wine	1.351	B.Y.
Human blood	1.354	B.Y.
Saturated aqueous solution of alum	1.356	He.
Oil of box-wood	1.356	B.Y.
Ether	1.358 1.374	W. B.Y.
Albumen	1.360	W.
White of a hen's egg	1.361 1.359	Br. B.Y.
Brandy	1.360	B.Y.
Rum	1.360	B.Y.
Oil of ambergris	1.368 1.379	Br. B.Y.
Alcohol	1.37	W.
Ditto (S. G. 0.866)	1.370	N.
Ditto	1.371	C.
Ditto (rectified spirits)	1.372	He.
Ditto	1.374	Br.
Ditto	1.377	B.Y.
Saturated solution of salt	1.375	C.
Muriatic acid (? S. G.)	1.376	Br.
Ditto (S. G. 1.134)	1.392	He.
Ditto	1.395	B.Y.
Ditto (strong)	1.401	Br.
Ditto (highly concentrated)	1.4098	Bi.
Oil of wine	1.379	B.Y.
Sweet spirit of nitre	1.384	He.
Cornea of a lamb	1.386	B.Y.
Malic acid	1.395	Br.
Pus	1.395	B.Y.
Nitrous (?) acid (? strength)	1.396 1.404	Br. B.Y.
Nitric acid (? strength)	1.406	Br.
Crystalline lens of the eye (human?) outer coat	1.3767	Br.
Ditto ditto middle coat	1.3786	Br.
Ditto ditto centre	1.3990	Br.
Ditto of the lamb's eye, outer coat	1.386	B.Y.
Ditto ditto middle coat	1.428	B.Y.
Ditto ditto centre	1.436	B.Y.
Ditto of the haddock's eye, outer coat	1.410	B.Y.
Ditto ditto middle coat		
Ditto ditto centre	1.439	B.Y.
Ditto of the ox	1.380 1.447	W.
Ditto ditto	1.463	Eul.
Ditto of the pigeon	1.406	B.Y.
Juice of orange peel	1.403	B.Y.
Solution of potash, S. G. 1.416, (ray E)	1.40563	Fr.
Nitric acid (S. G. 1.48)	1.410 1.410	B.Y. W.
Nitric acid	1.412	C.
Hydrate of soda melted by heat	1.411	B.Y.
Hydrophosphoric acid ditto	1.423	B.Y.
Phosphoric acid (fluid)	1.426	Br.

Gluten of wheat, dried	1.426	B. Y.
Fresh yolk of an egg	1.428	B. Y.
Sulphuric acid (S. G. 1.7)	1.429	N.
Ditto ditto (? S. G.)	1.430	He.
Ditto	1.435	W.
Ditto	1.440	Br.
Fluor spar	1.433	W.
	1.436	Br.
Oil of rhue	1.433	Br.
	1.449	B. Y.
Phosphorous acid	1.437	B. Y.
Hydrophosphoric acid, cold	1.441	B. Y.
	1.442	Br.
Spermaceti (melted)	1.446	W.
	1.454	B. Y.
Oil of wax	1.452	C.
Oil of wormwood	1.453	Br.
Bees wax, melted	1.453	B. Y.
Oil of chamomile	1.457	Br.
Ditto	1.476	B. Y.
Oil of lavender	1.457	Br.
Ditto	1.467	W.
Ditto	1.475	B. Y.
Alum	1.457	W.
Ditto (S. G. 1.714)	1.458	N.
Ditto	1.488	B. Y.
Tallow (melted)	1.460	W.
White wax (melted)	1.462	B. Y.
Oil of poppy	1.467	Br.
	1.483	B. Y.
Sulphate of magnesia (double? least refraction)	1.465	Br.
Borax, (S. G. = 1.714)	1.467	N.
	1.467	C.
	1.475	Br.
Oil of peppermint	1.468	W.
	1.473	B. Y.
Oil of rosemary	1.469	Br.
	1.472	B. Y.
Oil of spermaceti	1.470	Br.
	1.473	B. Y.
Oil of almonds	1.469	W.
	1.470	
Ditto	1.481	B. Y.
	1.483	Br.
Oil of turpentine, rectified	1.470	W.
Spirit of turpentine, (S. G. 0.874)	1.471	N.
Oil of turpentine	1.475	Br.
Ditto	1.476	B. Y.
Ditto (common)	1.476	W.
Ditto	1.482	C.
Ditto	1.485	B. Y.
Ditto (common)	1.486	He.
Ditto, S. G. = 0.885, (ray E)	1.47835	Fr.
Oil of olives	1.467*	N.
	1.469	W.
	1.470	Br.
	1.4705	He.
	1.476	B. Y.
Oil of bergamot	1.471	Br.
	1.473	B. Y.
Oil of beech, misprinted? oil of brick	1.471	Br.
Oil of brick, distilled from spermaceti	1.470	B. Y.
	1.471	Br.
Oil of juniper	1.473	Br.
Butter, cold	1.474	B. Y.
	1.480	W.
Palm oil	1.475	B. Y.
Oil of rape seed	1.475	B. Y.
		Br.
Naphtha	1.475	B. Y.
Essence of lemon	1.476	W.
Gum arabic (S. G. = 1.375)	1.476	N.
Oil of dill seed	1.477	Br.
Oil of thyme	1.477	Br.
Oil of cajeput	1.478	B. Y.
	1.483	Br.
Opal (partly hydrophanous)	1.479	Br.
Naples soap	1.479	B. Y.
Oil of mace, melted	1.481	B. Y.

* The S. G. of Newton's specimen was 0.913.

Oil of spearmint	1.481	Br.
	1.496	B. Y.
Oil of lemon	1.481	Br.
	1.489	B. Y.
Carbonate of potash (?)	1.482	Br.
Oil of pennyroyal	1.482	Br.
Ditto	1.485	B. Y.
Linseed oil (S. G. 0.932)	1.482	N.
Linseed oil	1.485	W.
Ditto	1.487	B. Y.
Oil of savine	1.482	Br.
Oil of juniper	1.482	B. Y.
	1.491	
Sulphate of ammonia and magnesia	1.483	Br.
Train oil	1.483	B. Y.
Oil of wormwood	1.485	B. Y.
Ditto	1.489	
Castor oil	1.485	B. Y.
	1.490	Br.
Florence oil	1.485	B. Y.
Oil of thyme	1.486	B. Y.
Oil of dill seed	1.487	B. Y.
Oil of feugreek (? fenugreek)	1.487	Br.
Ditto	1.488	B. Y.
Camphor	1.487	W.
	1.496	B. Y.
	1.500	C.
(S. G. = 0 996)	1.500	N.
Oil of hyssop	1.487	Br.
	1.495	B. Y.
Windsor soap	1.487	B. Y.
Obsidian	1.488	Br.
Iceland spar....weakest refraction	1.488	W.
	1.519	B.
Ditto .....strongest	1.657	W.
	1.665	Br.
Ditto .....ordinary index	1.6543	M.
Ditto .....extraordinary	1.4833	M.
Ditto .....(S. G. = 2.72)	1.667	N.
Sulphate of magnesia (? greatest refraction)	1.488	Br.
Nut oil (perhaps impure)	1.490	He.
Ditto	1.507	Br.
Oil of castor	1.490	Br.
Tallow (cold)	1.49—	W.
	1.492	B. Y.
Oil of carraway seed (carui seminis)	1.483	B. Y.
	1.491	Br.
Oil of marjoram	1.490	B. Y.
	1.491	Br.
Oil of nutmeg	1.491	B. Y.
	1.497	W.
Nut oil	1.491	B. Y.
	1.507	Br.
Oil of Angelica	1.491	B. Y.
	1.493	Br.
Bees wax, cold	1.492	B. Y.
Bees wax	1.507	B. Y.
Ditto 14° Reaum	1.5123	M.
Ditto, melting	1.4503	M.
Ditto, boiling	1.4416	M.
Ditto	1.542	W.
Ditto (white wax, cold)	1.535	W.
Sulphate of iron, greatest refraction	1.494	Br.
Balsam of sulphur	1.494	B. Y.
	1.497	Br.
Sulphate of potash	1.475	W.
	1.509	Br.
Honey	1.495	B. Y.
Rochelle salt (mean green rays)	1.4985	He.
Ditto (mean red)	1.4929	He.
Ditto (tartrate of potash and soda)	1.515	Br.
Treacle	1.500	W.
Yolk of an egg (dry)	1.500	B. Y.
Oil of beech nut	1.500	Br.
Oil of rhodium	1.500	Br.
	1.503	B. Y.
	1.505	Br.
Glass, plate and crown, various specimens:		
Ditto English plate	1.500	W.
Ditto French plate	1.504	W.
Ditto English plate (extreme red)	1.5133	He.
Ditto plate	1.514	Bos.

Glass, Dutch plate	1.517	W.
Ditto crown, common	1.525	W.
Ditto plate	1.526	Bos.
Ditto crown, a prism by Dollond, (extra red)	1.526	He.
Ditto plate	1.527	Br.
Ditto ditto	1.529	Bos.
Ditto crown, another prism by Dollond, (extra red)	1.5301	He.
Ditto Fraunhofer's crown, No. 13, (ray E,) S. G. 2.535	1.5314	Fr.
Ditto yellow plate, S. G. 2.52	1.532	C.
Ditto Fraunhofer's crown, No. 9, (ray E,) S. G. 2535	1.5330	Fr.
Ditto Radcliffe crown	1.533 1.536	W.
Ditto crown	1.534	Br.
Ditto plate	1.538	Bos.
Ditto ditto	1.542	Bos.
Ditto St. Gobin	1.543	W.
Ditto crown	1.544	Br.
Ditto old plate	1.545	W.
Ditto vulgar	1.550	N.
Ditto Fraunhofer's crown, M, S. G. 2.756, (ray E)	1.5631	Fr.
Ditto plate, (S. G. 2.76)	1.573	C.
Ditto bottle	1.582	Br.

N. B. It is probable that the more refractive specimens of this list are low flint glasses, containing lead.

Spermaceti, cold	1.503 1.535	B. Y. W.
Oil of pimento	1.503 1.507 1.510	B. Y. Br. B. Y.
Starch (dry)	1.504	B. Y.
Muriate of antimony, dry	1.504	B. Y.
Ditto, two days exposed	1.613	B. Y.
Oil of amber	1.505 1.507	W. B. Y.
Birdlime	1.506	B. Y.
Oil of sweet fennel seed	1.506 1.507	Br. B. Y.
Balsam of Copaiba	1.507 1.514 1.516 1.528	W. B. Y. Br.
Stilbite	1.508	Br.
Oil of cummin	1.508 1.578	Br. B. Y.
Scammony	1.510	B. Y.
Oil of mace	1.512 1.526	B. Y.
Gum Arabic	1.512	Br.
Ditto (not quite dry)	1.513	B. Y.
Ditto	1.514	W.
Human cuticle	1.514 1.517	W.
Nitre, greatest index	1.514	Br.
Ditto, least	1.335	Br.
Ditto	1.524	C.
"Niter" (?) S. G. 1.9	1.524	N.
Dantzic vitriol (sulphate of iron)	1.515	N.
Nadelstein from Faroe	1.5153	Br.
Mesotype, least index	1.516	Br.
Ditto, greatest	1.522	Br.
Sulphate of zinc, ordinary refraction	1.517	Br.
Myrrh	1.517 1.524	B. Y. Br.
Tartaric acid, least refraction	1.518	Br.
Ditto, greatest	1.529 1.575	Br. Br.
Gum dragon (Tragacanth)	1.520 1.66	Br. W.
Glass of borax 1, silex 2	1.522	Br.
Gum lac, or Shell lac	1.52+ 1.525 1.528	W. Br. B. Y.
Oil of sassafras	1.522 1.532 1.536 1.544	B. Y. Br. W. Eul.
Caoutchouc	1.524 1.534 1.557	W. Br. B. Y.

Selenite	1.525	W.
Ditto, greatest refraction	1.536	Br.
"A selenites, S. G. 2.252"	1.488	N.
Citric acid	1.527	Br.
Leucite	1.527	Br.
Canada balsam	1.528 1.532 1.549	W. B. Y. Br.
Balsam of Gilead	1.529	B. Y.
Crystalline of ox (dried) and of a fish	1.530	W.
Pitch	1.531 1.581 1.586 1.588	W. B. Y. Br. B. Y.
Sulphate of copper, least refraction	1.531	Br.
Ditto ditto, greatest	1.552	Br.
Olibanum	1.532 1.544	B. Y. Br.
Brazil pebble, (S. G. 2.62)	1.532	C.
Glass of phosphorus (fused phosphoric acid)	1.532	Br.
Solid phosphoric acid	1.544	Br.
Glass of borax (fused borax)	1.532	Br.
Manna	1.533	B. Y.
Ditto, burned	1.547 1.565	B. Y.
Arragonite, extraordinary index	1.5348	M.
Ditto, ordinary	1.6931	M.
Elemi	1.535 1.547 1.550	W. Br. B. Y.
Mastic	1.535 1.539 1.560	W. B. Y. Br.
Arseniate of potash	1.535	W.
Gum anime	1.535 1.546	W. B. Y.
Copal	1.535 1.549 1.553	W. Br. B. Y.
Oil of cloves	1.535 1.539	W. B. Y.
White sugar	1.535	W.
Ditto	1.541	B. Y.
Ditto (melted)	1.545	B. Y.
Ditto (after melting)	1.555	Br.
Felspar	1.536	Br.
Oil of cashew nut	1.536	B. Y.
Oil of aniseed	1.536 1.601	B. Y. Br.
Mellite	1.538	Br.
Ditto	1.556	Br.
Gum juniper	1.538 1.541	Br. B. Y.
Carbonate of barytes, least refraction	1.540	Br.
Box-wood	1.542	W.
Colophony	1.543	W.
Apophyllite, the variety which exhibits white and black rings (*Leucocyclite*)	1.5431	He.
Carbonate of strontia, least refraction	1.543	Br.
Ditto ditto greatest	1.700	Br.
Dichroite (iolite)	1.544	Br.
Petroleum	1.544	B. Y.
Sal gemmæ, S. G. 2.143 (rock salt)	1.545	N.
Ditto (rock salt)	1.557	Br.
Chio turpentine	1.545 1.557	B. Y. Br.
Gum sagapenum	1.545	B. Y.
Turpentine	1.545	B. Y.
Burgundy pitch	1.546 1.560	B. Y. Br.
Gum thus	1.546 1.554	B. Y. Br.
Oil of tobacco	1.547	Br.
Rock crystal	1.547	W.
Quartz, ordinary refractive index	1.5484	M.
Ditto, extraordinary	1.5582	M.
Amethyst	1.562	W.
Rock crystal (double)	1.562	Br.
Crystal of the rock (S. G. 2.65)	1.563	N.
Rock crystal	1.568 1.575	C.

Amber	1.547	W.
Ditto, (S. G. 1.04)	1.556	N.
Resin	1.548	B. Y.
	1.552	B. Y.
	1.559	Br.
Guiacum	1.550	B. Y.
Glue, nearly hard	1.553	B. Y.
Chalcedony	1.553	Br.
Comptonite	1.553	Br.
Opium	1.559	B. Y.
	1.57—	W.
HyposulphAte of lime (mean red)	1.5611	He.
Ditto, mean yellow green	1.566	He.
Dragon's blood	1.562	B. Y.
Horn	1.565	Br.
	1.58—	W.
Pink, coloured glass	1.570	Br.
Assafœtida	1.575	B. Y.
Flint glass (various specimens)	1.576	Br.
	1.578	He.
	1.583	W.
Ditto, a prism by Dollond (extreme red)	1.584	He.
Ditto, (extreme red)	1.585	He.
Ditto, another specimen	1.586	W.
Ditto	1.590	Bos.
Ditto	1.593	Bos.
Ditto	1.594	Bos.
Ditto	1.596	Br.
Ditto, a prism by Dollond (extreme red)	1.601	He.
Ditto ditto, marked "heavy," (extreme red)	1.602	He.
Ditto, another specimen	1.604	Br.
	1.604	Bos.
Ditto	1.605	M.
Ditto, Fraunhofer's No. 3 (ray E)	1.6145	Fr.
Ditto, another variety	1.616	Br.
Ditto ditto	1.625	Bos.
Ditto, Fraunhofer's No. 30 (ray E)	1.6374	Fr.
Ditto ditto No. 23 (ray E)	1.6405	Fr.
Ditto ditto No. 13 (ray E)	1.6420	Fr.
Anhydrite, ordinary index	1.5772	Bi.
Ditto, extraordinary	1.6219	Bi.
Gum ammoniac	1.578	B. Y.
	1.592	Br.
HyposulphIte of lime, least refraction	1.583	He.
Ditto, greatest	1.628	He.
Balsam of styrax	1.584	Br.
Emerald	1.585	Br.
Benzoin	1.586 to 1.596	W.
Oil of cinnamon	1.589	B. Y.
	1.604 to 1.632	B. Y.
Tortoise shell	1.591	Br.
Balsam of Peru	1.593	B. Y.
	1.597	Br.
	1.605	B. Y.
Guiacum	1.596	W.
	1.600	B. Y.
	1.619	Br.
Beryl	1.598	Br.
Balsam of Tolu	1.60—	W.
	1.610	B. Y.
	1.627	B. Y.
	1.628	Br.
Ruby red glass	1.601	Br.
Essential oil of bitter almonds	1.603	Br.
Meionite	1.606	Br.
Purple coloured glass	1.608	Br.
Resin of jalap	1.608	B. Y.
HyposulphIte of strontia, least refraction	1.608	He.
Ditto ditto greatest	1.651	He.
Colourless topaz	1.6102	Bi.
Bluish topaz (cairngorm)	1.624	Br.
Brazilian topaz, ordinary index	1.6325	Bi.
Ditto ditto, extraordinary	1.6401	Bi.
Blue topaz, Aberdeen	1.636	Br.
Yellow topaz	1.638	Br.
Red topaz	1.652	Br.
Green coloured glass	1.615	Br.
Castor	1.620	B. Y.
	1.626	Br.
Sulphate of barytes, ordinary	1.6201	Bi.
Ditto, extraordinary	1.6352	M.
Sulphate of barytes	1.6468	M.
Ditto ditto ordinary refraction (along the axis) for yellow green rays	1.6460	He.
Ditto, another specimen, ditto, red rays	1.6459	He.
Ditto ditto for yellow green	1.6491	He.
A "pseudo topazius" (S. G. 4.27) sulphate of baryta	1.643	N.
Sulphate of barytes	1.646	W.
Ditto ditto double, greater refraction	1.664	Br.
Oil of cassia	1.624	B. Y.
	1.631	B. Y.
	1.641	Br.
Muriate of ammonia	1.625	Br.
Aloes	1.634	B. Y.
Opal coloured glass	1.635	Br.
Euclase, ordinary index	1.6429	Bi.
Ditto, extraordinary	1.6630	Bi.
Sulphate of strontia	1.644	Br.
Hyacinth red glass	1.647	Br.
Mother of pearl	1.653	Br.
Spargelstein	1.657	Br.
Epidote, least refraction	1.661	Br.
Ditto, greatest	1.703	Br.
Tourmaline	1.668	Br.
Cryolite, least refraction	1.668	Br.
Ditto, greatest	1.685	Br.
Chloruret of sulphur	1.67—	He.
Nitrate of bismuth, least refraction, about	1.67—	He.
Ditto, greatest, about	1.89—	He.
Sulphuret of carbon	1.678	Br.
Orange coloured glass	1.695	Br.
Boracite	1.701	Br.
Glass tinged red with gold	1.715	Br.
Glass, lead 1, flint 2	1.724	Br.
Deep red glass	1.729	
Nitrate of silver, least refraction	1.729	Br.
Ditto, greatest	1.788	Br.
Glass, lead 3, flint 4	1.732	Br.
HyposulphIte of soda and silver, least refraction	1.735	He.
Ditto, greatest	1.785	He.
Axinite	1.735	Br.
Nitrate of lead	1.758	Br.
Cinnamon stone	1.759	Br.
Chrysoberyl	1.760	Br.
Spinelle	1.756	He.
	1.761	Br.
	1.812	W.
Felspar	1.764	Br.
Sapphire, (white)	1.768	W.
Ditto, (blue)	1.794	Br.
Rubellite	1.768	He.
	1.779	Br.
Ruby	1.779	Br.
Jargon (orange coloured)	1.782	Br.
Glass, lead 1, flint 1 (Zeiher)	1.787	Zei.
Pyrope	1.792	Br.
Labrador hornblende	1.80±	He.
Muriate of antimony (variable) about	1.8	W.
Arsenic	1.811	W.
Carbonate of lead, least refraction	1.813	Br.
Ditto ditto greatest	2.084	Br.
Borate of lead, fused and cooled (extreme red)	1.866	He.
Sulphate of lead	1.925	Br.
Glass, lead 2, sand 1	1.987	W.
Zircon	1.95	W.
Ditto, least refraction	1.961	Br.
Ditto, greatest	2.015	Br.
Sulphur (Hauy)	1.958	Ha.
Ditto	2.008	B. Y.
Ditto	2.04	W.
Ditto, native	2.115	Br.
Ditto, melted	2.148	Br.
Calomel	1.970	Br.
Tungstate of lime, least refraction	1.970	Br.
Ditto, greatest	2.129	Br.
Glass of antimony	1.980	W.
	2.216	Br.
Glass, lead 3, flint 1 (by Zeiher)	2.028	Z.
Scaly oxide of iron	2.1—	Y.
Silicate of lead, atom to atom, extreme red	2.123	He.
Phosphorus	2.125	B. Y
	2.224	Br.
	2.260	Br.

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Nitrite of lead (biaxal, ? quadro-nitrite) in six-sided prisms, ordinary refraction		2.322	He.	Chromate of lead	least refraction	2.479	Br.
Diamond (S. G. = 3.4)		2.439	N.			2.500	Br.
Ditto		2.470	Br.			2.503	Br.
Ditto (brown coloured)		2.487	Br.		greatest	2.926	Br.
Ditto (examined by Rochon)		2.755	Ro.			2.974	Br.
Plumbago	from	2.04	W.	Octohedrite		2.500	Br.
	to...	2.44	W.	Realgar, artificial		2.549	Br.
				Red silver ore		2.564	Br
				Mercury (probable, see Art. 594)		5.829	

1117. Remarks on the Table of Refractive Indices.

In casting our eyes down the foregoing Table, we cannot but be struck with the looseness and vagueness of those results which refer to bodies whose chemical nature is in any respect indeterminate. The refractive indices assigned to the different oils, acids, &c. though no doubt accurately determined for the particular specimens under examination, are yet, as scientific data, deprived of most of their interest from the impossibility of stating precisely what *was* the substance examined. Most of the fixed oils are probably (as appears from the researches of Chevreul) compounds, in very variable proportions of two distinct substances, a solid, concrete matter, (stearine,) and a liquid, (elaine,) and it is presumeable, that no two specimens of the same oil agree in the proportions. This is, probably, peculiarly the case with the oil of anise seed, which congeals almost entirely with a very moderate degree of cold. An accurate reexamination of the refractive and dispersive powers of natural bodies of strictly determinate chemical composition, and identifiable nature, though doubtless a task of great labour and extent, would be a most valuable present to optical science. Fraunhofer's researches have shown to what a degree of refinement the subject may be carried, as well as the important practical uses to which it may be applied. The high refractive power of oil of cassia, accompanied by a corresponding dispersion, has led Dr. Brewster to conceive the existence in it of some peculiar chemical element not yet cognisable by analysis. The low refractions of the oils of box-wood and ambergris are not less remarkable. It is among the artificial salts, however, that the widest field is open for the application of precise research, and one in which a rich harvest of important results would, in all probability, amply repay the trouble of the investigation, whether considered in an optical, a chemical, or a crystallographical point of view.

1118. Table of intrinsic Refractive Powers.

The fraction $\mathrm{P} = \frac{\mu^2 - 1}{s}$, where μ is the refractive index, and s the specific gravity of the medium, expresses (in the doctrine of emission) the intrinsic refractive energy of its molecules, *supposing the ultimate atoms of all bodies equally heavy.* The following results have been stated by various authors, as its values for bodies most widely differing in their chemical and mechanical relations.

I. *Gases, taking the value of* P *for atmospheric air as unity.* (From Biot's *Précis Elémentaire*, ii. 224.)

Oxygen	0.86161	Azote	1.03408	Carburetted hydrogen	2.09270
Air	1.00000	Muriatic gas	1.19625	Ammonia	2.16851
Carbonic acid	1.00476	Supercarburetted hydrogen	1.81860	Hydrogen	6.61436

II. *Direct values of* P *given by the formula.*

Those marked Dulong are computed from the refractive indices of Dulong in the last table.

Tabasheer	0.0976	Brewster.	Muriatic acid glass	0.5514	Dulong.	Vapour of sulph. ether	0.9138	Dulong.
Cryolite	0.2742	Brewster.	Sulphuric acid	0.6124	Newton.	Protophosphuretted hydr.	0.9680	Dulong.
Fluor spar	0.3426	Brewster.	Calcareous spar	0.6424	Malus.	Ammonia	1.0032	Dulong.
Oxygen	0.3799	Dulong.		0.6536	Newton.	Rectified spirits of wine	1.0121	Dulong.
Sulphate of barytes	0.3829	Dulong.	Sal gem	0.6477	Newton.	Carbonate of potash	1.0227	Brewster.
	0.3979	Newton.	*Muriate of soda	1.2086	Brewster.	Chromate of lead	1.0436	Brewster.
Sulphurous acid gas	0.44548	Dulong.	Alum	0.6570	Newton.	Olefiant gas	1.0654	Dulong.
Nitrous gas	0.44911	Dulong.	Nitric acid	0.6676	Brewster.	*Muriate of ammonia	1.1290	Brewster.
Air	0.4528	Dulong.	Borax	0.6716	Newton.	Carburetted hydrogen	1.2204	Dulong.
	0.4530	Biot.	Niter	0.7079	Newton.	Camphor	1.2551	Newton.
	0.5208	Newton.	*Nitre	1.1962	Brewster.	Olive oil	1.2607	Newton.
Carbonic acid	0.45372	Dulong.	Hydrocyanic acid	0.7366	Dulong.	Linseed oil	1.2819	Newton.
Azote	0.4734	Dulong.	Ruby	0.7389	Brewster.	Bees wax	1.3308	Malus.
Chlorine	0.48133	Dulong.	Dantzic vitriol,(sul. iron)	0.7551	Newton.	Spirit of turpentine	1.3222	Newton.
Glass of antimony	0.4864	Newton.	Muriatic ether (vapour)	0.7552	Dulong.	Amber	1.3654	Newton.
Nitrous oxide	0.5078	Dulong.	Brazilian topaz	0.7586	Brewster.	Octohedrite	1.3816	Brewster.
Phosgen	0.5188	Dulong.	Rain water	0.7845	Newton.	Diamond	1.4566	Newton.
Selenite	0.5386	Newton.	Flint glass (mean)	0.7986	Brewster.	Realgar	1.6666	Brewster.
Carbonic oxide	0.5387	Dulong.	Cyanogen	0.8021	Dulong.	Ambergris	1.7000	Brewster.
Quartz	0.5415	Malus.	Sulphuretted hydrogen	0.8419	Dulong.	Mercury (probable)	2.4247	
Rock crystal	0.5450	Newton.	Gum Arabic	0.8574	Newton.	Sulphur	2.2000	Brewster.
	0.6536	Brewster.	Vapour of sulphuret of carbon	0.8743	Dulong.	Phosphorus	2.8857	Brewster.
Vulgar glass	0.5436	Newton.				Hydrogen	3.0953	Dulong.

1119. Remarks on this Table.

The results marked with an asterisk in this table have probably originated in some miscalculation. As hydrogen stands highest in this scale, so it is probable that fluorine, should we ever obtain it in an insulated state, would prove the lowest. The optical properties of tabasheer, in all points of view, are strange anomalies.

It will be observed, that the function $\frac{\mu^2 - 1}{s}$ only expresses the intrinsic refractive power on the hypothesis of the infinite divisibility of matter, and the equal gravitating power of every infinitesimal molecule. But if, as modern Chemistry indicates, material bodies consist of a finite number of atoms, differing in their actual weight for every differently compounded substance, the intrinsic refractive energy of the atoms of any given medium will be the product of the above function by the atomic weight. This will alter totally the order of media from what obtains in the foregoing table. Thus, the weight of the atom of hydrogen being the least, and that of mercury one among the

Light. greatest in the chemical scale, such multiplication will depress the rank of the former, and exalt that of the latter, so as to separate them entirely from the proximity they now hold. A distinction, too, will require to be regarded between compound and simple atoms. But as these considerations are peculiar to the system of emission, we shall not prosecute them farther in detail Part IV.

The dispersive powers of bodies afford another very interesting and distinctive character. Of these, Dr. Brewster, in his Treatise on *New Philosophical Instruments*, has given the following extensive table, almost entirely from his own observation.

TABLE OF DISPERSIVE POWERS.

1120. Table of Dispersive Powers. Column 1 contains the name of the medium; column 2 the value of the function $\frac{\delta\mu}{\mu - 1}$; column 3, that of $\mu\,\delta$ simply, $\delta\,\mu$ being the difference of refractive indices of extreme red and violet rays.

Dispersive Powers.	$\frac{\delta\mu}{\mu-1}$	$\delta\,\mu$.	Author.	Dispersive Powers.	$\frac{\delta\mu}{\mu-1}$	$\delta\,\mu$.	Author.
Chrom. lead, greatest estimated	0.400	0.770	B.	Oil brick	0.046	0.021	B.
Ditto greatest exceeds	0.296	0.570	B.	Flint glass, (Boscov. lowest)	0.0457		B.
Realgar, melted, different kind	0.267	0.394	B.	Nitric acid	0.045	0.019	B.
Chrom. lead, least refraction	0.262	0.388	B.	Oil lavender	0.045	0.021	B.
Realgar melted	0.255	0.374	B.	Balsam of sulphur	0.045	0.023	B.
Oil cassia	0.139	0.089	B.	Tortoise shell	0.045	0.027	B.
Sulphur after fusion	0.130	0.149	B.	Horn	0.045	0.025	B.
Phosphorus	0.128	0.156	B.	Canada balsam	0.045	0.024	B.
Balsam Tolu	0.103	0.065	B.	Oil marjorum	0.045	0.022	B.
Balsam Peru	0.093	0.058	B.	Gum olibanum	0.045	0.024	B.
Carb. lead, greatest	+0.091	+0.091	B.	Nitrous acid (?)	0.044	0.018	B.
Barbadoes aloes	0.085	0.058	B.	Cajeput oil	0.044	0.021	B.
Oil aniseed	0.074	0.044	B.	Oil hyssop	0.044	0.022	B.
Balsam styrax	0.069	0.039	B.	Oil rhodium	0.044	0.022	B.
Guiacum	0.066	0.041	B.	Pink coloured glass	0.044	0.025	B.
Carb. lead, least refraction	0.066	0.056	B.	Oil savine	0.044	0.021	B.
Oil cummin	0.065	0.033	B.	Oil poppy	0.044	0.020	B.
Gum ammoniac	0.063	0.037	B.	Jargon, greatest refraction	0.044	0.045	B.
Oil Barbadoes tar	0.062	0.032	B.	Mur. acid	0.043	0.016	B.
Oil cloves	0.062	0.033	B.	Copal	0.043	0.024	B.
Green glass	0.061	0.037	B.	Nut oil	0.043	0.022	B.
Sulphate lead	0.060	0.056	B.	Burgundy pitch	0.043	0.024	B.
Deep red glass	0.060	0.044	B.	Oil turpentine	0.042	0.020	B.
Oil sassafras	0.060	0.032	B.	Oil rosemary	0.042	0.020	B.
Opal coloured glass	0.060	0.038	B.	Felspar	0.042	0.022	B.
Resin	0.057	0.032	B.	Glue	0.041	0.022	B.
Oil sweet fennel seed	0.055	0.028	B.	Balsam capivi	0.041	0.021	B.
Oil spearmint	0.054	0.026	B.	Oil nutmeg	0.041	0.021	B.
Orange glass	0.053	0.042	B.	Stilbite	0.041	0.021	B.
Rock salt	0.053	0.029	B.	Amber	0.041	0.023	B.
Flint glass, (Boscov. greatest)	0.0527		Bos.	Oil peppermint	0.040	0.019	B.
Caoutchouc	0.052	0.028	B.	Spinelle	0.040	0.031	B.
Oil pimento	0.052	0.026	B.	Carb. lime, greatest refraction	0.040	0.027	B.
Flint glass	0.052	0.032	B.	Oil rape seed	0.040	0.019	B.
Deep purple glass	0.051	0.031	B.	Bottle glass	0.040	0.023	B.
Oil Angelica	0.051	0.025	B.	Gum elemi	0.039	0.021	B.
Oil thyme	0.050	0.024	B.	Sul. iron	0.039	0.019	B.
Oil fen (? fenu) greek	0.050	0.024	B.	Diamond	0.038	0.056	B.
Oil wormwood	0.049	0.022	B.	Oil olives	0.038	0.018	B.
Oil pennyroyal	0.049	0.024	B.	Gum mastic	0.038	0.022	B.
Oil carraway	0.049	0.024	B.	White of egg	0.037	0.013	B.
Oil dill seed	0.049	0.023	B.	Oil rhue	0.037	0.016	B.
Oil bergamot	0.049	0.023	B.	Gum myrrh	0.037	0.020	B.
Flint glass	0.048	0.029	B.	Beryl	0.037	0.022	B.
Chio turpentine	0.048	0.028	B.	Obsidian	0.037	0.018	B.
Gum thus	0.048	0.028	B.	Ether	0.037	0.012	B.
Flint glass	0.048	0.028	B.	Selenite	0.037	0.020	B.
Oil lemon	0.048	0.023	B.	Alum	0.036	0.017	B.
Oil juniper	0.047	0.022	B.	Oil castor	0.036	0.018	B.
Oil chamomile	0.046	0.021	B.	Sulphur copper	0.036	0.019	B.
Gum juniper	0.046	0.025	B.	Crown glass, very green	0.036	0.020	B.
Carb. strontia, greatest refrac.	0.046	0.032	B	Gum Arabic	0.036	0.018	B.

Dispersive Powers.	$\frac{\delta\mu}{\mu-1}$	$\delta\mu$.	Author.	Dispersive Powers.	$\frac{\delta\mu}{\mu-1}$	$\delta\mu$.	Author.
Sugar, cooled from fusion ..	0.036	0.020	B.	Sulphuric acid	0.031	0.014	B.
Jelly fish (*medusa æquora*) body	0.035	0.013	B.	Apophyllite (leucocyclite)....	0.031	0.017	He.
Water.....	0.035	0.012	B.	Tartaric acid	0.030	0.016	B.
Aqueous humour, haddock eye	0.035	0.012	B.	Borax.....	0.030	0.014	B.
Vitreous humour, haddock eye	0.035	0.012	B.	Axinite	0.030	0.022	B.
Citric acid ..,.....	0.035	0.019	B.	Alcohol	0.029	0.011	B.
Rubellite	0.035	0.027	B.	Sulph. barytes	0.029	0.019	B.
Leucite	0.035	0.018	B.	Tourmaline.....	0.028	0.019	B.
Epidote	0.035	0.024	B.	Crown glass, Leith, (Robison,) cited by Brewster....	0.027		Rob.
Common glass, Boscovich's highest, (Brewster)	0.0346		Bos.	Carb. strontia, least refraction	0.027	0.015	B.
Garnet	0.033	0.027	B.	Rock crystal	0.026	0.014	B.
Common glass, Boscovich s lowest, cited by Brewster ..	0.033		Bos.	Emerald.....	0.026	0.015	B.
				Carb. lime, least refraction ..	0.026	0.016	B.
Pyrope	0.033	0.026	B.	Blue sapphire	0.026	0.021	B.
Chrysolite	0.033	0.022	B.	Bluish topaz, cairngorm	0.025	0.016	B.
Crown glass	0.033	0.018	B.	Chrysoberyl	0.025	0.019	B.
Oil ambergris	0.032	0.012	B.	Blue topaz, Aberdeenshire ..	0.024	0.025	B.
Oil of wine	0.032	0.012	B.	Sulph. strontia	0.024	0.015	B.
Phosphoric acid, solid prism .	0.032	0.017	B.	Fluor spar	0.022	0.010	B.
Plate glass	0.032	0.017	B.	Cryolite	0.022	0.007	B.

1121. Respecting the results in this table, the remark applied to that of refractive indices may be yet more strongly urged. The whole stands in need of a radical reinvestigation. Those only, however, who have had some experience of the difficulties in the way of a strict scientific examination of dispersive powers, can appreciate either the labour of such a task, or the merit of Dr. Brewster in his researches, which we must not be understood as in the slightest degree depreciating by this remark. But the refinements of modern science are every day carrying us beyond all that could be contemplated in its earlier stages, and it is matter of congratulation, rather than disappointment, to every true philosopher, to see his methods replaced by others more powerful, and his results rendered obsolete by the more exact conclusions of his successors. What is now chiefly wanted is a knowledge of the whole series of refractive indices for the several definite rays throughout the spectrum, under uniform circumstances, and for all media whose chemical and other characters are sufficiently definite and constant to enable us to identify and reproduce them in the same state, at all times. The researches of Fraunhofer and Arago have shown that accuracy in the determination of refractive indices sufficient for the purpose, *may* be attained, and we trust, therefore, that this great desideratum will not long remain unsupplied.

Remarks on the Table of Dispersive Powers.

1122. To the substances in the table many important remarks apply. *In general*, high refractive is accompanied by high dispersive power; but exceptions are endless, especially among the precious stones, of which diamond affords a striking instance. Particular bodies seem to carry their dispersive as well as their refractive powers with them into their compounds, and that more *evidently*, because by the peculiar mode in which the dispersion is represented, the state of condensation is eliminated. Thus, fluorine, and even oxygen, appear to exercise a very lowering influence on the dispersive powers of their compounds, while hydrogen, sulphur, and especially lead, act with great energy in the opposite sense. The contrast between the oils of ambergris and cassia, is at least as remarkable in point of dispersive as of refractive power. The following experiment would seem to point out the hydrogen of the latter oil, as the principle to which its extraordinary dispersion is due, and is otherwise instructive, as exemplifying strongly the independence of the two powers *inter se*. A stream of chlorine was passed through oil of cassia till it refused to act any farther. The oil was at first greatly deepened in colour, but as the action proceeded, it changed to a much lighter ruddy yellow, which it retained till the action was complete, (and which in a few days changed to a fine rose red.) Copious fumes of muriatic acid gas were given off during the whole process, indicating the abstraction of abundance of hydrogen, and at length the oil was converted into a viscous mass, drawing out into long threads, having entirely lost its peculiar perfume, and acquired a pungent, penetrating scent, and an acrid, astringent taste, totally unlike its former aromatic flavour. It was inflammable, though less than before, burning with a flame green at the edges, indicating the presence of chlorine. Its refractive power was very little diminished. A drop being placed in the angle of two glass plates, and close to it a drop of unaltered oil of cassia, the spectrum of a line of light was viewed *at once with the same eye* through both the media. They still formed a continuous line, the spectrum of the unaltered oil being more refracted by only about one-fourth the breadth of that of the *altered* specimen. But the dispersive power of the latter was most remarkably diminished, being brought below not only that of the unaltered oil, but even below that of flint glass. When the dispersion of the unaltered oil was corrected by flint glass, that of the altered was found to be much more than corrected; and when the angle of the glass plates was such that the dispersion of the latter was just corrected by a prism of Dollond's "heavy" flint, whose refracting angle = about 25°, the uncorrected spectrum of the former was about equal to that of the flint prism. The dispersion, then, had been diminished to half its former amount, while the refraction had suffered hardly any appreciable change. (October 7, 1825.)

Experiment on oil of cassia.

1123. The angle of complete polarization of a ray reflected at the surface of a medium, affords a most valuable character in mineralogy, as it gives at once an approximation to the refractive index, sufficient in a great variety

Light. Part IV.

Use of the polarizing angle as a physical character.

Action of crystallized surfaces on reflected light.

of cases to decide between two substances, which might be otherwise confounded together, and inasmuch as it can be measured on any single surface sufficiently polished to give a regular reflexion, thus enabling us to apply this character to minute fragments, or to specimens set as jewels, or otherwise too precious to be sacrificed; to opaque bodies, and to a variety of other cases where a direct measure of the refraction would be impracticable. It has not escaped the acute and careful observation of Dr. Brewster, that the polarizing angle on the surfaces of crystallized media is not absolutely the same in all planes of incidence; and the deviation, though excessively small when the natural reflexion is used, becomes very sensible, and even enormous, when the reflexion is weakened by covering the surface with a cement of a refraction approaching that of the medium, so as to allow only those rays to reach the eye which have penetrated, as it were, to some minute depth, and undergone some part of the action of the crystal *as such*. The point is among the most curious and interesting in the doctrine of reflexion, and we regret that our limits, as well as the obscurity still hanging over it, and which it will require much elaborate research to dissipate, prevent our devoting a section to it, but we must be content to refer the reader to an excellent paper on the subject by that Philosopher, *Philosophical Transactions*, 1819.

Table of angles between the optic axes of crystals.

1124. The angles included between the optic axes of biaxal crystals is a physical character of the first rank, both on account of its distinctness, its extent of range, (indifferently over the whole quadrant,) and its immediate and intimate connection with the state in which the molecules of the crystals subsist, and what may, loosely speaking, be termed their structure. It is, however, a character by no means easily determined: both axes rarely lying within one field of view, capable of being examined through natural surfaces, and requiring, in almost all cases, the production of artificial sections; at least, this is the only safe way for observations of the tints, for the angles at which, in a thin parallel plate, the several successive orders of colours are produced in situations remote from the axes, are for the most part far too vague to lead to any accurate conclusion as to the position of these lines within the plate; not to speak of the sources of fallacy highly coloured, or dichroite, crystals obviously present. With these considerations before us, we cannot but be struck with surprise and admiration at the unwearied assiduity, which could produce, almost unassisted, a table of results so extensive and so valuable as the following.

Table of the Inclinations of the Optic Axes in various Crystals.

I. Uniaxal Crystals. *Inclination* = 0.

Negative Class.

Carbonate of lime, (Iceland spar.)	Corundum.	Idocrase, (Vesuvian.)	Arseniate of potash.
Carbonate of lime and magnesia, (bitter spar.)	Sapphire,	Wernerite.	Muriate of lime.
Carbonate of lime and iron, (brown spar.)	Ruby.	Mica from Kariat.	Muriate of strontia.
Tourmaline.	Emerald.	Phosphate of lead.	Subphosphate of potash.
Rubellite.	Beryl.	Phosphato-arseniate of lead.	Sulphate of nickel and copper.
	Apatite.	Hydrate of strontia.	

Positive Class.

Zircon.	Tungstate of zinc.	Apophyllite.	Hydrate of magnesia.
Quartz.	Titanite.	Sulphate of potash and iron.	Ice.
Oxide of iron.	Boracite.	Superacetate of copper and lime.	

Unclassed.

Hyposulphate of lime. Oxysulphate of iron.

II. Biaxal Crystals.

Names of crystals.	Character of the principal axis according to Dr. Brewster's system.	Inclination of optic axes.	Names of crystals.	Character of the principal axis according to Dr. Brewster's system.	Inclination of optic axes.
Sulphate of nickel, certain specimens ..	+	3° 0′	Mica	−	45° 0′
Carbonate of lead	−	5 15	Lepidolite	−	45 0
Sulphato-carbonate of lead	−	——	Benzoate of ammonia	+	45 8
Carbonate of strontia	−	6 56	Sulphate of magnesia and soda	..	46 49
Carbonate of baryta	−	——	Sulphate of ammonia	+	49 42
Nitrate of potash	−	5 20	Brazilian topaz (Brewster and Biot)	..	49 or 50°
Mica, certain specimens	−	6 0	Sugar	−	50° 0′
Talc	−	7 24	Sulphate of strontia	+	50 0
Mother of pearl.....	−	11 28	Muriosulphate of magnesia and iron	−	51 16
Hydrate of baryta	−	13 18	Sulphate of ammonia and magnesia..	+	51 22
Mica, certain specimens	−	14 0	Phosphate of soda	−	55 20
Arragonite.....	−	18 18	Comptonite...	+	56 6
Prussiate of potash (? Ferrocyanate) ..	+	19 24	Sulphate of lime	+	60 0
Mica, certain specimens	..	25 0	Oxynitrate of silver	+	62 16
Cymophane	+	27 51	Iolite.....	−	62 50
Anhydrite	+	28 7	Felspar	−	63 0
Borax	+	28 42	Topaz (Aberdeenshire).....	+	65 0
Mica, various specimens examined by M. Biot	..	30 0	Sulphate of potash	+	67 0
	..	31 0	Carbonate of soda	−	70 1
	..	32 0	Acetate of lead.....	−	70 25
	..	34 0	Citric acid	+	70 29
	..	37 0	Tartrate of potash.....	−	71 20
Biaxal apophyllite.....	−	35 8	Tartaric acid	−	79 0
Sulphate of magnesia	−	37 24	Tartrate of potash and soda.....	+	80 0
Sulphate of barytes	+	37 42	Carbonate of potash	..	80 30
Spermaceti (about)	+	37 40	Cyanite.....	+	81 48
Tineal (native borax)	−	38 48	Chlorate of potash	..	82 0
Nitrate of zinc (estimated)	..	40 0	Epidote, about	..	84 19
Stilbite	+	41 42	Muriate of copper	..	84 30
Sulphate of nickel	+	42 4	Peridot	..	87 56
Carbonate of ammonia.....	−	43 24	Crystallized Cheltenham salts	..	88 14
Sulphate of zinc	−	44 28	Succinic acid, estimated at about ..	..	90 0
Anhydrite (examined by Biot).....	..	44 41	Sulphate of iron, about	..	90 0

Light. Among crystals with one axis, Dr. Brewster has enumerated the Idocrase, or Vesuvian, and correctly. Had he noticed, however, in the specimens examined by him the very striking inversion of the tints of Newton's scale exhibited in the rings of that now before us, he would doubtless have made mention of it. We insert here the scale of colours exhibited by a plate cut from the specimen in question, (a fine large crystal,) as affording another remarkable case in addition to that of ~~the hyposulphate of lime, and~~ the several varieties of uniaxal apophyllite already mentioned, of such inversion. Part IV. 1125. Remarks. Inverted tints of Vesuvian.)

Table of the tints exhibited by a plate of Vesuvian, thickness = 0.11035 inch, cut a little obliquely to a perpendicular to the axis.

Angle of Incidence.	Ordinary Image.	Extraordinary Image.	$n =$	Angle of Refraction ϱ.
+ 66° +′	No light passed	No light passed.		
+ 66 0	Brick red	Dull pale green.		
+ 64 0	Orange red	Fine blue green.		
+ 60 0	Tolerable orange pink	Fine bluish green.		
+ 52 0	Pale yellow pink	Pale yellowish green.		
+ 47 0	Pink, with a dash of purple..	Pretty bright yellow.		
+ 42 0	Pale neutral purple	Good yellow	$\frac{1}{2}$	− 25° 56′
+ 37 0	Bluish white	Yellow less bright.		
+ 30 0	Very pale yellowish white....	Sombre brownish yellow.		
+ 15 0	Yellowish white.....	Very sombre yellow brown ..		
+ 10 0	Yellowish white.	Almost totally extinct	0	− 6 31 ±
+ 3 0	Yellowish white.....	Very sombre purplish brown		
± 0 0	Yellowish white.....	Dusky brownish yellow.		
− 9 0	Bluish white	Rather dull yellow.		
− 12 0	Dull purplish blue	Bright yellow	$\frac{1}{2}$	+ 7 48
− 16 0	Ruddy purple	Pale yellow.		
− 19 0	Pink, verging to brick red ..	Imperfect green.		
− 22 0	Yellowish red	Tolerable bluish green.		
− 26 0	Yellow, inclining to orange ..	Rich greenish blue.		
− 28 0	Bright yellow	Blue purple.		
− 28 30	Bright yellow	Neutral purple	1	+ 18 10
− 29 0	Bright yellow	Ruddy purple.		
− 30 0	Yellow green.....	Crimson.		
− 32 0	Good green	Good pink.		
− 35 0	Greenish blue	Orange pink.		
− 37 30	Blue purple	Pale yellow.		
− 38 30	Neutral purple	Pale yellow	$\frac{3}{2}$	+ 24 0
− 39 15	Ruddy purple	Greenish yellow.		
− 41 30	Good pink.....	Good green.		
− 45 0	Pink yellow	Fine greenish blue.		
− 47 20	Yellowish white	Blue purple.		
− 47 30	Yellowish white	Neutral purple	2	+ 28 48
− 48 0	Very pale green.....	Ruddy purple.		
− 49 30	Fine green.....	Good pink.		
− 53 0	Fine blue green.....	Orange pink.		
− 54 0	Greenish blue	Yellow.		
− 54 —	No light passed.....	No light passed.		

The first ring, it will be observed, in calculating from this table, is contracted beyond what is due to the law of the sines, probably from the section examined not passing precisely over their common centre, and gives a polarizing power greater than that deduced from the angles corresponding to $n = 1$, $n = \frac{3}{2}$, $n = 2$, all which agree in assigning 41.35 nearly as the measure of the power in question. See Art. 1126.

It follows from this series, that of the two images formed by double refraction in Vesuvian, and other similar crystals, the *most refracted* should be the *least dispersed*, a peculiarity we have not yet had an opportunity of verifying by direct observation. It follows, however, immediately from the theory of the rings above delivered, since the smaller the diameters of the rings for any coloured ray, the greater the separation of its pencils by double refraction. Hence, in the present case, the red rays will be separated by a greater interval than the violet in the two spectra; and, consequently, the least refracted spectrum will be the longest. In the variety of apophyllite exhibiting white and black rings, (*leucocyclite*) the two dispersions should be almost exactly equal, and the only difference between the two spectra ought to consist in a slight variation in the proportional breadths of the several coloured spaces in them.

Another very important optical character is the *intensity* of the polarizing, or doubly refractive energy. This may be concluded by measuring the actual angular separation of the images; but this is usually too small to 1126.

Light. Polarizing powers considered as a physical character of media.

admit of being determined with sufficient precision, in such very imperfect specimens as are usually subjected to examination for the purpose of identification; and a much better course is to make the tint developed at a perpendicular incidence, by a plate of given thickness, in a direction at right angles to both the optic axes, the object of determination. This tint (which we shall term the equatorial tint) may be derived immediately from observations of tints at any angle, by the formula

$$N = \frac{n}{t} \cdot \frac{\cos \rho}{\sin \theta \,.\, \sin \theta'},$$

where N is the tint in question, numerically expressed as usual, and where n is the tint, (also similarly expressed) developed at an angle of incidence whose corresponding angle of refraction is ρ, on a plate whose thickness is t, (expressed in English inches and decimals) and where θ, θ' are the angles made by the ray in traversing the plate with the two axes. This value of N is the same with $\frac{1}{h}$ in the equation of Art. 907. The following list of a very few substances will suffice to show the great range the value of N admits, and its consequent utility as a physical character, considerations which we hope will induce observers to extend the list itself, as well as to give it all possible exactness.

Uniaxal Crystals.

	For mean yellow rays.	
	$N =$	$h = \frac{1}{N}$
Iceland spar	35801	0.000028
Hydrate of strontia (assuming $\mu = 1.5$)	1246	0.000802
Tourmaline	851	0.001175
Hyposulphate of lime	470	0.002129
Quartz	312	0.003024
Leucocyclite (uniaxal apophyllite, 1st variety)	109	0.009150
Camphor	101	0.009856
Vesuvian	41	0.024170
Uniaxal apophyllite, 2d variety	33	0.030374
Ditto, 3d variety	3	0.366620

Biaxal Crystals.

	For mean yellow rays.	
	$N =$	$h = \frac{1}{N}$
Nitre	7400	0.000135
Anhydrite (angle between axes $= 43° 48'$)	1900	0.000526
Mica (angle between axes $= 45°$)	1307	0.000765
Sulphate of baryta	521	0.001920
Heulandite (white;—angle between axes $= 54° 17'$)	249	0.004021

1127. Use of polarized light in detecting complex structures. But the phenomena of refraction, reflexion, and polarization, may not only be applied by the aid of these and similar tables of registered results, to the examination and identification of substances in the gross, they are also of use in detecting peculiarities of structure in individual specimens, or in certain species which would otherwise escape observation. The singular structure of amethyst has been already explained, and a variety of cases of hemitropism might be noticed, in which the juxtaposition of the parts is rendered evident by the test of polarized light. Of these, however, by far the most curious and interesting are those in which the juxtaposed parts combine to form a regular whole, and to produce a species of pseudo-crystal, built up as it were of several individuals, arranged with a regard to symmetry, and forming a structure of more or less complication. Such instances have been more particularly noticed in nitre, arragonite, topaz, apophyllite, sulphate of potash, analcime, harmotome, &c.

1128. Compound crystals of nitre. Arragonite. The usual form of the crystals of nitre, when large and well developed, is the regular hexagonal prism; but a section of this, cut at right angles to the axis, is very commonly found to consist of two or more portions, in which the optic meridians are 60° inclined to each other; but the plane of division often intersects one of the lateral faces of the prism, without any visible external mark of a breach of continuity, so that but for the test of polarized light, the macled structure would never be discerned. The phenomena of arragonite, in this respect, are very similar to those of nitre.

1129. Topaz. If a plate of Brazilian topaz, cut at right angles to the axis of the rhombic prism in which it crystallizes, be examined by polarized light, it will occasionally be found to consist of a central rhomb, surrounded by a border in which the optic meridians of the alternate sides are inclined at $\frac{1}{4}$ of a right angle to that of the central compartment, and $\frac{1}{2}$ a right angle to each other. In consequence, when such a rhombic plate is held with its long diagonal in the plane of primitive polarization, two opposite sides of the border appear bright, the other two black, and the central compartment of intermediate brightness. Such specimens often exhibit the phenomena of dichroism in the central compartment, while the border is colourless in all positions.

1130. Tesselite. Fig. 223. But it is in the apophyllite of the variety named by Dr. Brewster, Tesselite, that this enclosure of one crystal in a case as it were of another, is exhibited in the most regular and extraordinary manner. In one of the varieties of this singular body, whose form is the right rectangular prism with flat summits, slices taken off from either summit were found by him to be of uniform structure; but, when these were detached, every subsequent slice was

Light. found to consist of a rectangular border enclosing no less than nine several compartments, arranged as in fig. 223, and separated from each other, and from the border, by delicate lines or films as there marked. Each of these compartments possesses its own peculiar crystallographic structure, and polarizes its peculiar tints, the law of symmetry being observed. In some specimens the triangular spaces *p q r s* were wanting, while in others they seem to have consisted of two portions, separated by an imaginary prolongation of the line joining their obtuse angles with the central lozenge. Part IV

The terminal plates, the central lozenge, and the minute stripes dividing the compartments from each other (which are sections of laminæ or films parallel to the axis of the crystal, and running its whole length) consist of that uniaxal variety, in speaking of which we have used the term leucocyclite, from the whiteness of its rings. The rectangles R V, S T, (with the exception of the portions occupied by the lozenge and partitions) consist of a *biaxal* medium, having its axes 34° inclined to each other, and its optic meridian parallel to the axis of the prism, and passing through the diagonals R V, S T of these rectangles. The other rectangles are composed of a similar medium, but with its optic meridian at right angles to the former, or passing through the diagonals R T, S V.

1131. Another variety. Fig. 224.

A still more remarkable and artificial structure has been observed by Dr. Brewster, in a variety of the Faroe apophyllites of a greenish white hue. When a complete prism of this variety is exposed to polarized light, with its axis in 45° of azimuth, the light being transmitted perpendicularly through two opposite sides, the pattern represented in fig. 224 is seen, in which the central curvilinear area is red, and its complements to the surrounding rectangle green. The squares immediately adjacent on either side in the direction of the axis are also vivid red in their centres, fading into white, while the rest of the pattern consists in a most brilliant succession of red, green, and yellow, bands, for a coloured figure of which we must refer the reader to the original most curious and interesting memoir, (*Edinburgh Transactions*, vol. ix. part ii.) where, as also in the *Edinburgh Philosophical Journal*, vol. i. he will find the phenomena described in full detail.

1132. Sulphate of potash.

The sulphate of potash offers another very remarkable example of compound structure. This salt occurs in hexagonal prisms, and occasionally in bipyramidal dodecahedrons. But besides these forms it also occurs in rhombic prisms of 114° and 66°. These Dr. Brewster found to have two axes, while the hexagonal prisms have but one; thus affording another instance of dimorphism in addition to those of arragonite, sulphur, &c. On examining the dodecahedrons, however, he found them to consist of six equilateral triangular prisms, of the biaxal variety, grouped together, and having their optic meridians all converging to the common axis; the molecules being so disposed in each opposite pair of individuals as to make the angle between the opposite faces of either pyramid (114°) equal to the obtuse angle of the rhomboid.

1133. Analcime.

The structure and mode of action of the analcime, described by Dr. Brewster in vol. x. of the *Edinburgh Transactions*, part i. p. 187, are so extremely singular, that it is difficult to say whether it should be regarded as a grouped crystal, consisting of independent portions adhering together, or as a mass the distribution of the ether in whose parts is governed by a general and uniform law; the latter, however, is probably the truth. The form of this crystal is the icositetrahedron, contained by twenty-four similar and equal trapezia, and may be regarded as derived from the cube by the truncation of each of its angles by three planes symmetrically related to the edges including it. If we conceive from the centre of this cube, (in its natural situation with respect to the derived figure) planes to pass through each of the edges, and through each of the diagonals of the six faces, they will divide the cube into twenty-four irregular tetrahedra; and of these, all the faces which pass through *edges* of the cube will also pass through edges of the derived figure, while those which pass through *diagonals* of faces of the cube will also pass through diagonals of the faces of its derivative, bisecting their obtuse angles. Now it appears from Dr. Brewster's observations, that all the molecules *situated in any part of any one of these planes* are devoid of the power of double refraction and polarization; and that in proportion as a molecule is distant from all such planes, its polarizing power is greater. In this respect it differs entirely from all crystals hitherto examined, every particle of which, wherever situated, so long as they belong to one and the same crystalline system, being equally endued with the polarizing virtue. Nor is there a closer analogy between the mode of action in question, and that of unannealed glass and similar bodies; for in these a change of external form is always accompanied with a change of the polarizing powers, while in the analcime each particular portion, whether detached from the mass, or in its natural connection with the adjacent molecules, possess the very same optical properties. The action too of the portions which possess a polarizing power is not related to axes given only in *direction*, and passing through every molecule, but to planes given both in direction and in *place* within the mass, (the planes above mentioned;) the tint developed at any point of a plate being as the square of the distance from the nearest of such planes, and the isochromatic lines being, in consequence, straight fringes of colour arranged parallel to the dark bands marked out by the intersection of such planes with the plate examined. The phenomena described are accompanied with a sensible double refraction. The reader is referred to the memoir already cited (which is one of the most interesting to which we can direct his attention) for further details: and to a work understood to be forthcoming from the pen of the eminent author here and so often before cited, on optical mineralogy, for what we are sure will prove a treasure of valuable information on every point connected with this important application of optical science.

§ XIV. *On the Colours of Natural Bodies.*

1134. It was our intention to have devoted a considerable share of these pages to the explanation of such natural phenomena as depend on optical principles, but the great length to which this essay has already extended, renders it necessary to confine what we have to say on such subjects within very narrow limits, and to points of promi-

nent importance. Among these there is certainly none more entitled to consideration than the phenomena of colour, as exhibited by natural objects, which strike us wherever we turn our eyes, and it is impossible to pass in total silence the theory devised by Newton to account for them; a theory of extraordinary boldness and subtilty, in which great difficulties are eluded by elegant refinements, and the appeal to our ignorance on some points is so dexterously backed by the weight of our knowledge on others, as to silence, if not refute, objections which at first sight appear conclusive against it. The postulates on which this theory rests are essentially as follows:

Newton's theory of the colours of natural bodies.

1135. Postulates. 1. All bodies are porous; the pores or intervals vacant of ponderable matter, occupying a very much larger portion of the whole space filled by the body, than the solid particles of which it essentially consists.

1136. 2. These solid particles have a certain size (and perhaps figure) essential to them as particles of that particular medium, and which cannot be changed by any mechanical action, or by any means not involving a change in the chemical nature or condition of the medium. They are, in short, the ultimate *atoms;* to break which, is to destroy their essence, and resolve them into other forms of matter, having other properties.

1137. 3. These atoms are perfectly transparent, and equally permeable to light of all refrangibilities, which, having once passed their surfaces, is in the act of pursuing its course through their substances.

Newton, indeed, makes his atoms only "in some measure transparent." But he never refers to this limitation, and his theory depends essentially on their *perfect* transparency, as is indeed obvious from his account of opacity, which is contained in the next postulate.

1138. Cause of opacity. 4. *Opacity in natural bodies arises from the multitude of reflexions caused in their internal parts.*

It is obvious, therefore, that unless we admit a cause of opacity in *atoms* different from that which, on this hypothesis, causes it in their aggregates constituting natural bodies, the former cannot be otherwise than absolutely pellucid, since no reflexions can take place where there are no intervals, and no change of medium. Of the sufficiency of this cause, either in natural bodies or atoms, however, we confess there does appear to us some room for doubt, as it seems difficult so to conceive these internal reflexions, that the rays subjected to them shall be *all* and *for ever* retained, entangled as it were, and running their rounds from atom to atom, without a possibility of reaching the surface and escaping; which, were they to do, it is evident that every body so constituted, receiving a beam of light, would in fact only disperse it in all directions in the manner of a self luminous one.

1139. Origin of natural colours. 5. *The colours of natural bodies are the colours of thin plates, produced by the same cause which produces them in thin laminæ of air, glass, &c.* viz. the *interval* between the anterior and posterior surfaces of the atoms, which, when an odd multiple of half the length of a fit of easy reflexion and transmission for any coloured ray moving within the medium, obstructs its penetration of the second surface, and when an even, ensures it, (see Art. 655.) The *thickness*, therefore, of the atoms of a medium, and of the interstices between them, determines the colour they shall reflect and transmit at a perpendicular incidence. Thus, if the molecules and interstices be less in size than the interval at which total transmission takes places, or less than that which corresponds to the edge of the central black spot in the reflected rings, a medium made up of such atoms and interstices will be perfectly transparent. If greater, it will reflect the colour corresponding to its thickness.

1140. Objections. It may be objected to this, that all natural colours do not of necessity find a place in the scale of tints of thin plates, even those of bodies whose chemical composition is uniform; but to this we may answer, that the colours reflected from the first layer only of molecules next the surface ought to be *pure* tints, those from lower layers having to make their way to the eye through the upper strata, and thus undergoing other analyses, by transmissions and reflexions among the incumbent atoms. Besides which, whatever *shape* we attribute to the atoms, it is impossible that all rays shall penetrate them so as to traverse the same thickness of them, unless we regard them as mere *laminæ* without angles or edges, and of enormous refractive power.* The same answer must be made to the objection, equally obvious, that the transmitted tint ought to be in all cases complementary to the reflected one, and that therefore cases like that of leaf gold, opalescent glass, and infusion of *lignum nephriticum*, all which reflect one tint and transmit another, but in all which this condition is violated, form exceptions to the theory. But, in reality, the transmitted rays have traversed the whole thickness of the medium, and have therefore undergone, many more times, the action of its atoms, than those reflected, especially those near the first surface, to which the brighter part of the reflected colour is due.

1141. Apparent exceptions considered. The infusion of lignum nephriticum is a very singular case, and its peculiar properties have been explained by Dr. Young, on the supposition of minute particles of definite magnitude suspended in it. Though very transparent, it yet *reflects* a bluish green colour, while the light transmitted is yellow or wine-coloured, in this respect offering almost the exact converse of leaf gold. It is, however, no doubt a case of opalescence, and is exactly imitated by certain yellow glasses, in which a very visible thin film of opalescent matter near the surface reflects to the eye a bluish green tint, while yet the colour transmitted has the yellow tint belonging to the glass. The reflexion proceeds from particles which have nothing to do with the transmitted light.

1142. Case of transparent coloured media. But, in fact, the objection (as appears to us) is not yet fully answered. Transparent coloured media (*clear* liquids in which no floating particles exist,) have *no* reflected colour. When examined by pouring them into an opaque vessel, blackened internally and filled to the brim, and when the colourless reflexion from their upper surface is destroyed by reflexion in an opposite plane at the polarizing angle, it is seen at once that no light is reflected from within the medium, either near the surface, or at greater depths; and if this mode of examination be regarded as objectionable, as perhaps destroying the internal as well as external reflexion, it is equally satisfactory to observe, that the image of a white object reflected from the surface of a fluid in a black opaque vessel is always purely white, whatever be the colour of the reflecting fluid. We are not aware that the objection so put has been sufficiently considered, or even propounded. To us its weight appears considerable,

* Newton appears to have been fully aware of the necessity of taking this into consideration. Prop. vii. book ii. *Opt. versus finem.*

Light. and we cannot but believe that some other cause besides mere internal reflexions must interfere to prevent the complementary colour from reaching the eye; and that absorption, with its kindred phenomenon, or rather its extreme case, opacity, is not satisfactorily accounted for in this theory, but must rather be admitted as (at present,) an ultimate fact, of which the cause is yet to seek. Part IV.

1143. True and false colours. If this be granted, the colours of all bodies may be distinguished into *true*, viz., those which arise from rays which have actually entered their substance and undergone their absorptive action, (as the colours of powders of transparent coloured media, cinnabar, red lead, Prussian blue, those of flowers, &c.,) and *false*, or *superficial*, or those which originate obviously in the law of interference; thus, the variable colours of feathers, insects' wings, striated surfaces, oxidated steel, and a variety of cases to which the Newtonian doctrine strictly applies, for there is no denying that cases of colour, not *merely* superficial, do occur, in which the Newtonian doctrine, to say the least, is highly probable. To instance one or two only. If a few drops of an extremely weak solution of nitrate of silver be added to a very dilute solution of hyposulphite of lime, a precipitate is formed of an opalescent whiteness and extreme tenuity. If more of the nitrate be added, the precipitate increases in weight and aggregation, and at the same time changes its colour, becoming first yellow, then yellow brown, then a rich orange brown, then a purplish brown, and, finally, a deep brown black. The precipitate, meanwhile, continually acquires density, and, finally, sinks rapidly to the bottom. It is impossible, in this series, not to trace the tints of the first order of reflected rings, produced by the thickening of the minute particles in the act of aggregation, but equally impossible not to recognise the agency of a cause totally different, acting to increase the opacity of the compound by an absorptive action far superior to, and independent of, the action of the particles as thin plates. The phenomena of *Hematine*, described by Chevreul and cited by Dr. Brewster, (*Encyc. Edin. Optics*, p. 623; see also Biot, *Traité de Phys.* tom iv. p. 134, there referred to,) afford too close an approximation to the series of tints of the second order not to authorize a presumption that the Newtonian theory may apply to this case also. The diffused light and blue colour of the clear sky, affords another very satisfactory instance. This blue is, no doubt, a blue of the first order, reflected from minute aqueous particles in the air. The proof is, that at 74° distance from the sun, it is *completely polarized* in a plane passing through the sun's centre. Cases in which Newton's theory applies.

1144. Another objection. Answered. Another objection, no less obvious, to the Newtonian doctrine, has been successfully answered by Newton himself. A change of obliquity of incidence, it may be urged, should cause a change of colour, as a plate of given thickness reflects a different tint at oblique and perpendicular incidences. But this variation is less, the greater the refractive power of the medium; and as the refractive power increases with the density, that of the dense ultimate atoms of bodies must be exceeding great, so that the tint reflected from them will vary little with a change of incidence, (art. 669.) The colours of oxidated steel afford an excellent case in point. The refractive power of this oxide, though great, (2.1), is, doubtless, not to be compared with that of the ultimate atoms of bodies, yet the tints on the surface of blued steel vary but little with a change of obliquity. We may add, too, that the colour exhibited by any body of sensible magnitude, is in reality an average of the colours reflected from all its molecules at all possible incidences, so that no change of incidence ought to be expected to affect it.

1145. Newton's ideas of the size of the particles of bodies. Of the extreme tenuity of the ultimate molecules of bodies, Newton seems to have had but an inadequate idea, as he supposed that they might be seen through microscopes magnifying three or four thousand times.* We have viewed an object *without utter indistinctness*, through a microscope by Amici, magnifying upwards of three thousand times in linear measure, and had no suspicion that the object seen was even approaching to resolution into its primitive molecules. But it should rather seem that Newton regarded his colorific molecules as divisible *groupes* of atoms of a yet more delicate kind, and yet more dense, and these again as still further resolvable till the last stage of indivisibility be reached. M. Biot has given a striking, and, we may almost term it, picturesque account of this doctrine, in his *Traité de Physique*.

§ XV. *Of the Calorific and Chemical Rays of the Solar Spectrum.*

1146. It has long been a matter of everyday observation, that solar light exercises a peculiar influence in altering the colours of bodies exposed to it, either by deepening or discharging them, even when totally secluded from air, and that various metallic salts and oxides, especially those of silver, are speedily blackened and reduced when freely exposed to direct sunshine, or even to the ordinary light of a bright day. Whether these effects were owing to the heat of the rays, or to some other cause, remained long uninquired. The first step was

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* The passage, however, is in the highest tone of a refined philosophy, and, independent of its theoretic bearings, we extract it, as indicating a scrutinizing spirit of observation far beyond the age he lived in.

"In these descriptions I have been the more particular, because it is not impossible but that microscopes may at length be improved to the discovery of the particles of bodies on which their colours depend, if they are not already in some measure arrived to that degree of perfection. For if those instruments are or can be so far improved as with sufficient distinctness to represent objects five or six hundred times bigger than at a foot distance they appear to our naked eyes, I should hope that we might be able to discover some of the greatest of those corpuscles. And by one that would magnify three or four thousand times perhaps they might all be discovered, but those which produce blackness. In the mean while I see nothing material in this discourse that may rationally be doubted of, excepting this position: That transparent corpuscles of the same thickness and density with a plate, do exhibit the same colour. And this I would have understood not without some latitude, as well because those corpuscles may be of irregular figures, and many rays must be obliquely incident on them, and so have a shorter way through them than the length of their diameters, as because the straitness of the medium put in on all sides within such corpuscles may a little alter its motions or other qualities on which the reflection depends. But yet I cannot much suspect the last, because I have observed of some small plates of Muscovy glass which were of an even thickness, that through a microscope they have appeared of the same colour at their edges and corners where the included medium was terminated, which they appeared of in other places. However, it will add much to our satisfaction, if those corpuscles can be discovered with microscopes; which if we shall at length attain to, I fear it will be the utmost improvement of this sense. For it seems impossible to see the more secret and noble works of nature within the corpuscles, by reason of their transparency."

Light. Part IV.

Discoveries of Scheele. Sir W. Herschel.

made by Scheele, who ascertained that muriate of silver is much more powerfully blackened in the violet rays than in any other part of the spectrum. (*Traité de l'Air et du Feu*, § 66.) The experiments of Sir W. Herschel, on the heating power of the several prismatic rays, on the other hand, which appeared in 1800, showed satisfactorily that the more refrangible rays possess very little heating power, the calorific effect being at its maximum for the extreme red rays, and even extending considerably beyond the limits of the spectrum in that direction. This remarkable discovery, which established the independence of the heating and illuminating effects of the solar rays, led Professor Ritter, of Jena, in 1801, to examine whether a similar extension beyond the limits of the visible spectrum might not also have place in the chemical or deoxidating rays, and on exposing muriate of silver in various points within and without the spectrum, he found the maximum of effect to lie beyond the visible violet rays, the action being less in the violet itself, still less in the blue, and diminishing with great rapidity as he proceeded towards the less refrangible end. Dr. Wollaston independently arrived at the same conclusion.

Ritter.

1147. Calorific, luminous, and chemical rays.

The solar rays, then, possess at least three distinct powers: those of heating, illuminating, and effecting chemical combinations or decompositions, and these powers are distributed among the differently refrangible rays, in such a manner as to show their complete independence on each other. Later experiments have gone a certain way to add another power to the list—that of exciting magnetism. Without calling in question the accuracy of the observations which are directed to establish this point, we may be permitted to hope that further researches will, ere long, explain the causes of failure in those numerous cases where such effects have not been produced.

1148. All obey the same optical laws.

The calorific rays appear, from experiments of Berard, to obey the laws of polarization and double refraction, like those of light. Those of interference could not be made without excessive difficulty. In the case of the chemical rays, the same difficulty is not experienced; and Dr. Young, and after him, by more delicate means, M. Arago, have satisfactorily demonstrated that these conform to the same laws of interference, whether polarized or otherwise, that are obeyed by the luminous rays similarly circumstanced. Thus, a set of fringes formed by the interference of two solar pencils with a common origin, being kept very steadily projected for a long time on one and the same part of a sheet of paper rubbed with muriate of silver, a series of black lines became traced on it, the intervals of which were smaller than those of the dark and luminous fringes formed by homogeneous violet light.

Chemical rays interfere like luminous ones.

1149. Wollaston's observations on guiacum.

Dr. Wollaston having observed that gum guiacum is turned green by exposure to solar light in contact with air, took two specimens of paper coloured with a yellow solution of this gum in alcohol, and exposed one of them to air and sunshine, the other to air in the dark. The former was turned perceptibly green in five minutes, and the change was complete in a few hours, while the latter was no way discoloured after many months. He then concentrated the violet rays on paper so coloured, by a lens, and the change was speedily performed, while in the most luminous there was no change of colour, and, in the red rays, the green colour was not only not produced, but when induced by exposure to the violet, was again destroyed, and the original yellow colour restored. This seems, however, to have been merely an effect of the heat, as the warmth from the back of a heated silver spoon discharged the green colour just as effectually.

1150. Effect of light on purple glass.

Mr. Faraday has observed that glass tinged purple with manganese, has its hue much deepened by the passage of solar light through it, and that two portions of the same plate, one preserved in the dark, the other exposed freely, after some time differ materially in intensity of colour.

1151. Other effects of solar light.

The direct action of solar light, or, possibly, of its heat also, produces other chemical effects, such as the immediate combination of the elements of phosgen, the explosion of an atomic mixture of chlorine and hydrogen, and other phenomena, all indicative of powers resident in this wonderful agent, of which we have but a very imperfect notion at present. The green colour of plants, and the brilliant hues of flowers, depend entirely on it. Tansies which had grown in a coal pit, were found totally destitute either of colour or of their peculiar and powerful flavour, and the bleaching and sweetening of celery by the exclusion of light, is another familiar instance of the same cause. How far the differently coloured rays are concerned in these effects, has never yet been *accurately* investigated, though attempts have been made; but we hope, from the distinguished ability of an eminent individual who has recently taken up this most interesting inquiry, that our stock of knowledge will soon receive material accessions.

1152. We cannot close this Essay without an expression of regret, that the Memoir of Professor Airey, on the Spherical Aberration of the Eyepieces of Telescopes, just on the point of publication in the *Transactions of the Cambridge Philosophical Society*, reached us too late to allow of our attempting to condense its valuable contents, and we can only recommend it to the notice of our readers in lieu of, and in preference to, anything we could ourselves say on that subject. A similar expression of regret applies to the interesting "Theory of Systems of Rays," by Professor Hamilton of Dublin, a powerful and elegant piece of analysis, communicated to the Royal Irish Academy in 1824, and only now in the course of impression, but of which enough has reached us, by the kindness of its Author, to make us fully sensible of the benefit we might have derived from its perusal at an earlier period of our undertaking.

Slough, December 12, 1827.

J. F. W. HERSCHEL

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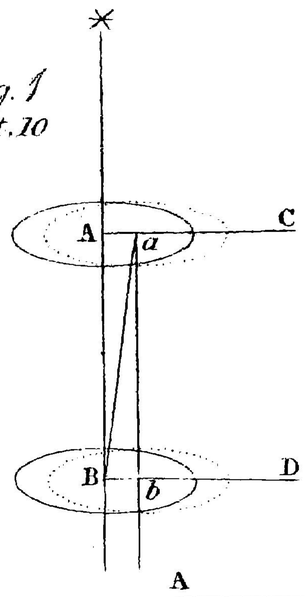
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LIGHT

Fig. 1 Art. 10



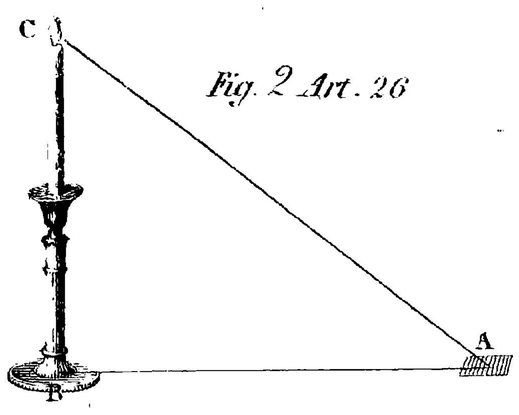


Fig. 2 Art. 26

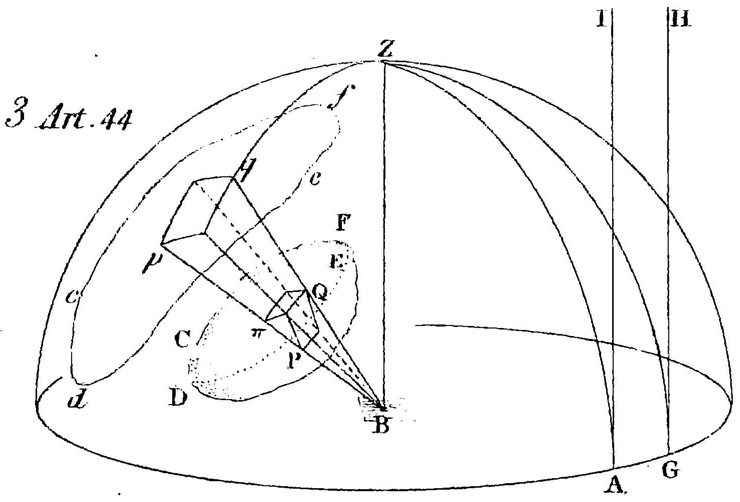


Fig. 3 Art. 44

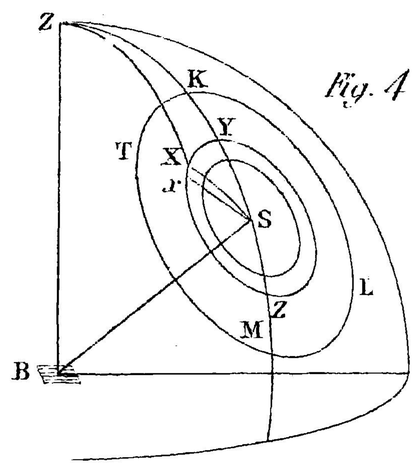


Fig. 4 Art. 53

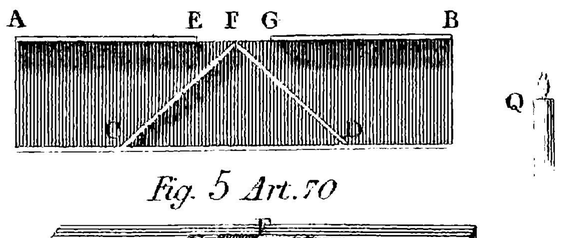


Fig. 5 Art. 70

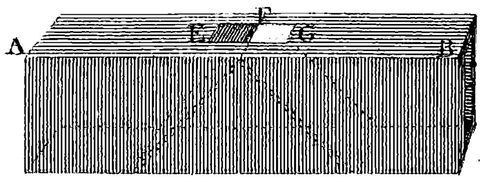
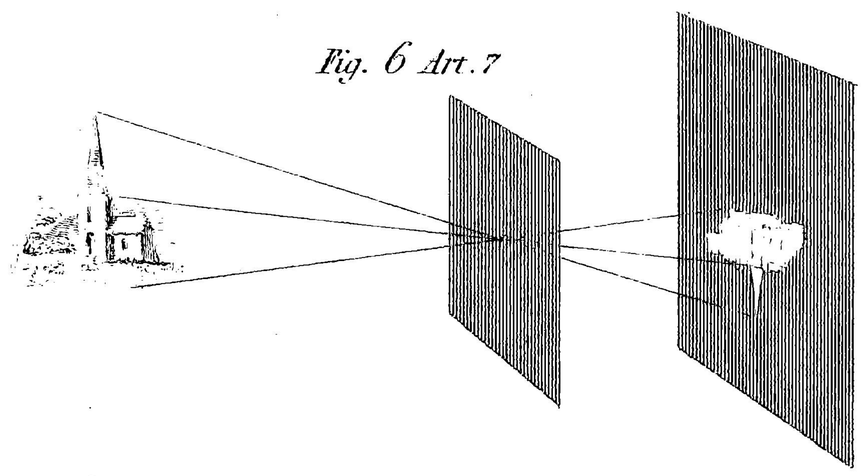


Fig. 6 Art. 7



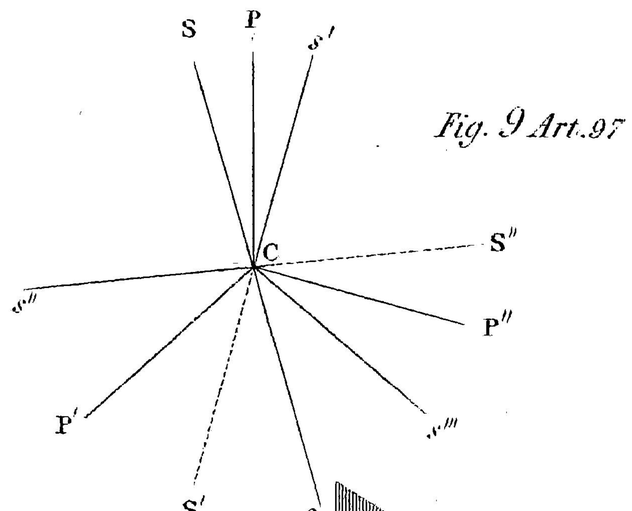


Fig. 9 Art. 97

Fig. 7 Art. 8 & 4

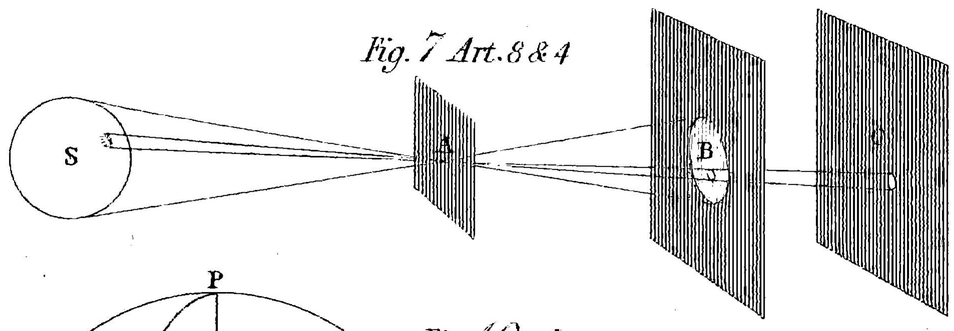


Fig. 8 Art. 77

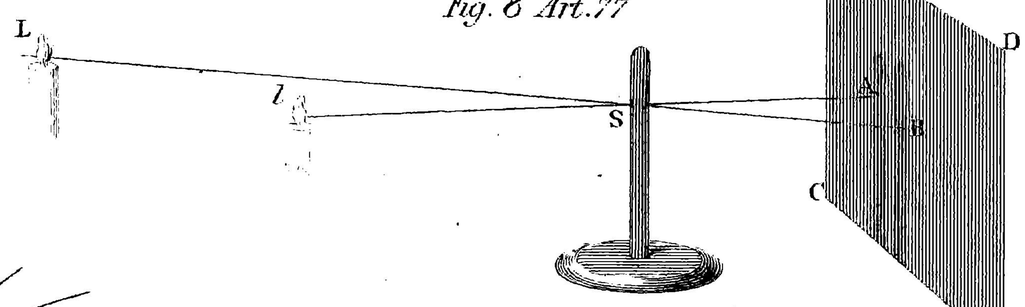


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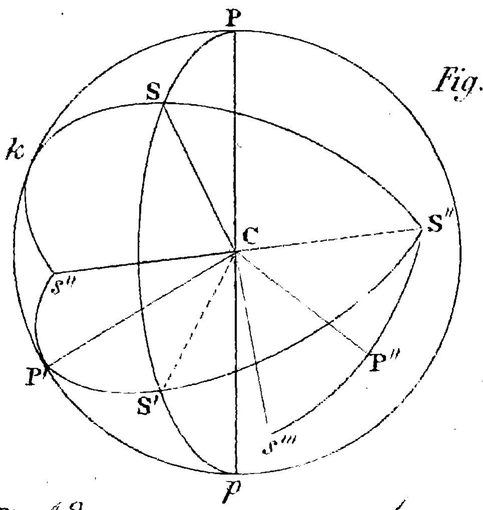


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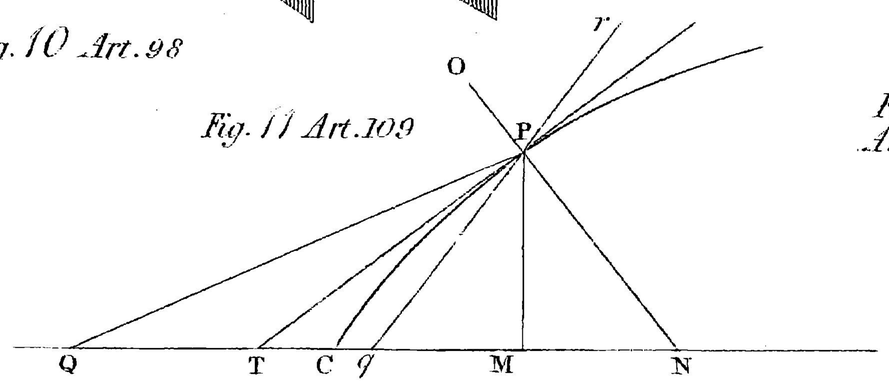


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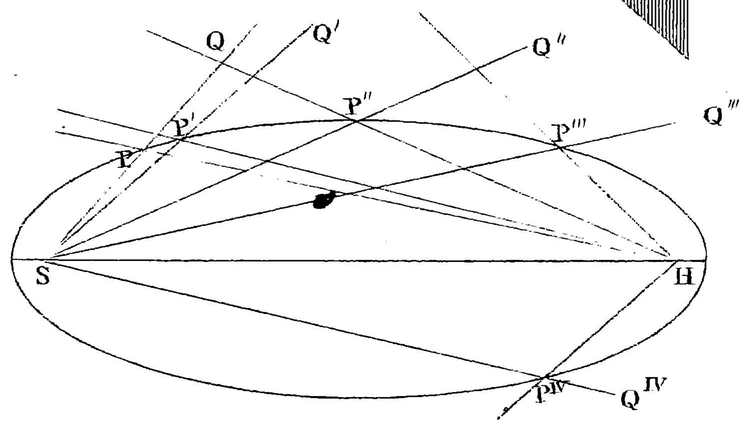


Fig. 13 Art. 119

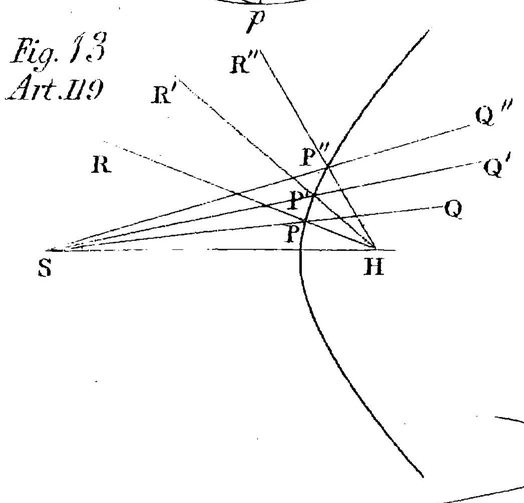


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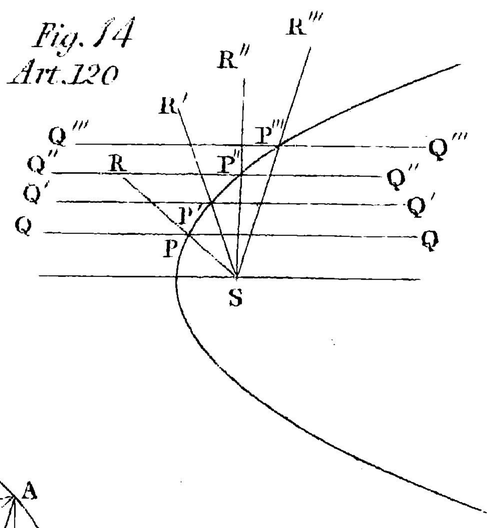


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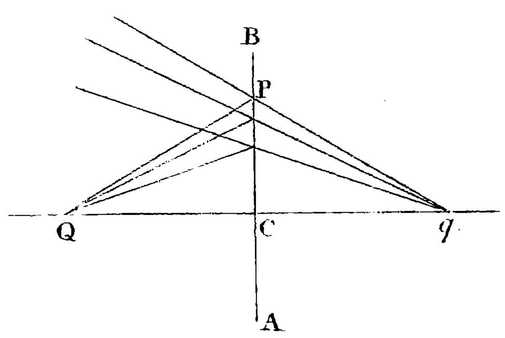


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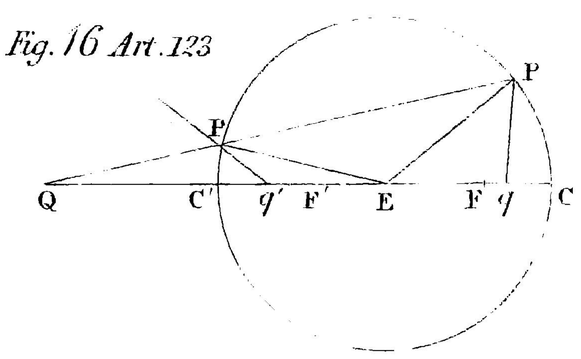


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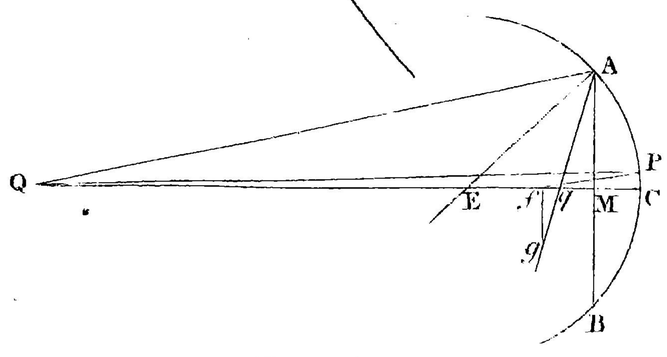


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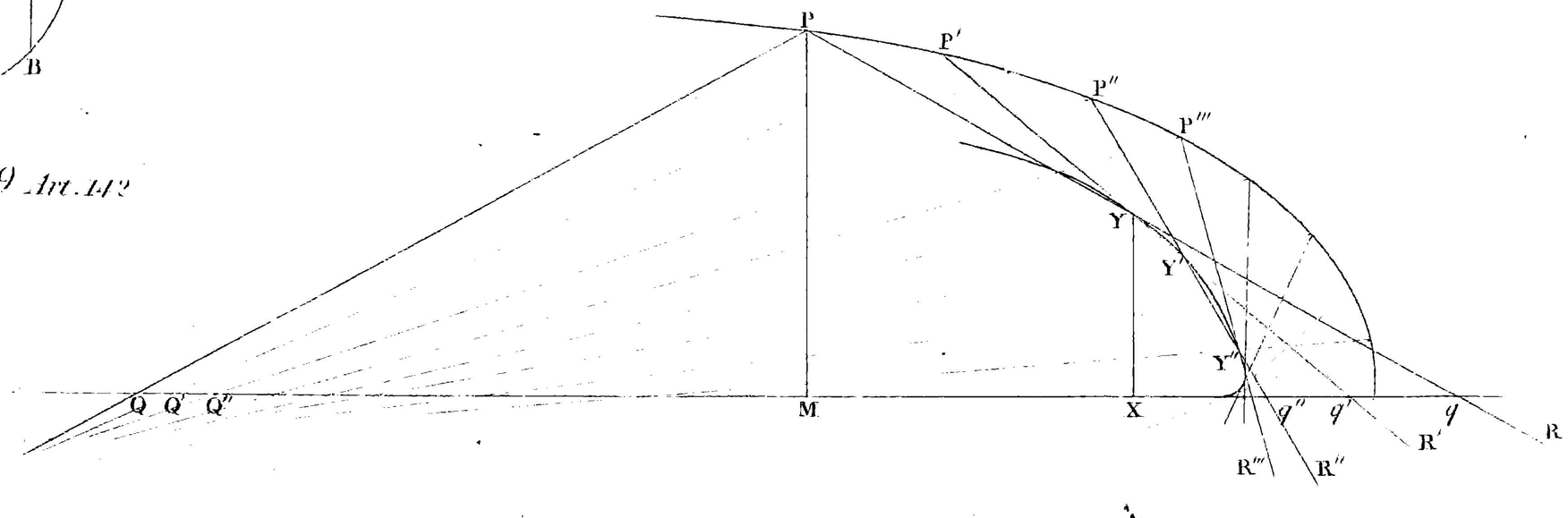
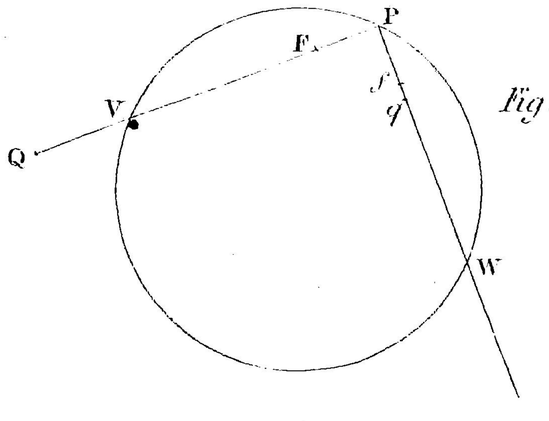


Fig. 19 Art. 142



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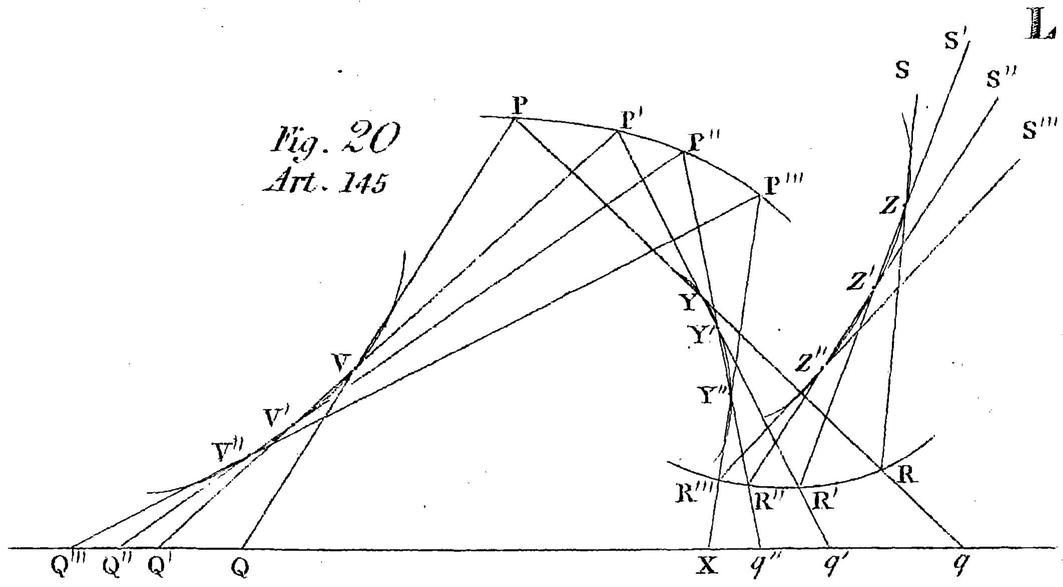


Fig. 20
Art. 145

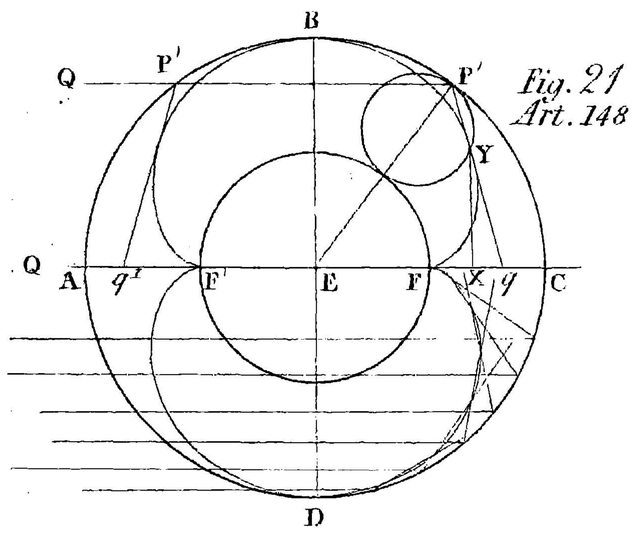


Fig. 21
Art. 148

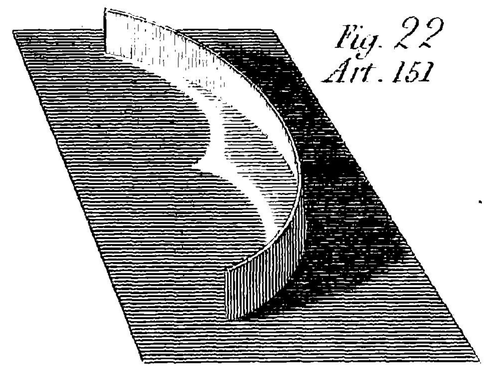


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Art. 151

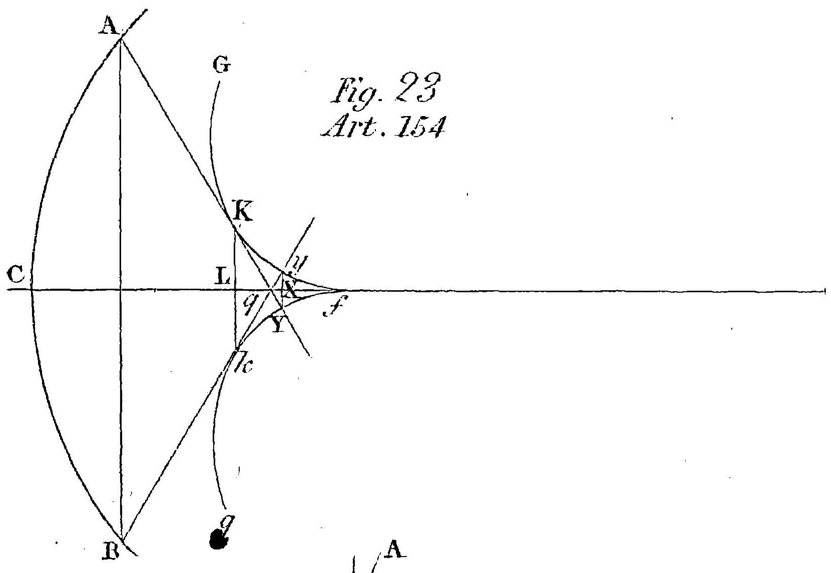


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Art. 154

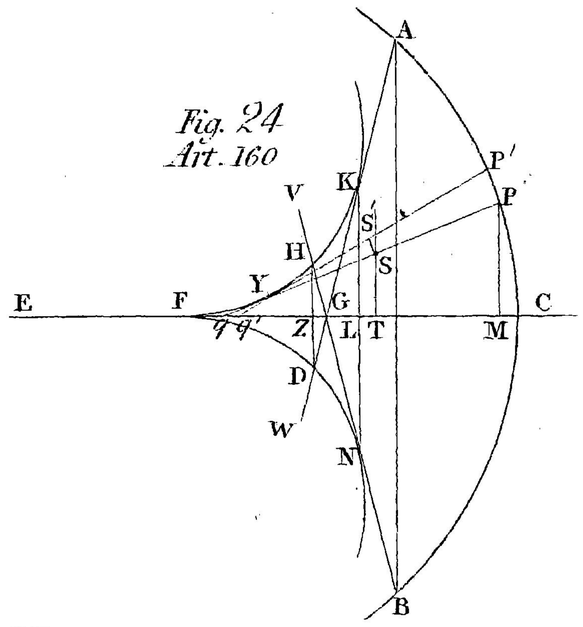


Fig. 24
Art. 160

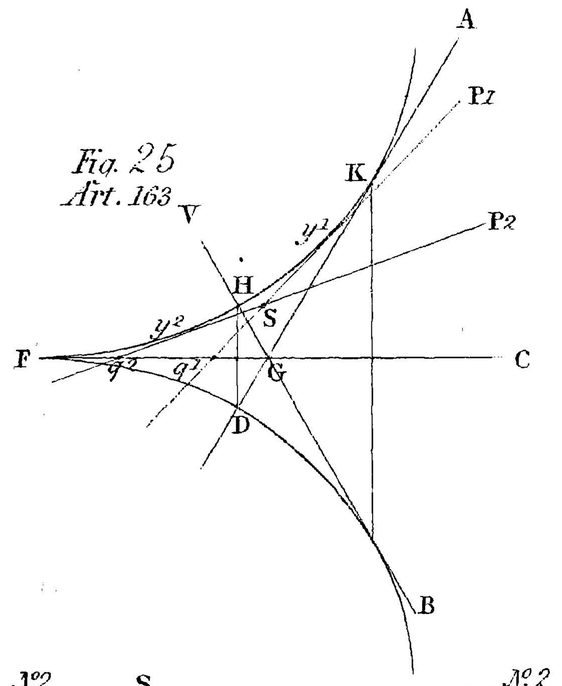


Fig. 25
Art. 163

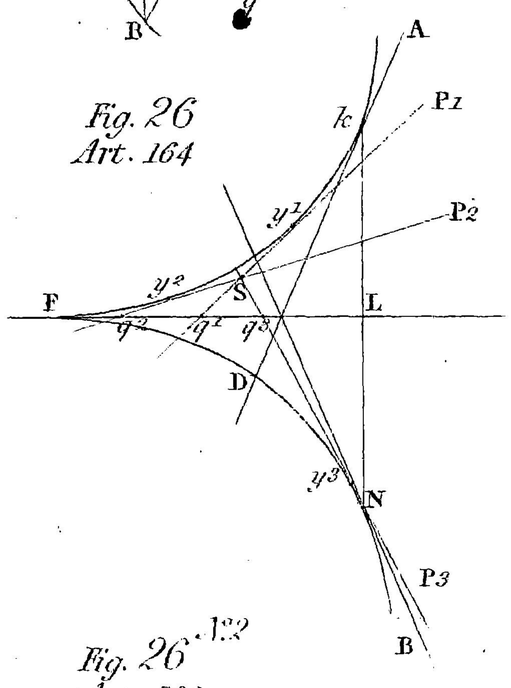


Fig. 26
Art. 164

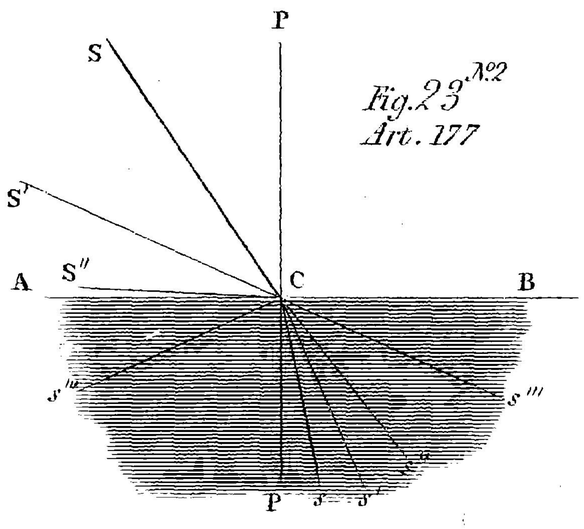


Fig. 23 №2
Art. 177

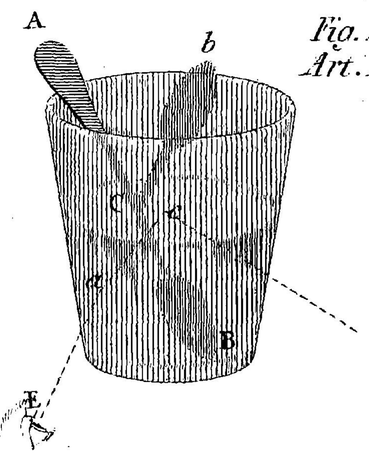


Fig. 24 №2
Art. 184

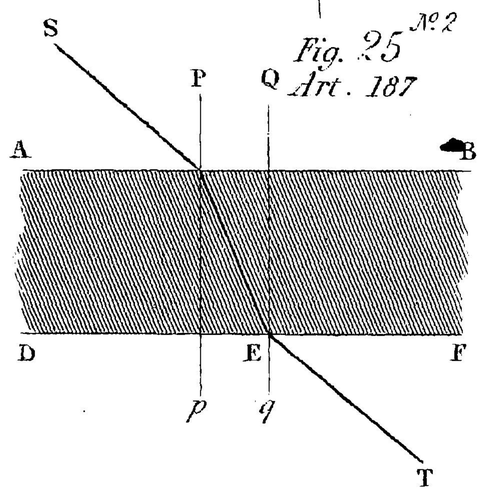


Fig. 25 №2
Art. 187

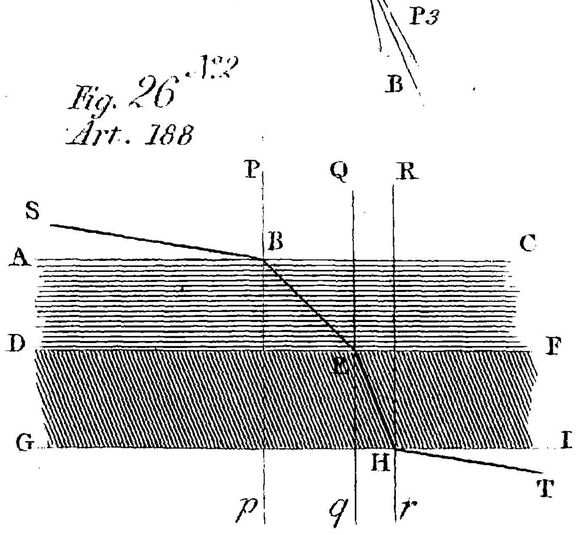


Fig. 26
Art. 188

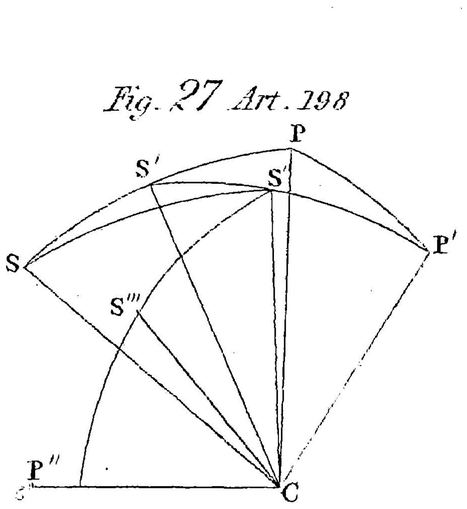


Fig. 27 Art. 198

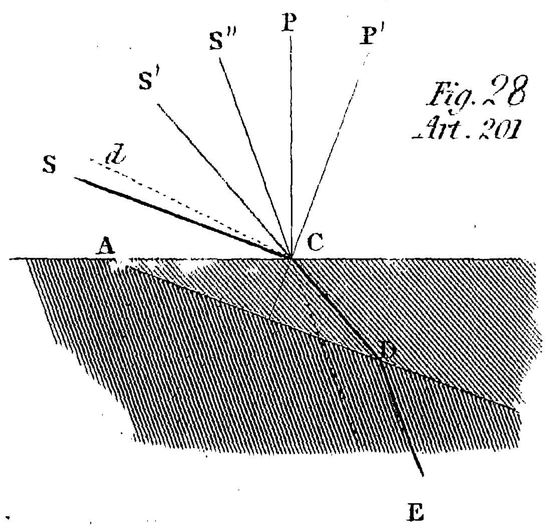


Fig. 28
Art. 201

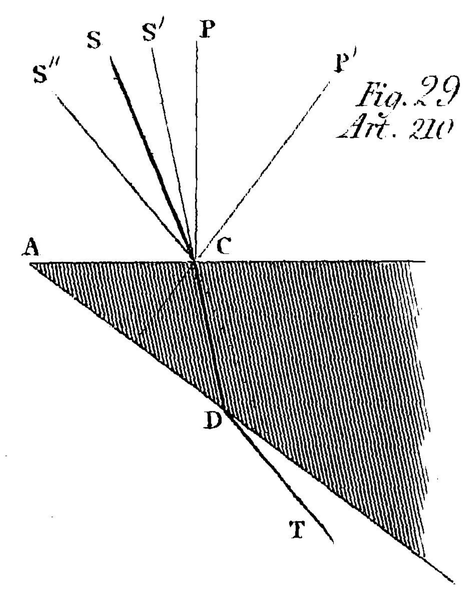


Fig. 29
Art. 210

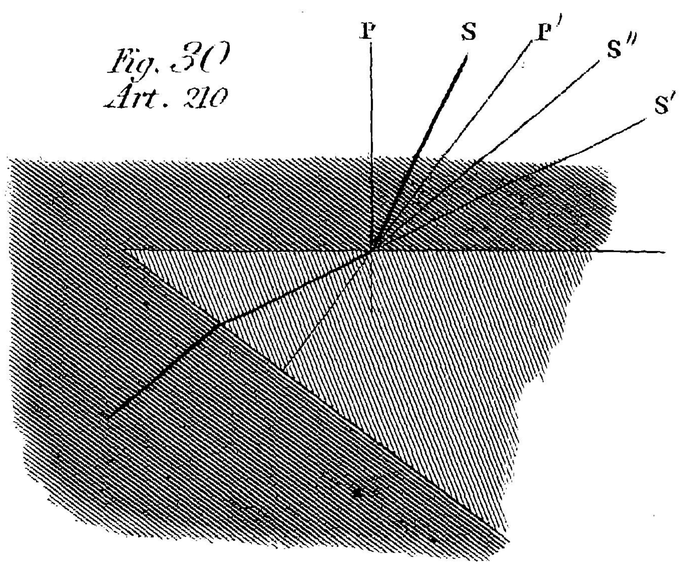


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Art. 210

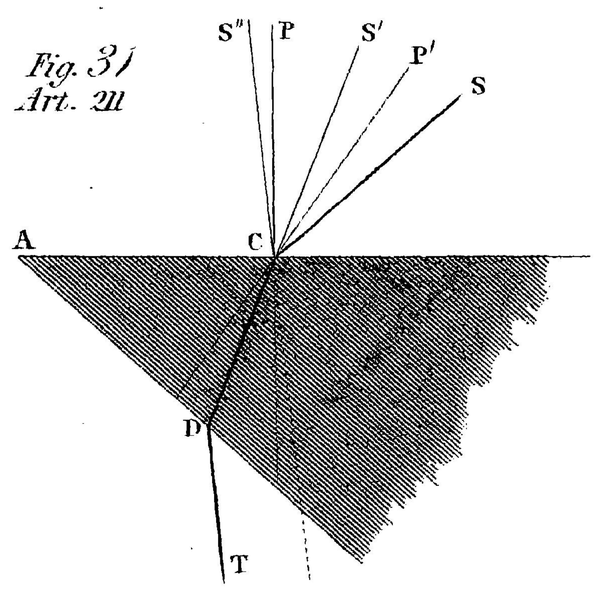


Fig. 31
Art. 211

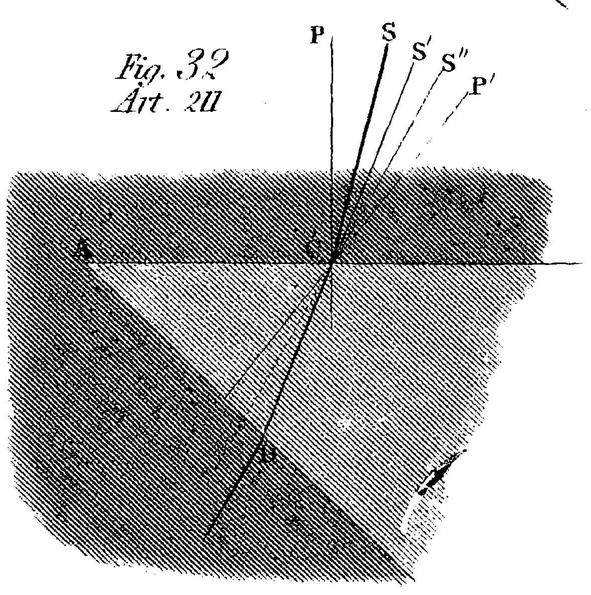


Fig. 32
Art. 211

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Fig. 33. Art. 221.

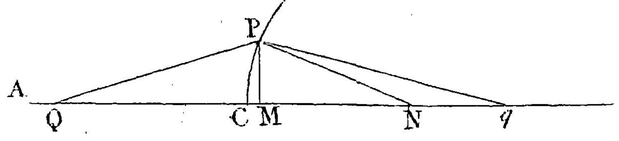


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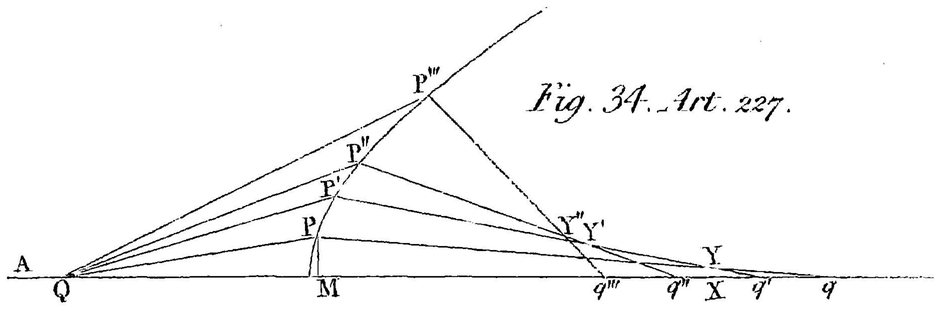


Fig. 36. Art. 233 & 303.

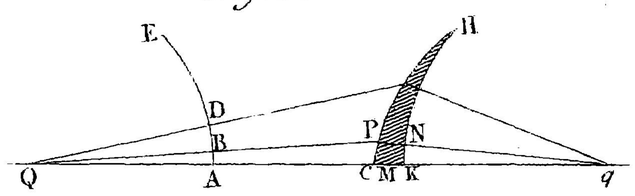


Fig. 35.
Art. 231.

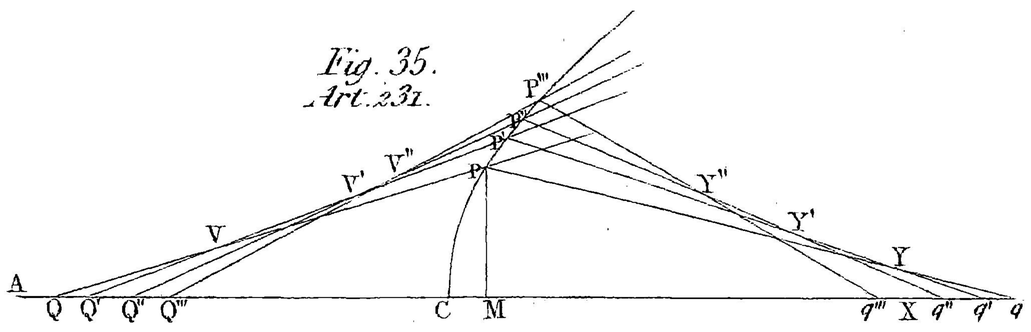


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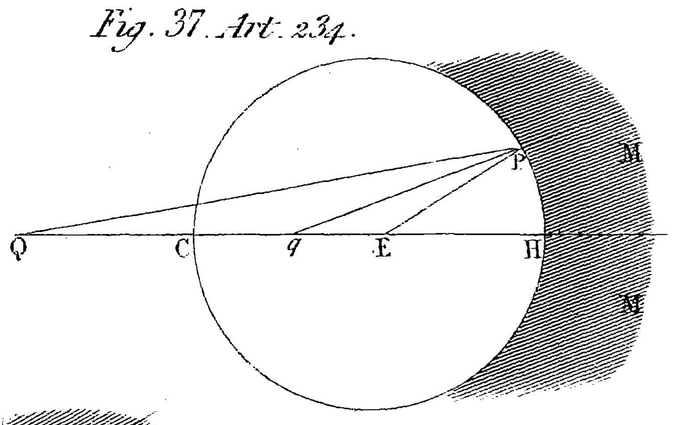


Fig. 39. Art. 238.

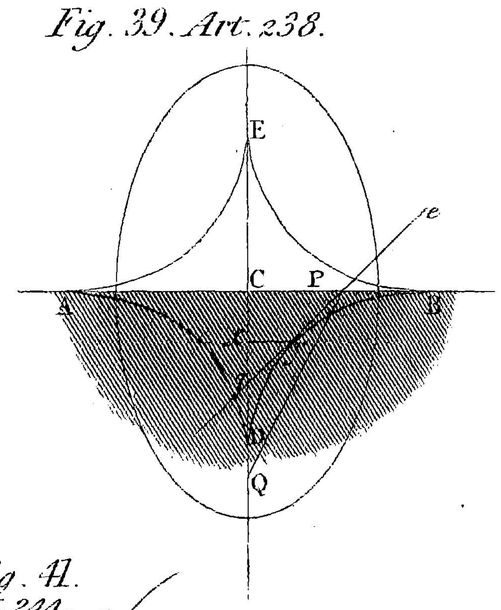


Fig. 38. Art. 236 & 303.

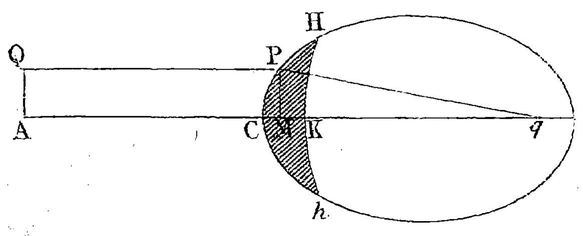


Fig. 40.
Art. 238.

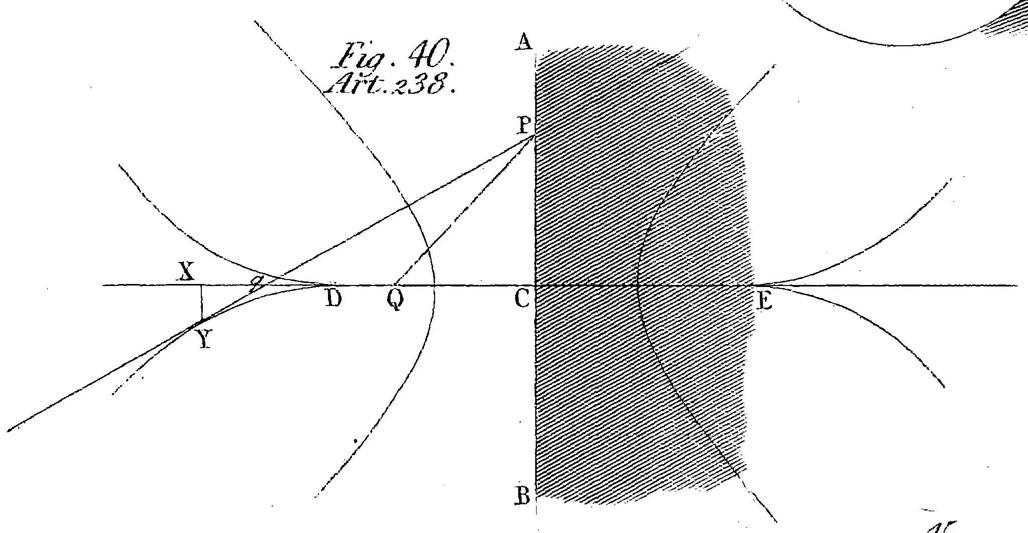
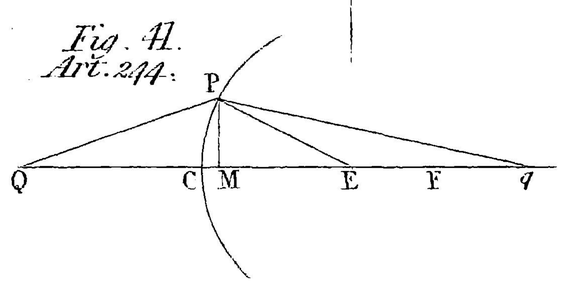


Fig. 41.
Art. 244.



Article 259.

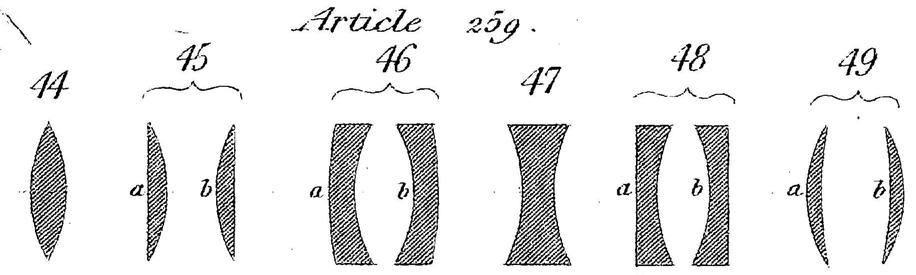


Fig. 42. Art. 247.

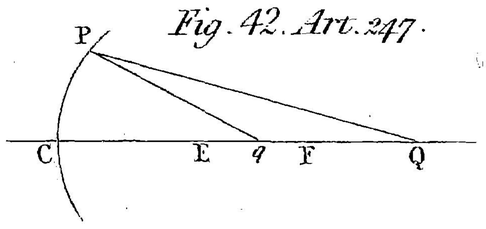


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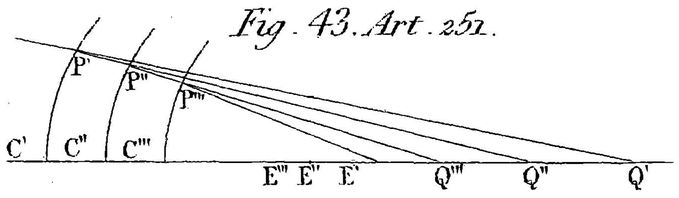


Fig. 50. Art. 269.

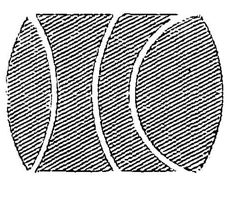


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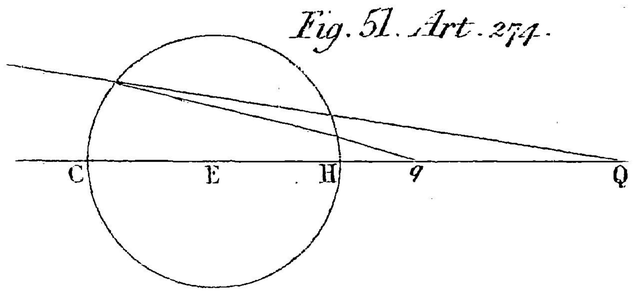


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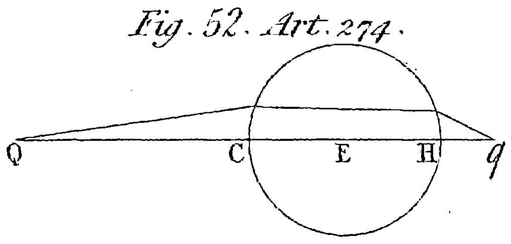
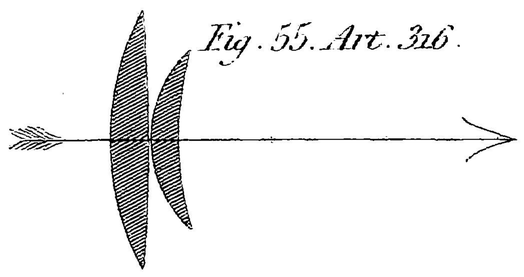


Fig. 55. Art. 316.



K

Fig. 53. Art. 287 & 304.

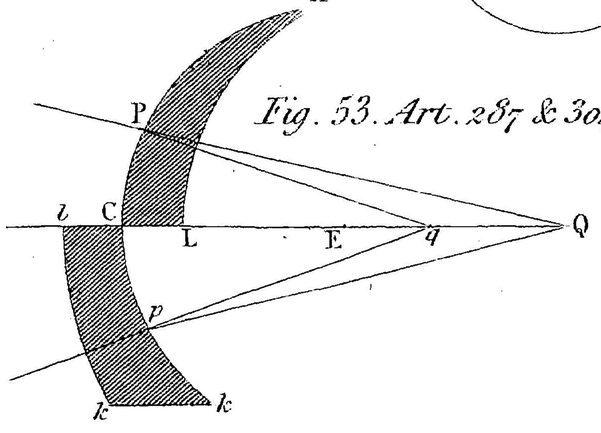


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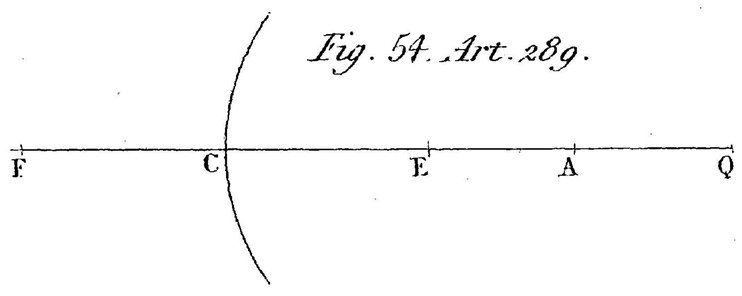


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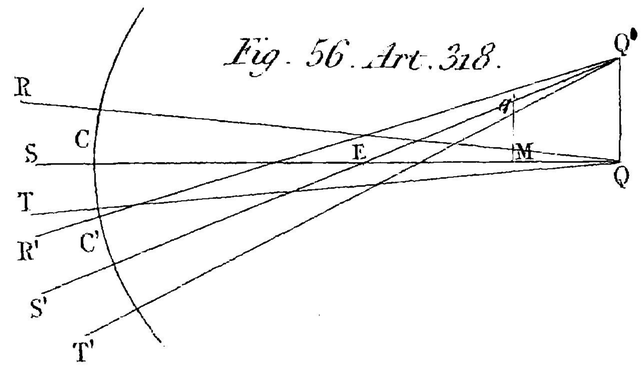


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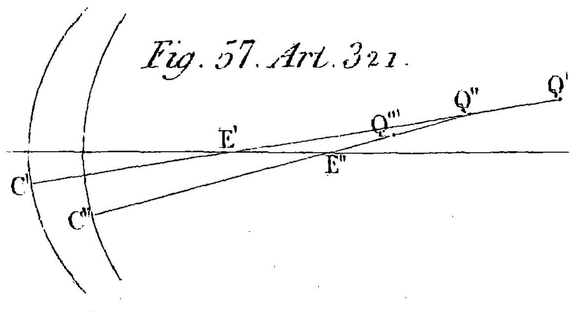


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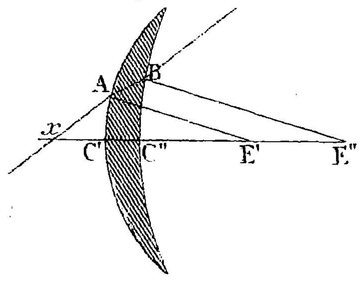


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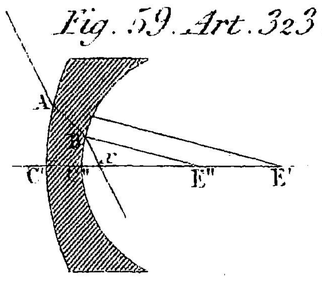


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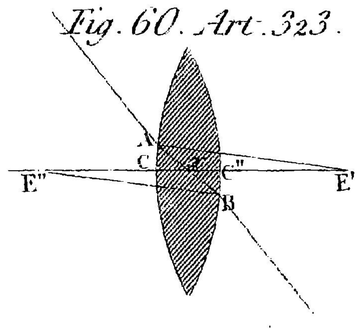
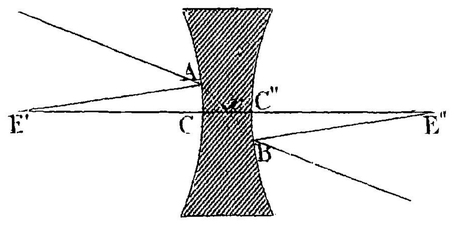


Fig. 61. Art. 323.



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Fig. 62. Art. 329.

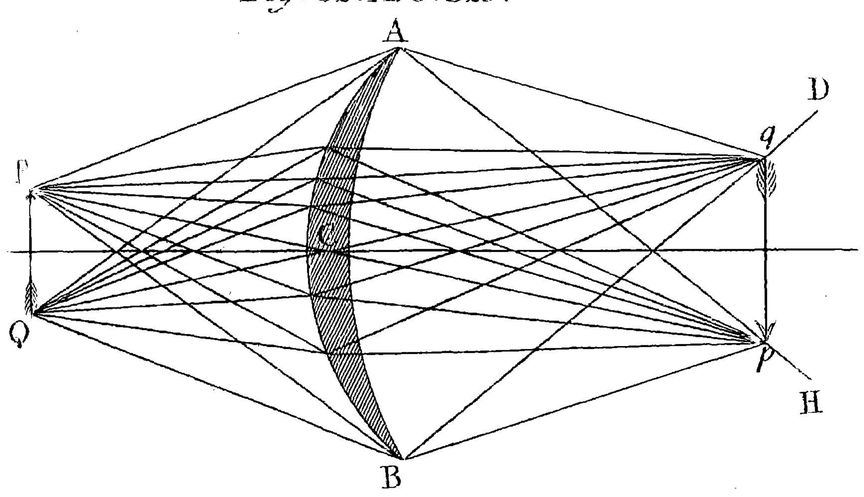


Fig. 63. Art. 330.

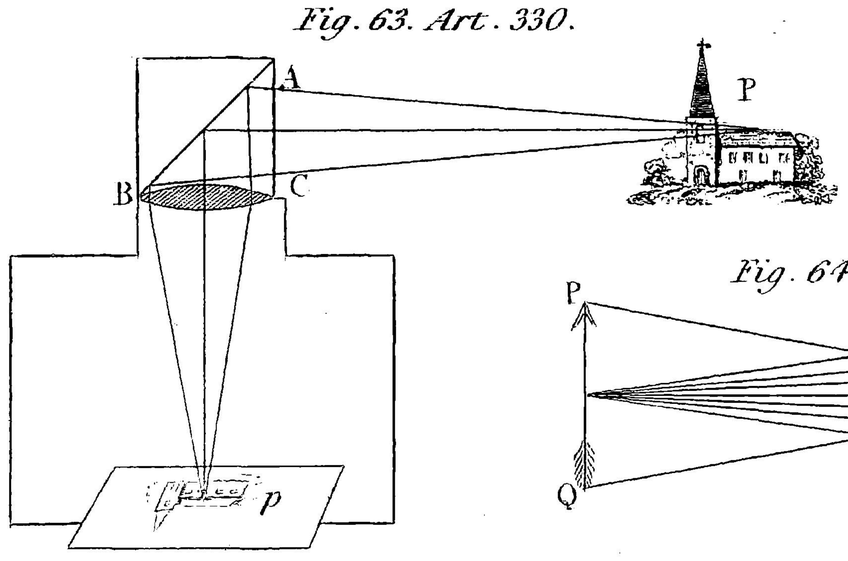


Fig. 64. Art. 333.

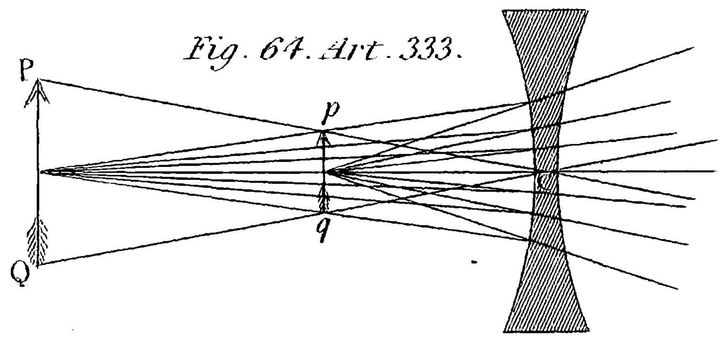


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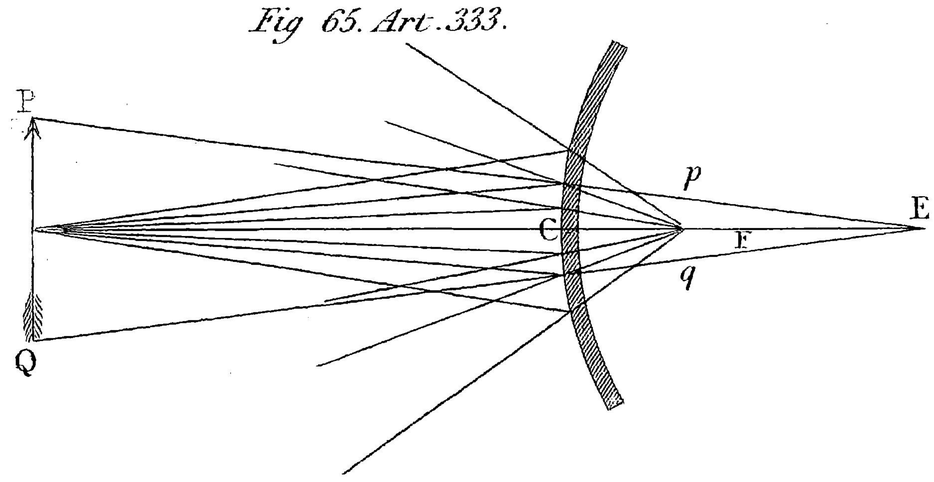


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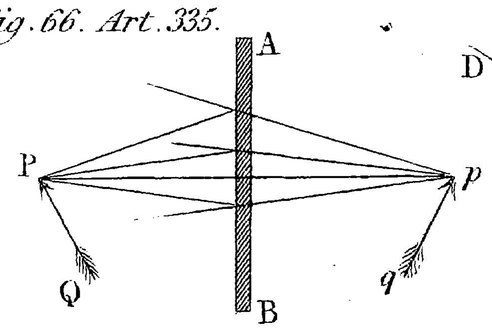


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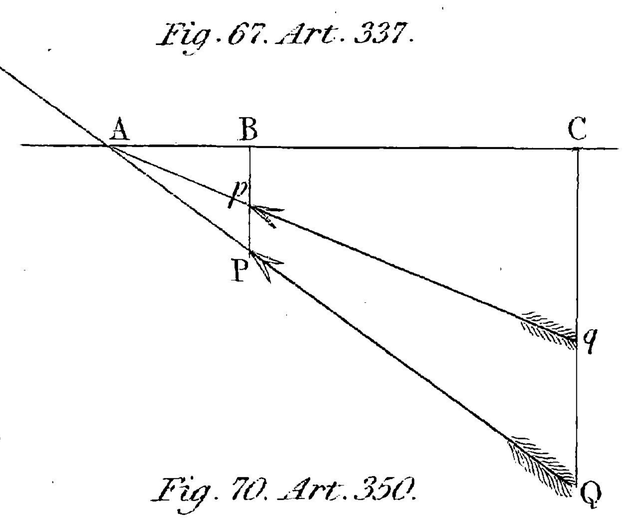


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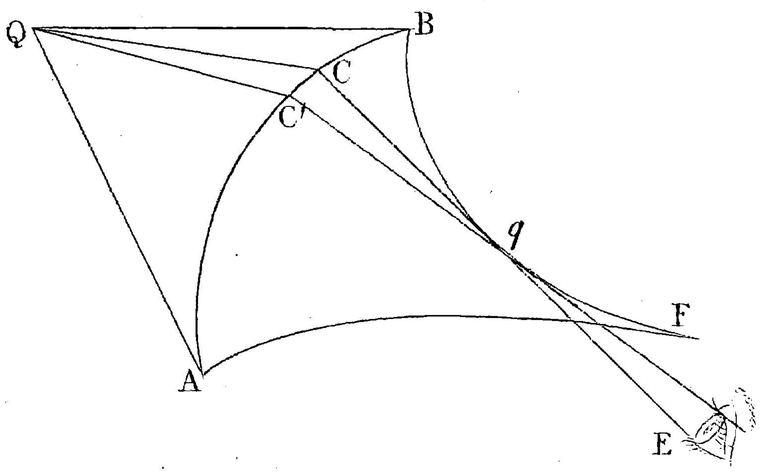


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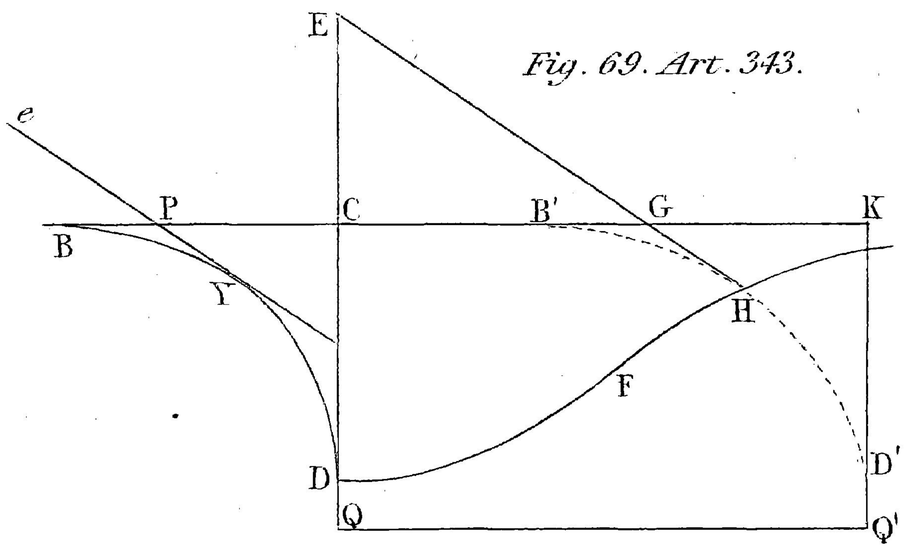


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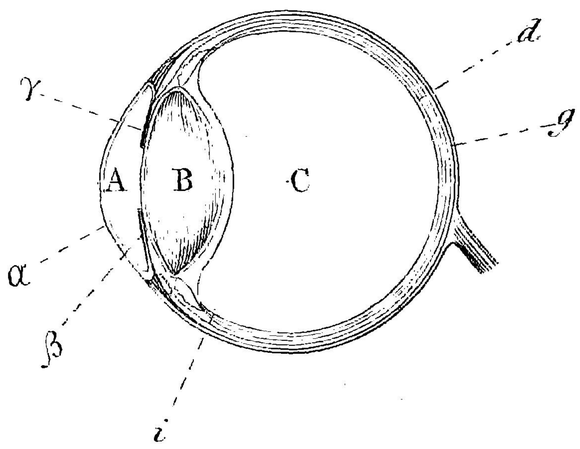


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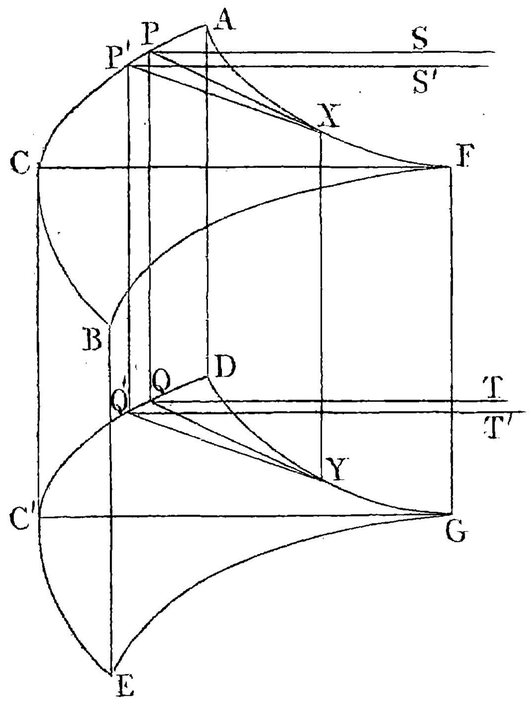


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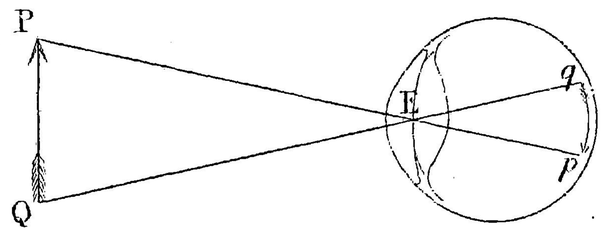


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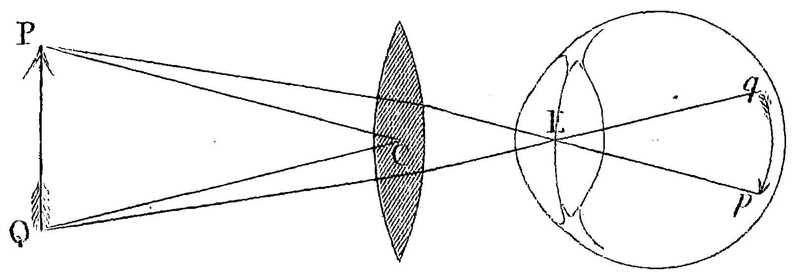


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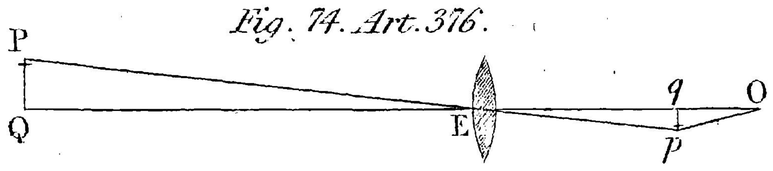


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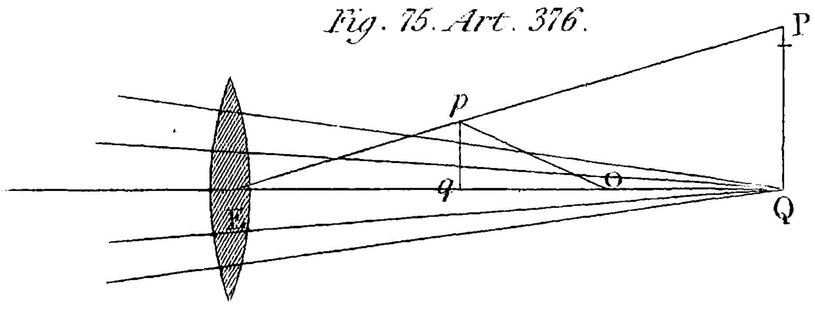


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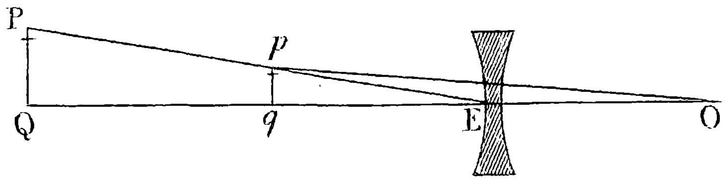


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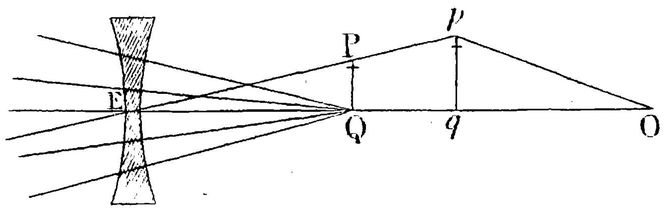
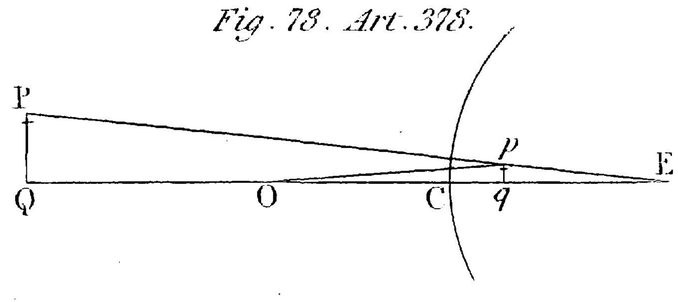


Fig. 78. Art. 378.



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J. W. Lowry. sculp.ᵗ

L I G H T

Plate 5

Fig. 79 Art. 378

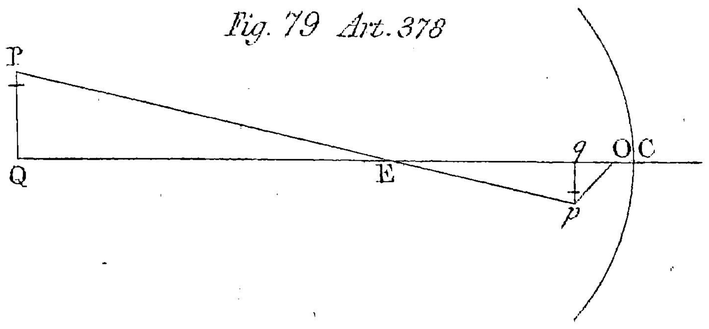


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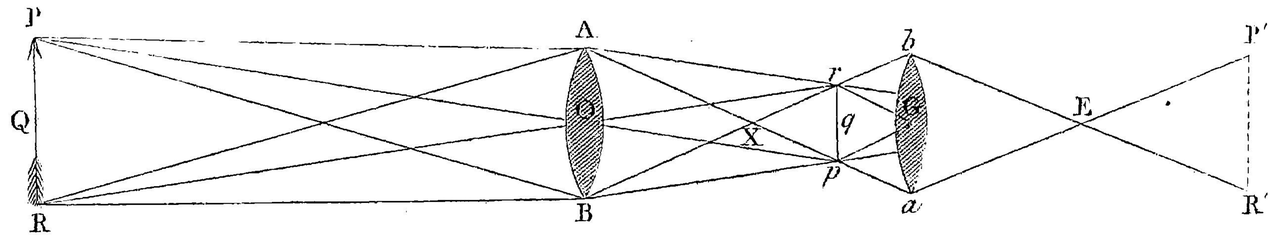


Fig. 81 Art. 387

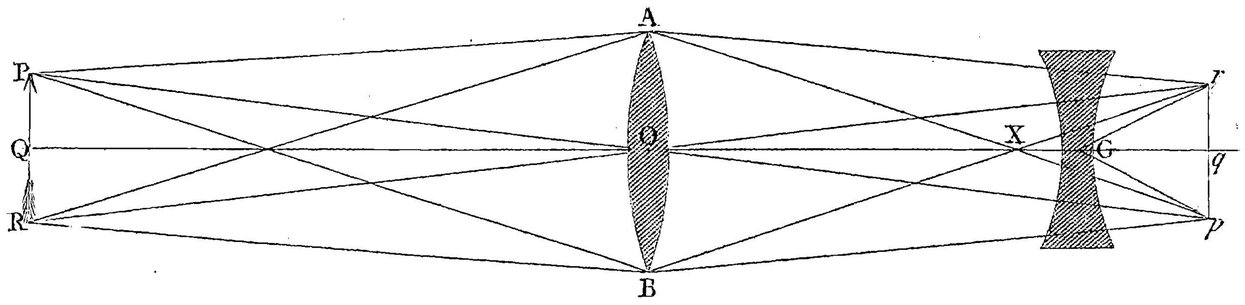


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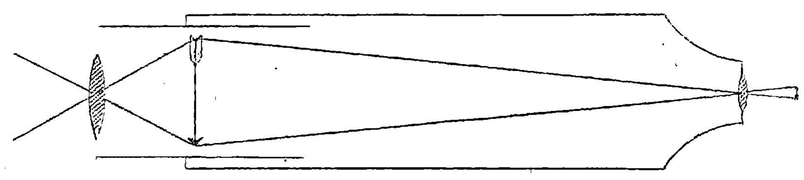


Fig. 83 Art. 391

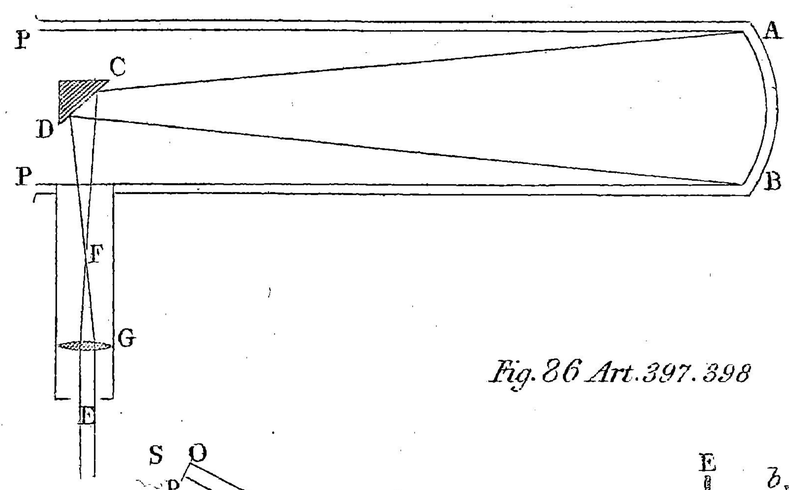


Fig. 84 Art. 392

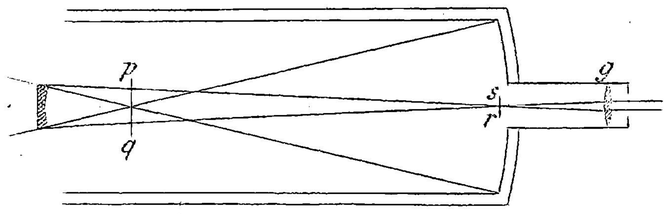


Fig. 85 Art. 395

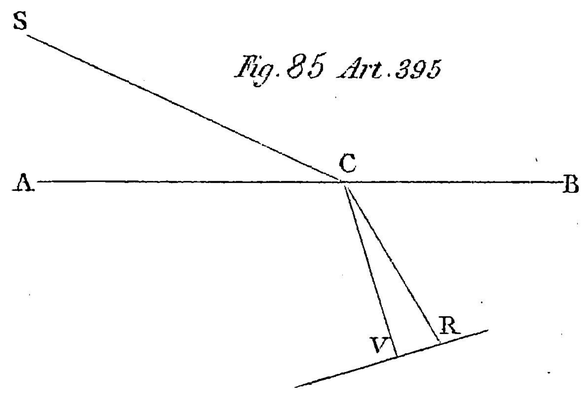


Fig. 86 Art. 397. 398

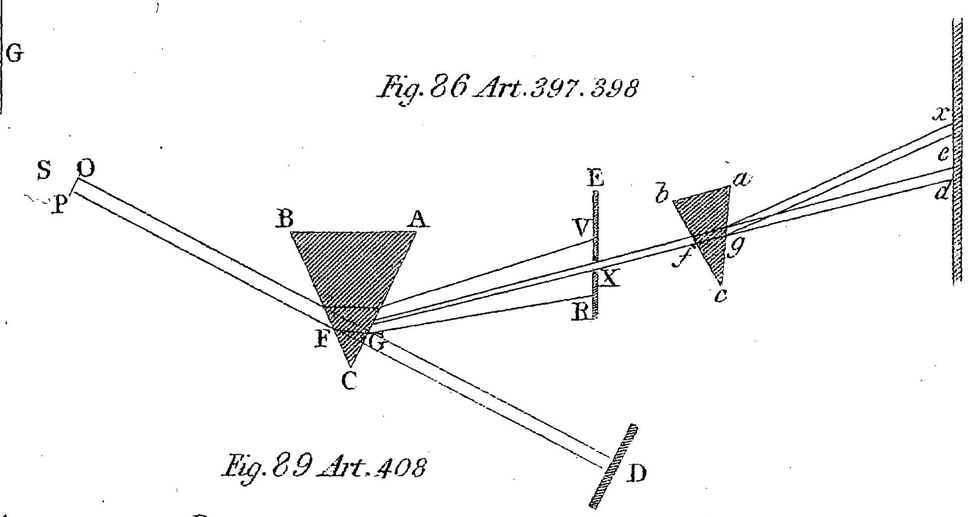


Fig. 87 Art. 406

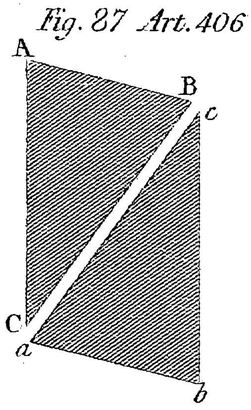


Fig. 88 Art. 406

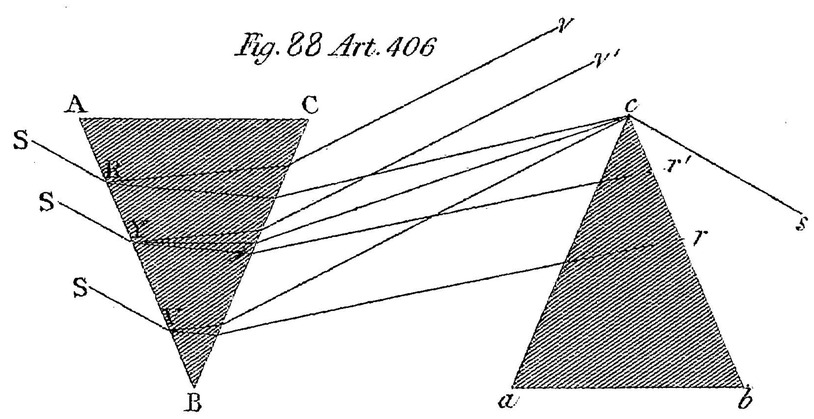


Fig. 89 Art. 408

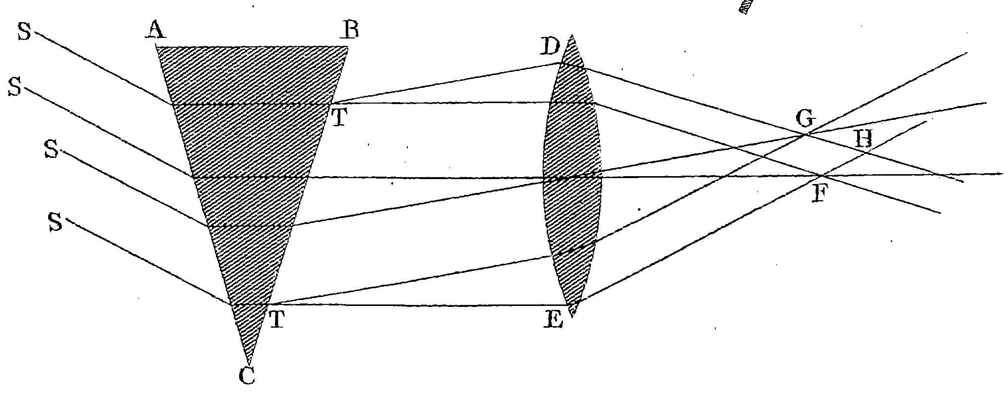


Fig. 89 N°2 Art. 412

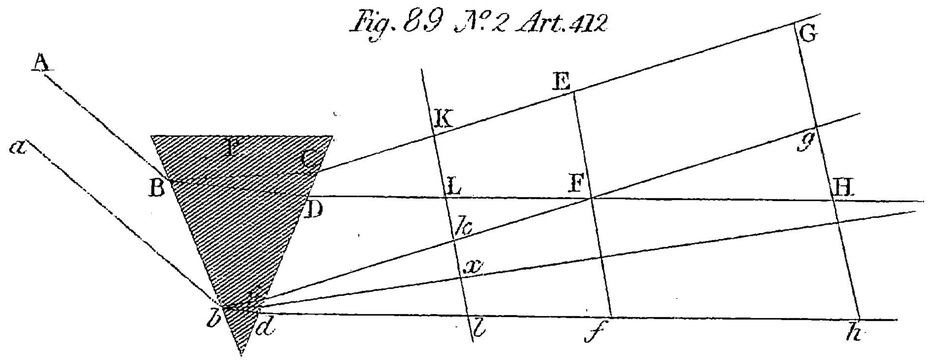


Fig. 91 Art. 413

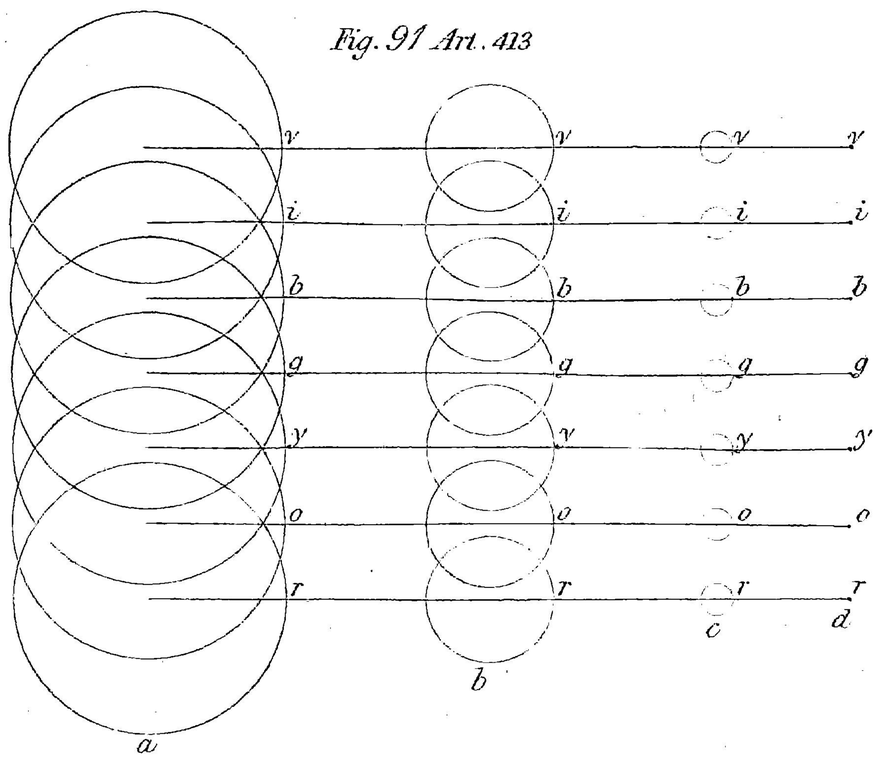


Fig. 92 Art. 415

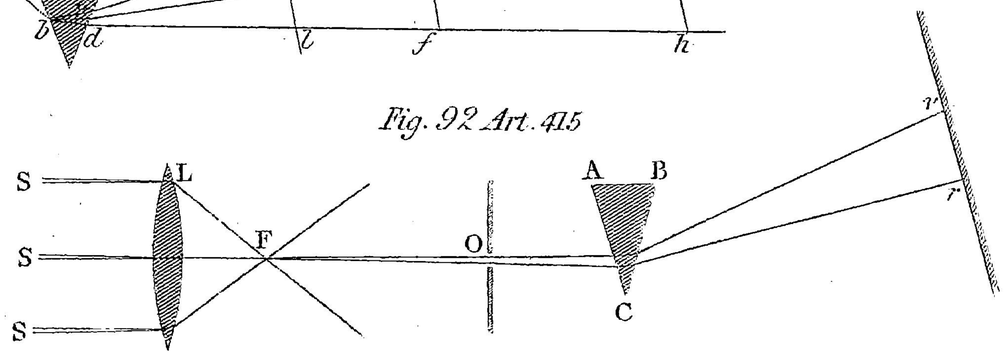


Fig. 90 Art. 413

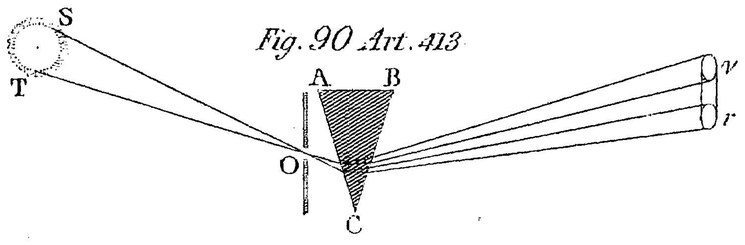
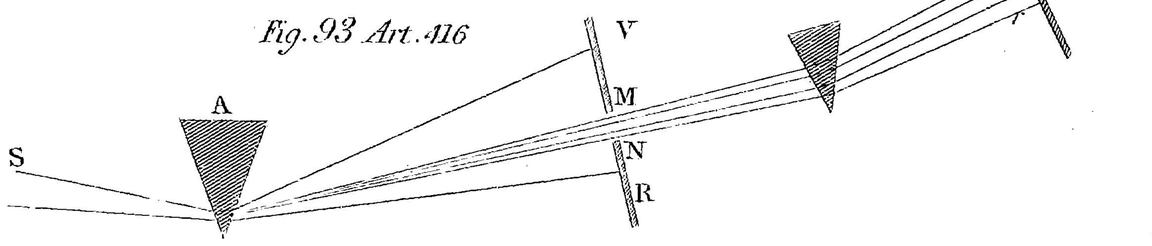


Fig. 93 Art. 416



Published as the Act directs May 1. 1827 by J. Mawman Ludgate Street.

J. Lowry Sculp.

LIGHT.

Fig. 94. Art. 419.

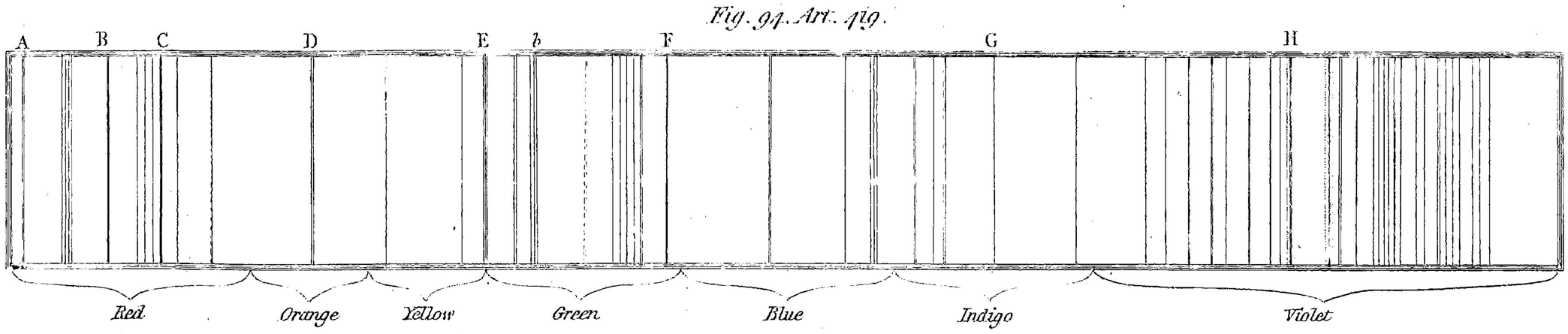


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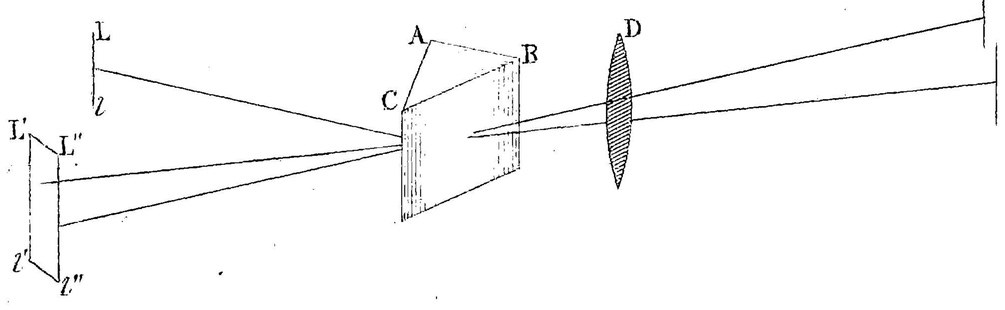


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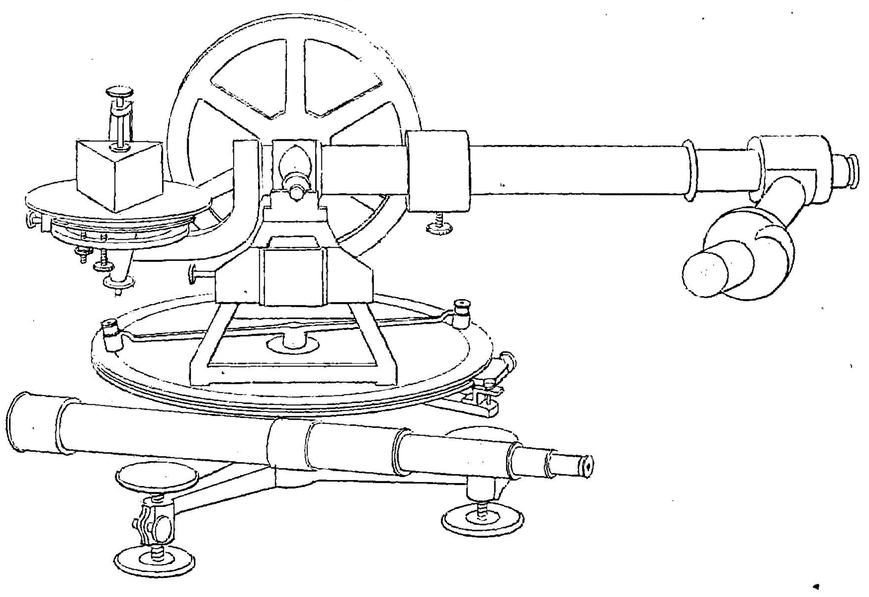


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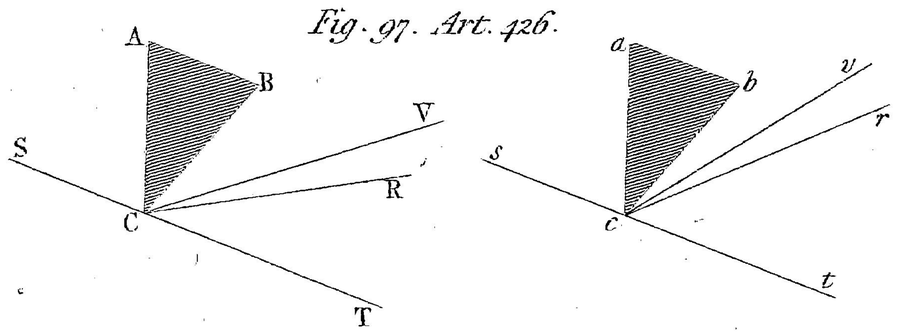


Fig. 98. Art. 426.

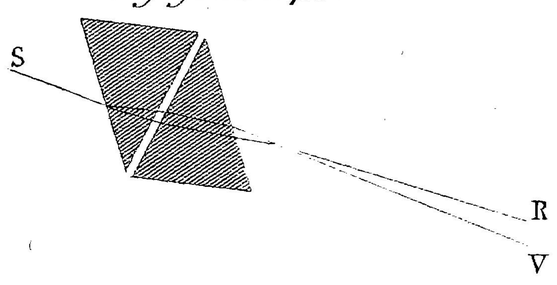


Fig. 99. Art. 429.

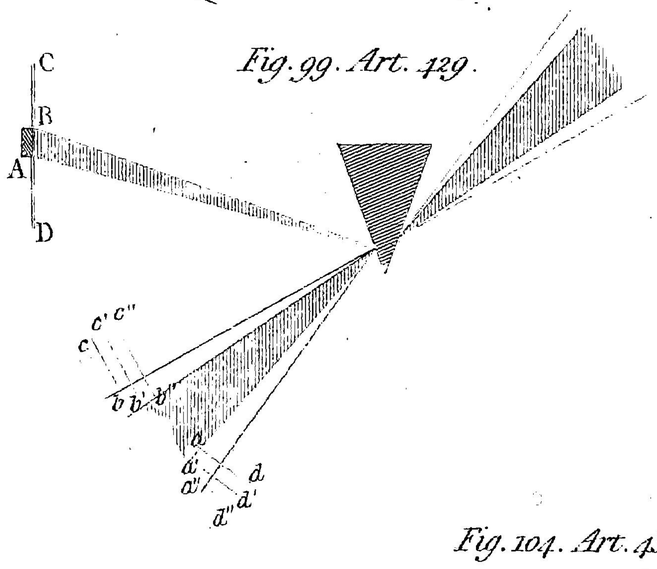


Fig. 100. Art. 431.

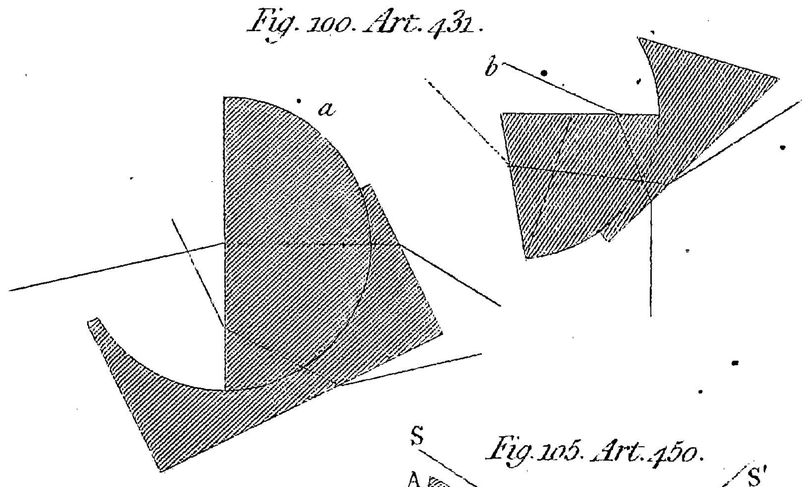


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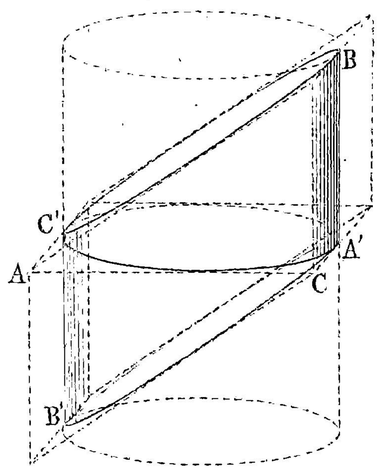


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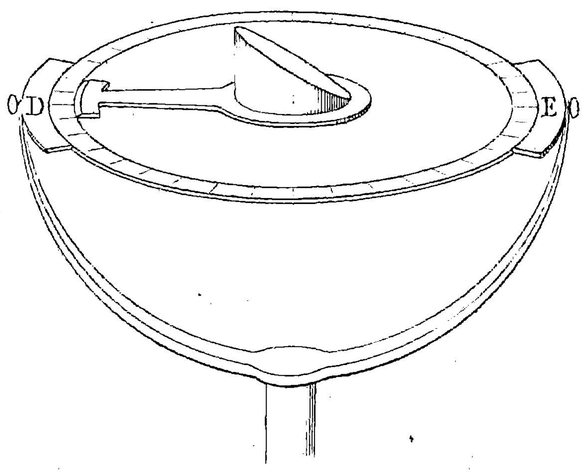


Fig. 103. Art. 434.

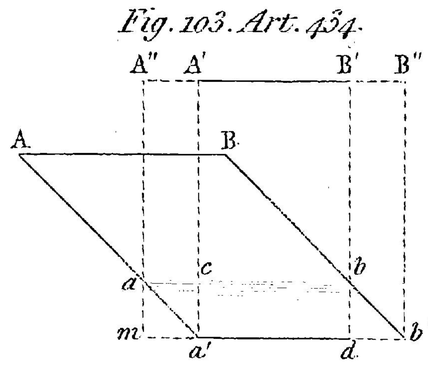


Fig. 104. Art. 435.

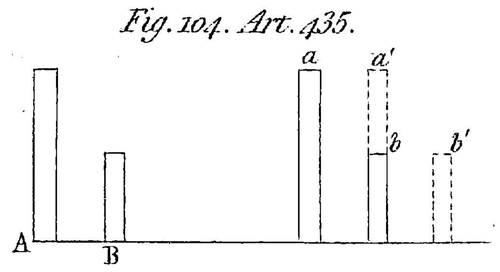


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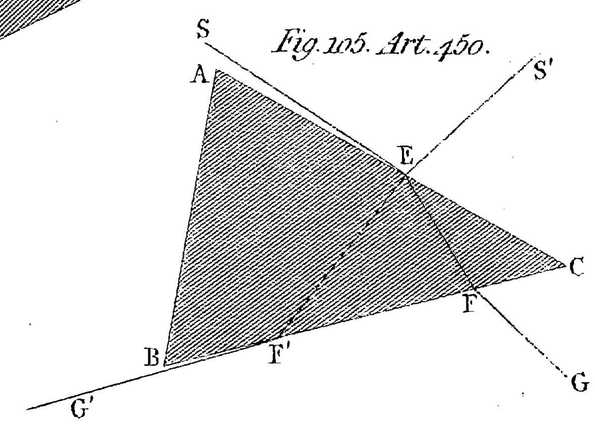


Fig. 106. Art. 451.

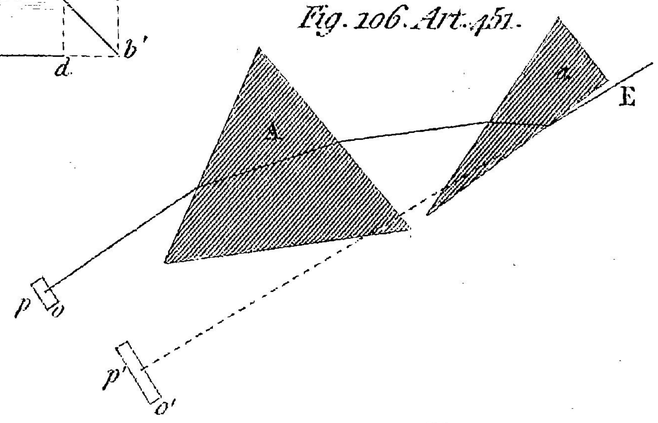


Fig. 107. Art. 457.

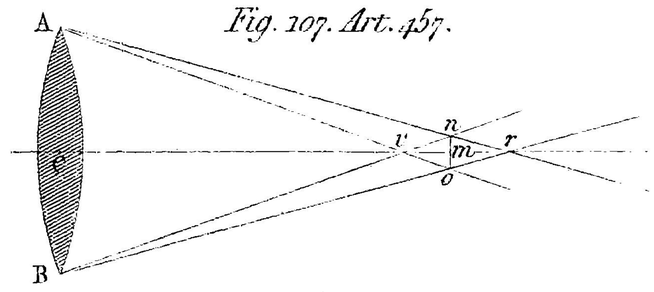


Fig. 108. Art. 470, 472.

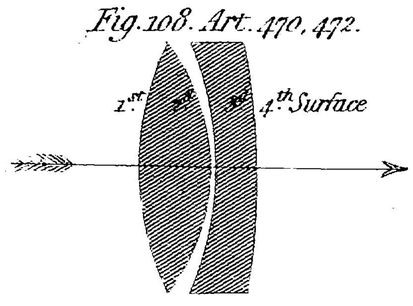


Fig. 111. Art. 476.

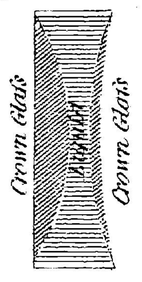


Fig. 109. Art. 475.

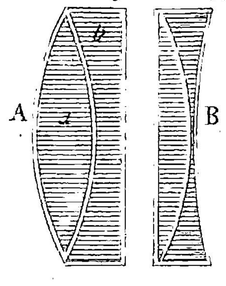
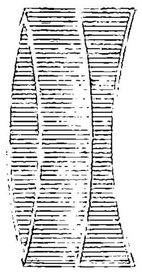


Fig. 110. Art. 475.



Published as the Act directs, April 1827 by J. Murman, Ludgate Street.

J. W. Lowry, sculpt.

LIGHT.

Fig. 112. Art. 490.

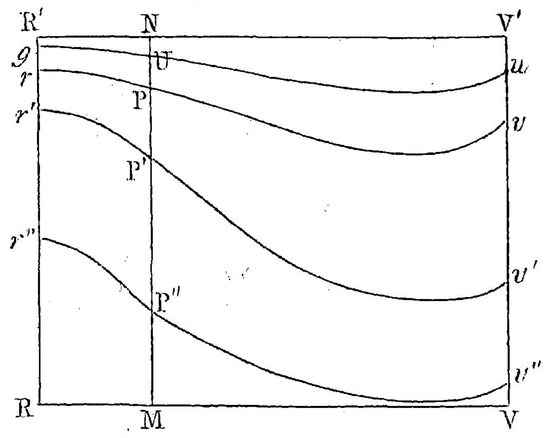


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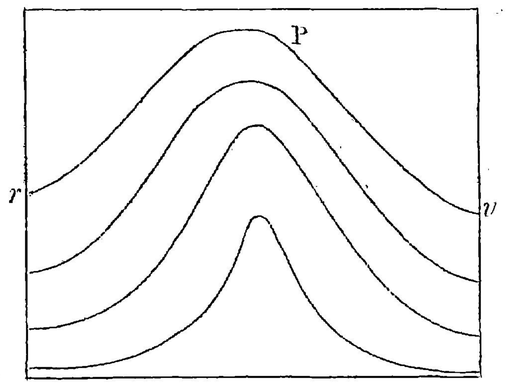


Fig. 114. Art. 494.

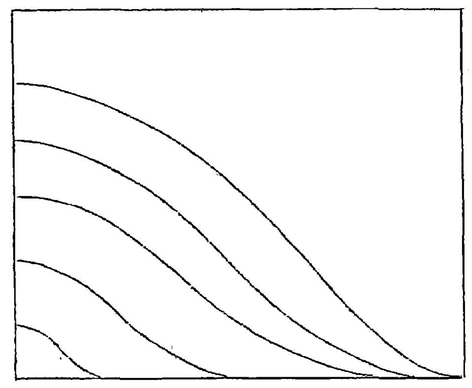


Fig. 115. Art. 495.

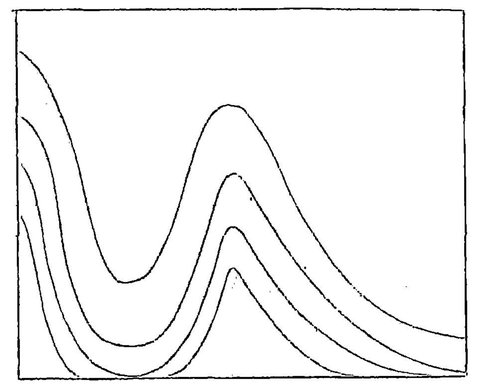


Fig. 116. Art. 496.

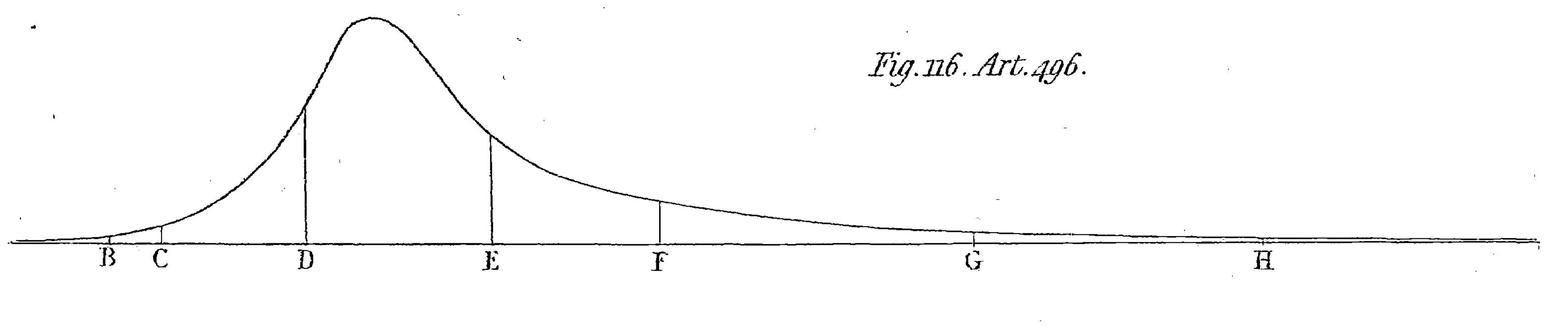


Fig. 117. Art. 497.

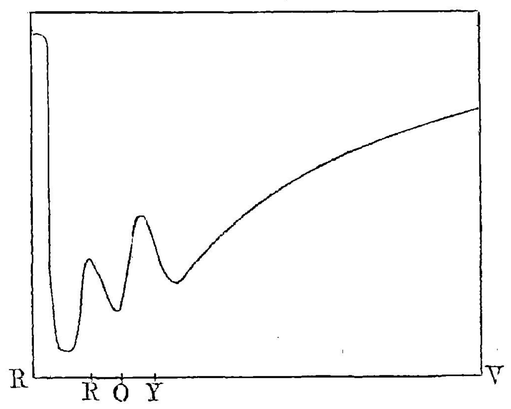


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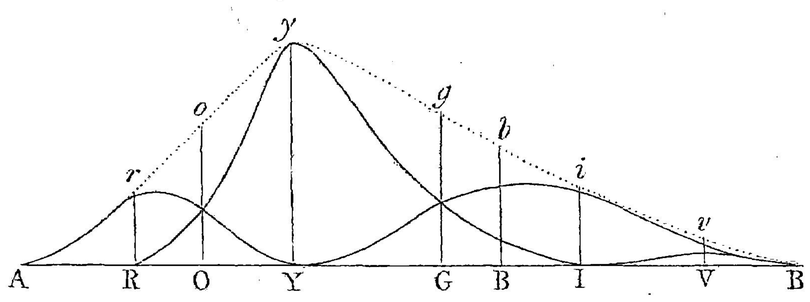


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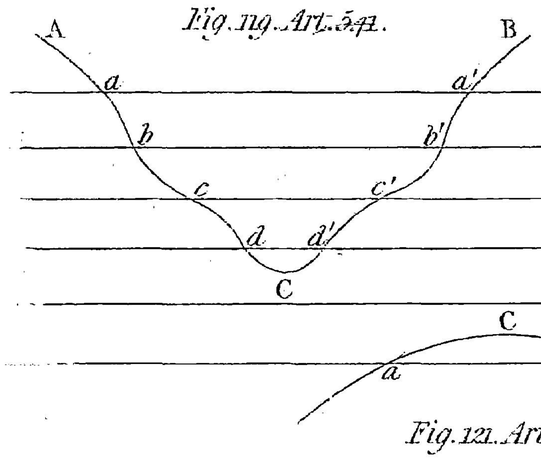


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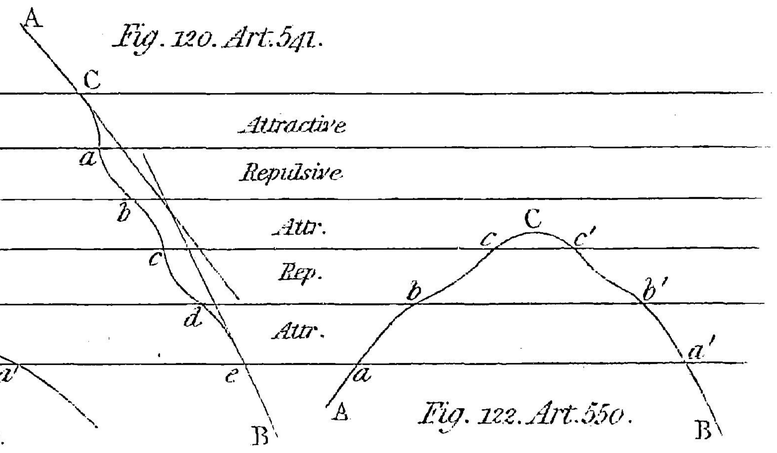


Fig. 121. Art. 550.

Fig. 122. Art. 550.

Fig. 123. Art. 550. 552.

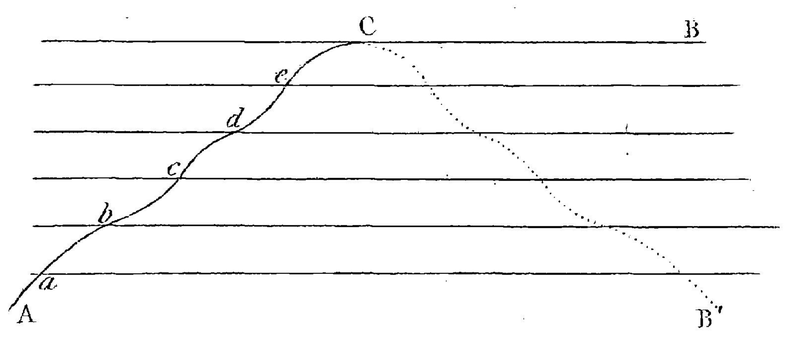


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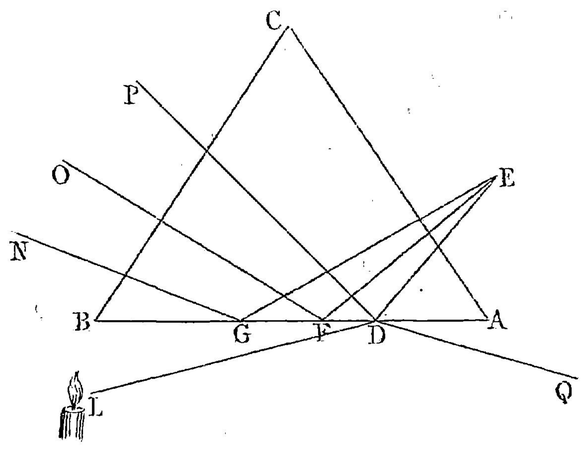


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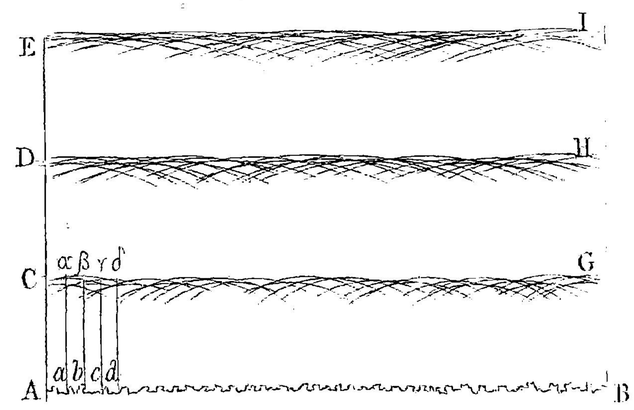


Fig. 126. Art. 579.

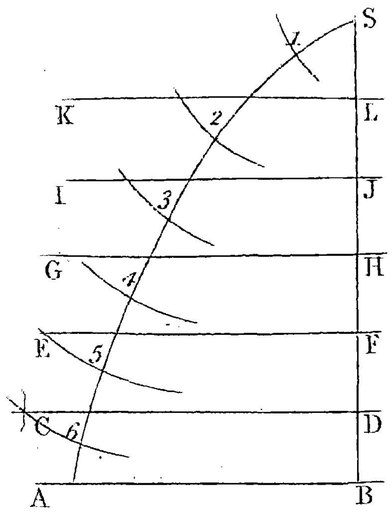


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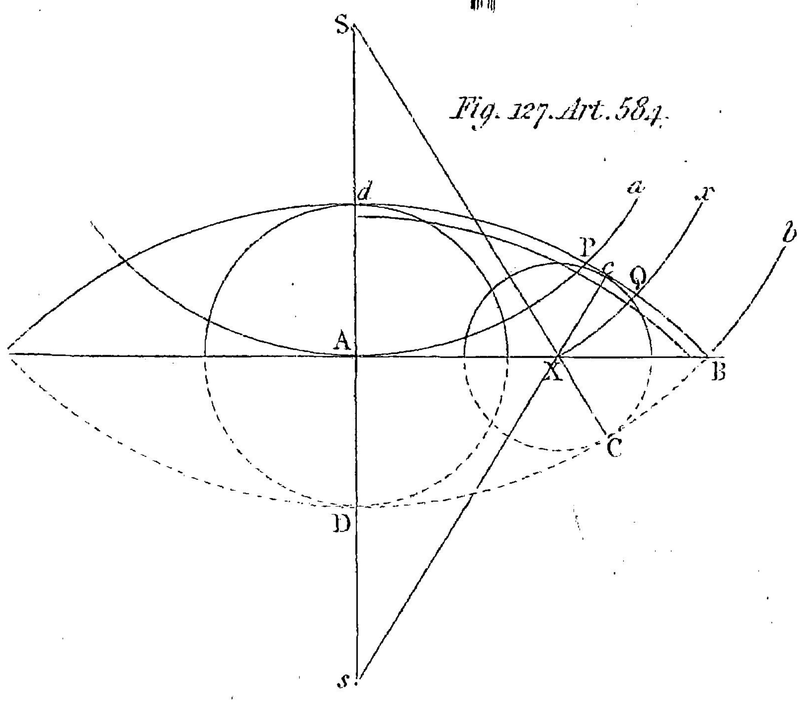


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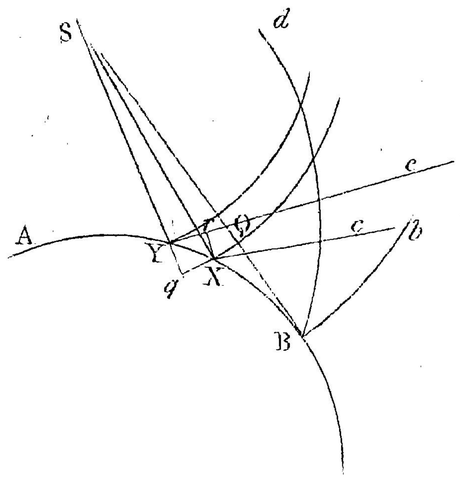
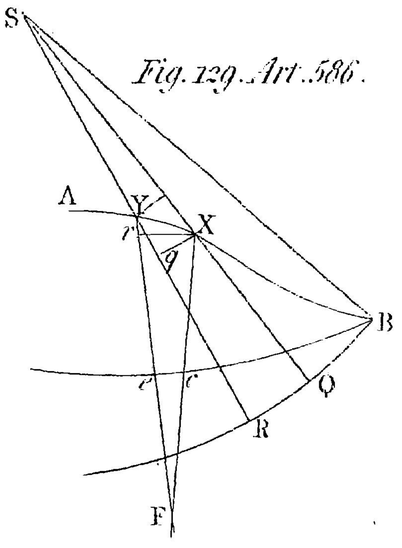


Fig. 129. Art. 586.



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Fig. 130. Art. 628.

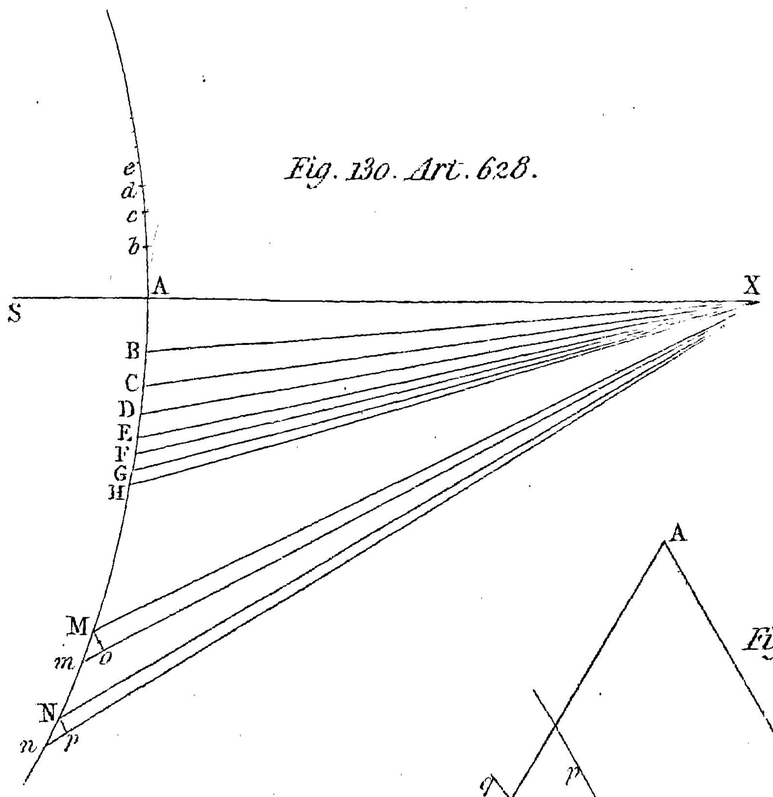


Fig. 131. Art. 463.

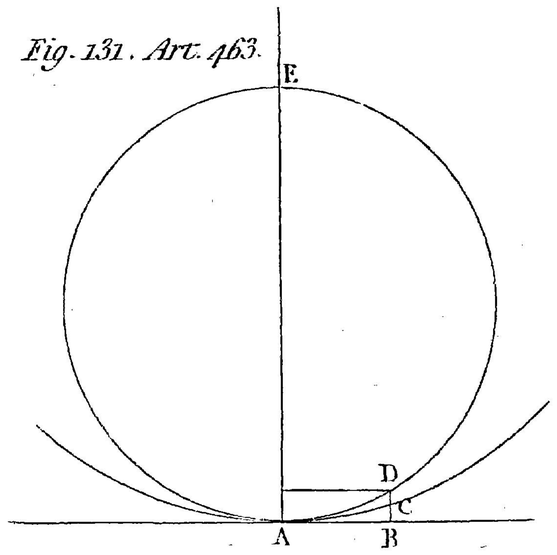


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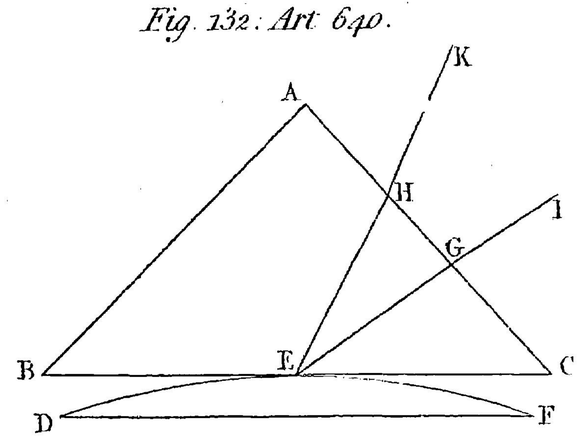


Fig. 133. Art. 641.

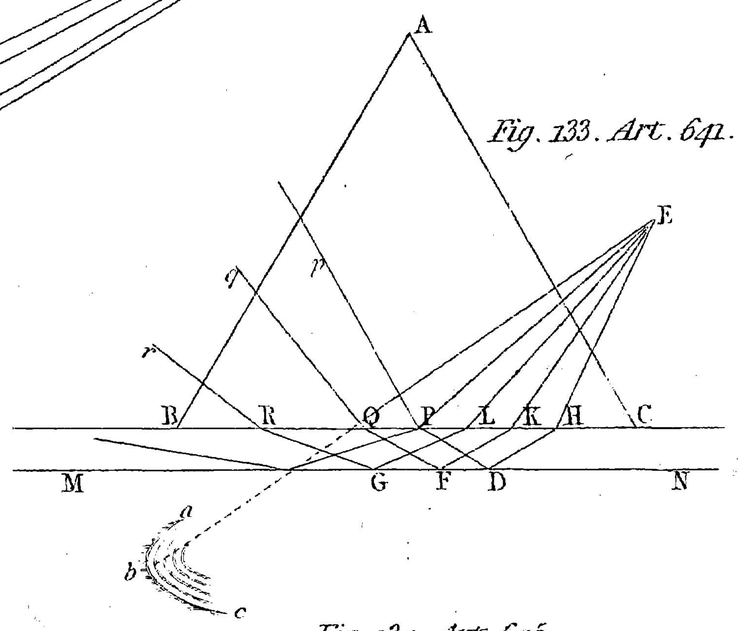


Fig. 134. Art. 645.

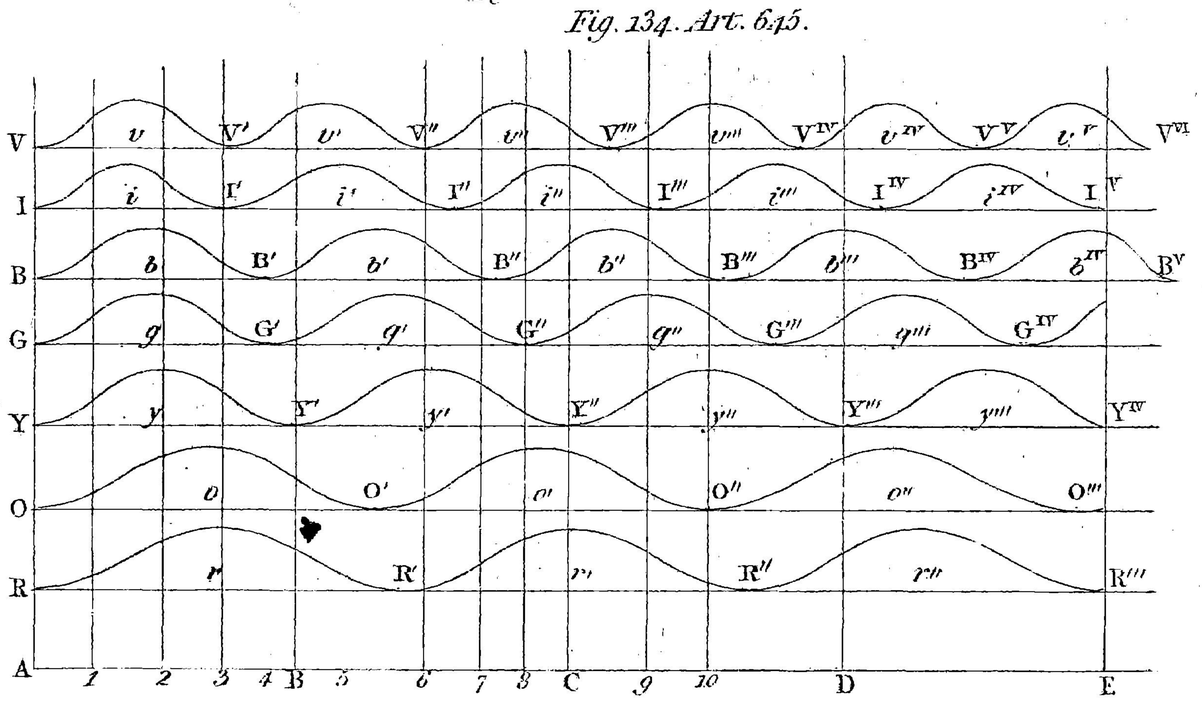


Fig. 135. Art. 664

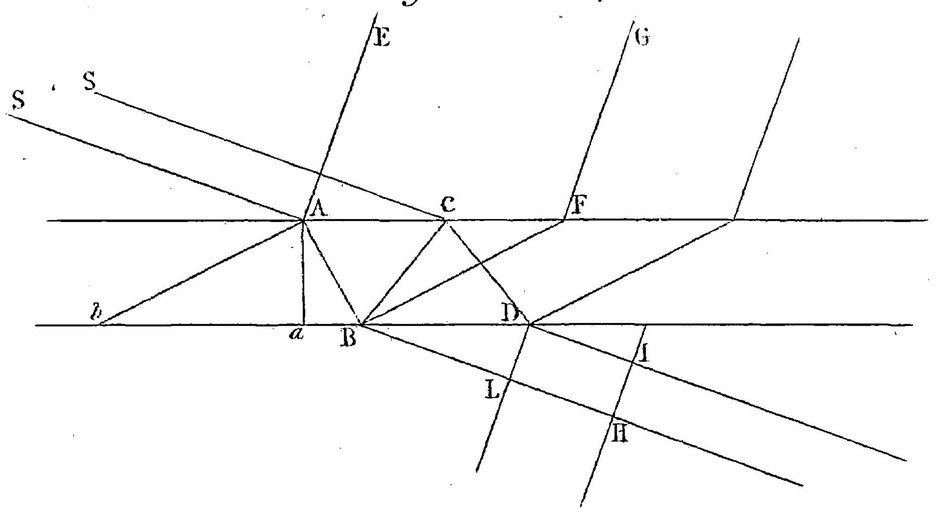


Fig. 136. Art. 679.

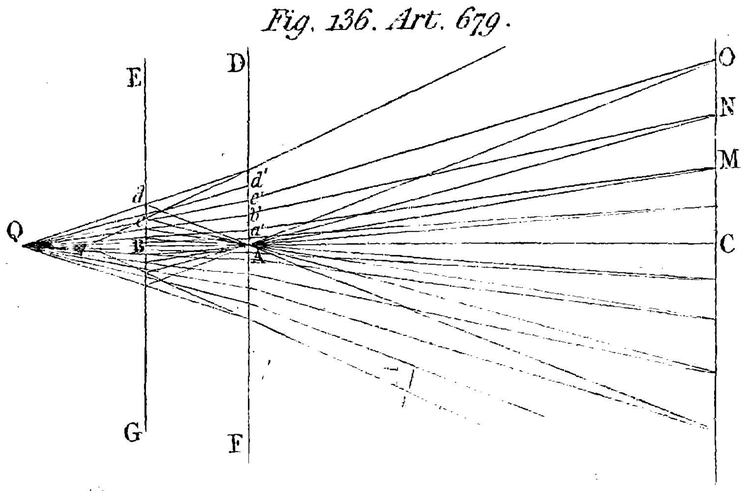


Fig. 137. Art. 682.

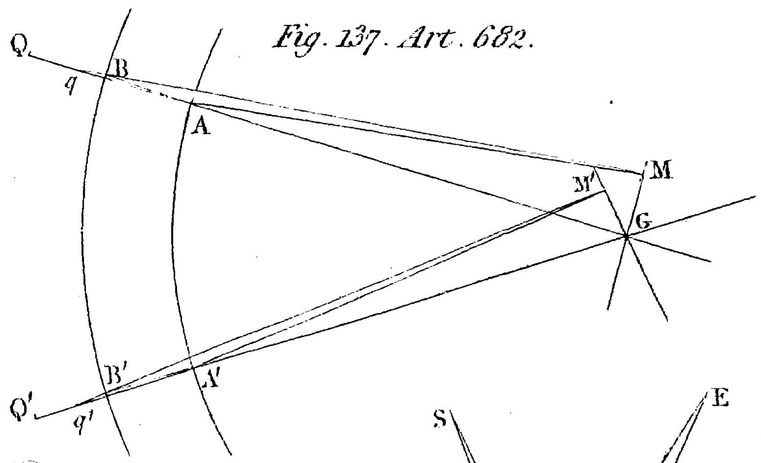


Fig. 138. Art. 691.

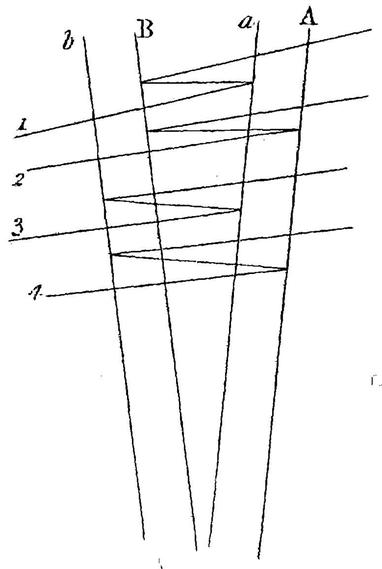


Fig. 139. Art. 693.

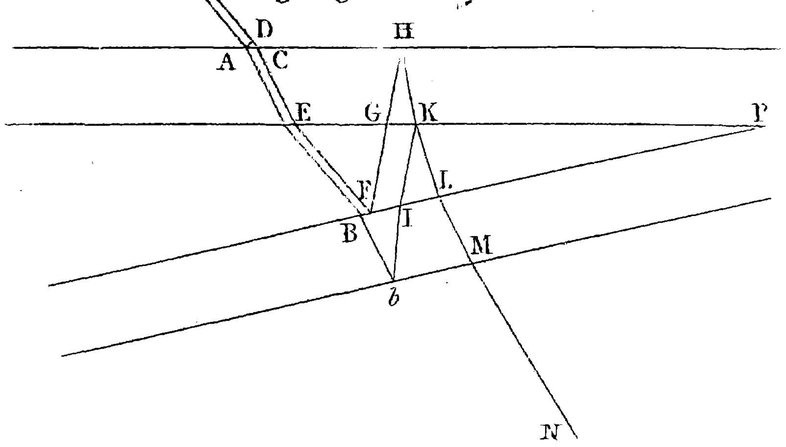


Fig. 140. Art. 694.

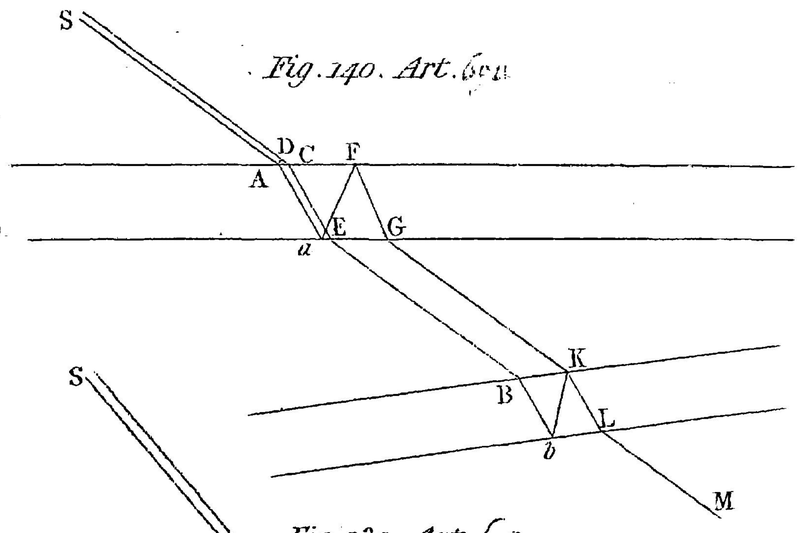
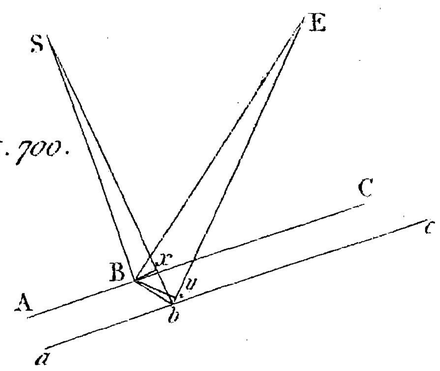


Fig. 141. Art. 700.



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Plate 9.

Fig. 142, Art. 711.

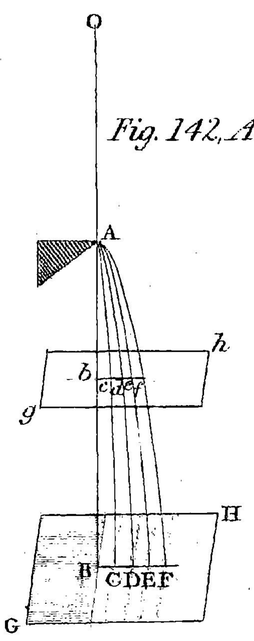


Fig. 143, Art. 713.

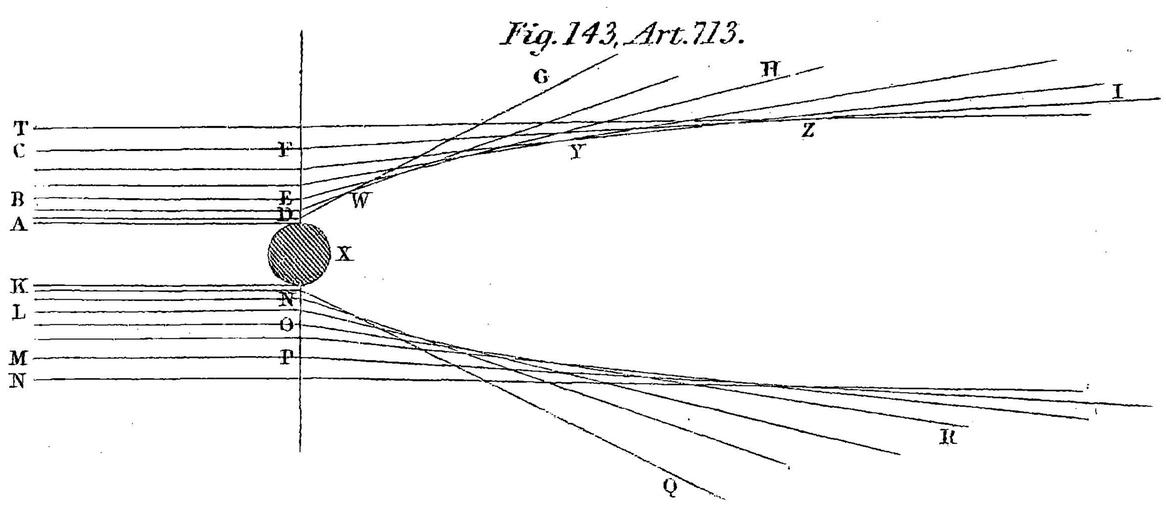


Fig. 145, Art. 718.

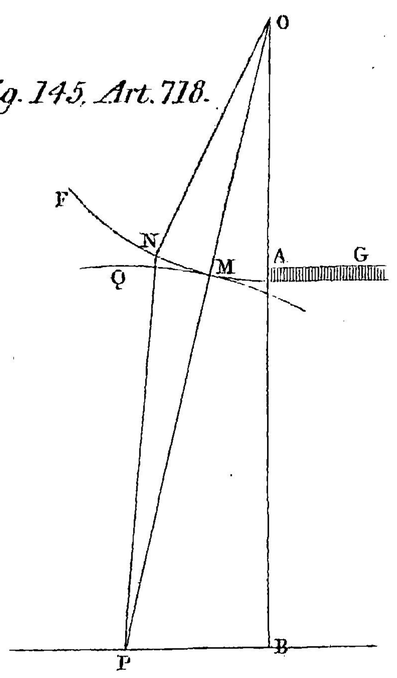


Fig. 144, Art. 714.

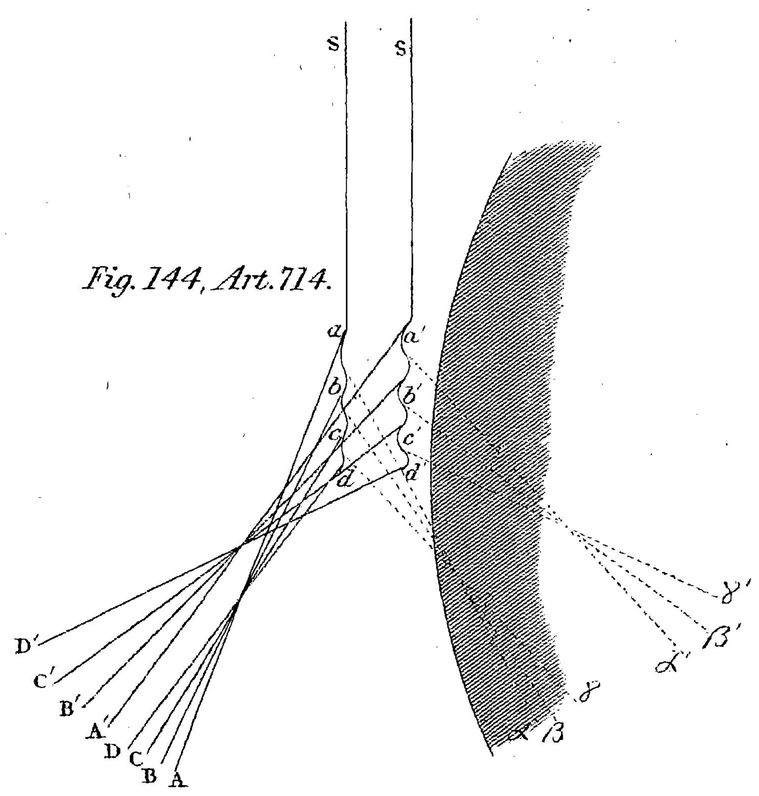


Fig. 146, Art. 726.

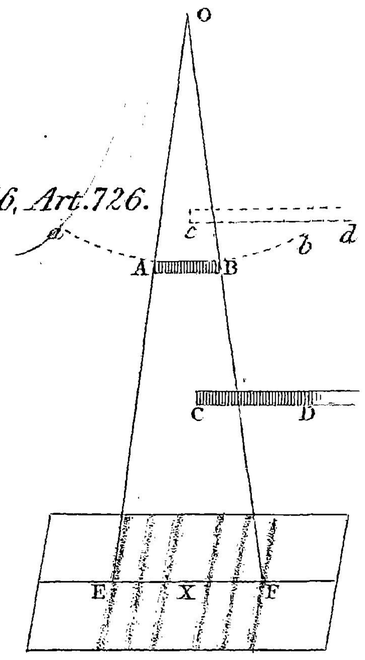


Fig. 147, Art. 735.

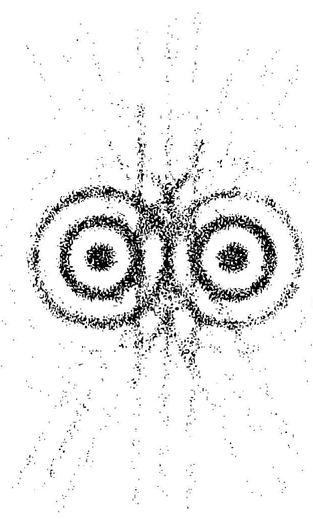


Fig. 149 Art. 736.

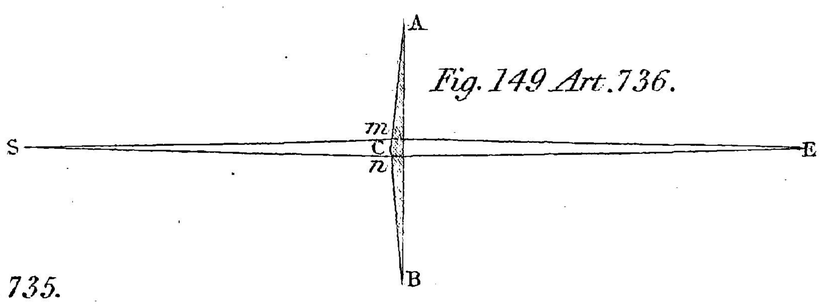


Fig. 148, Art. 735.

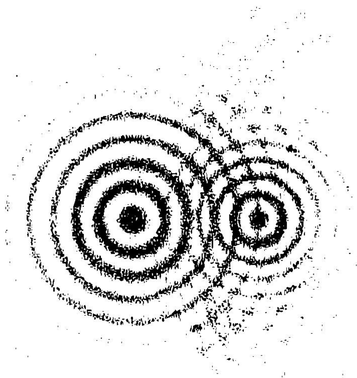


Fig. 150, Art. 736.

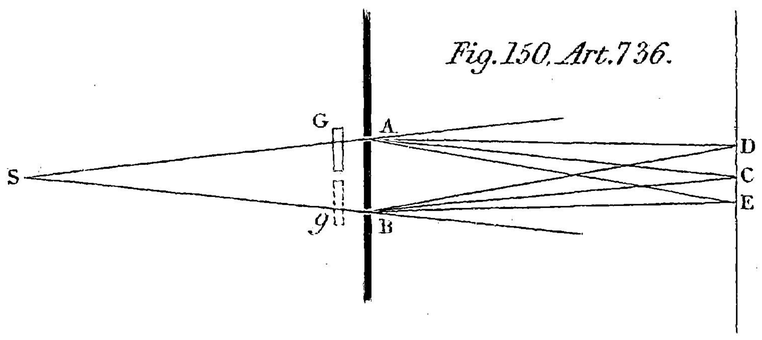


Fig. 151, Art. 755.

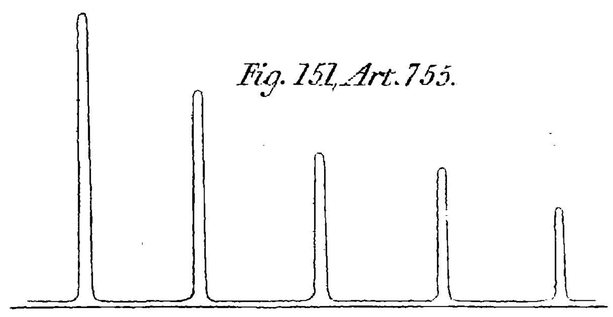


Fig. 151. N.° 2. Art. 765.

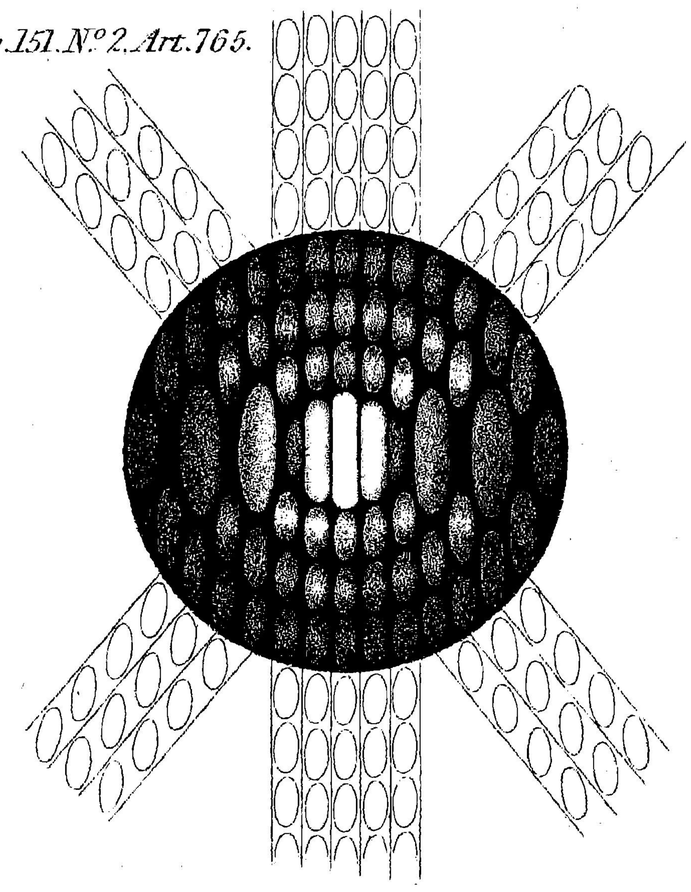


Fig. 152, Art. 770.

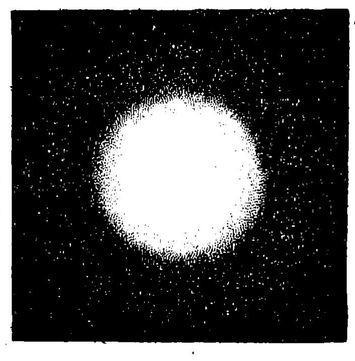


Fig. 153, Art. 770.

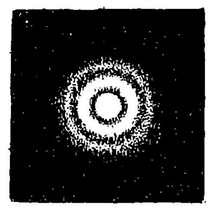


Fig. 154, Art. 770.

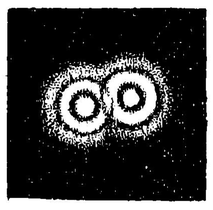


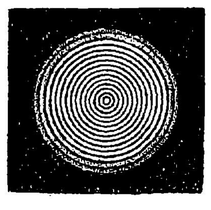
Fig. 155, Art. 770.



Fig. 156, Art. 770.



Fig. 157 Art. 770.



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Fig. 151. N°3. Art. 742.

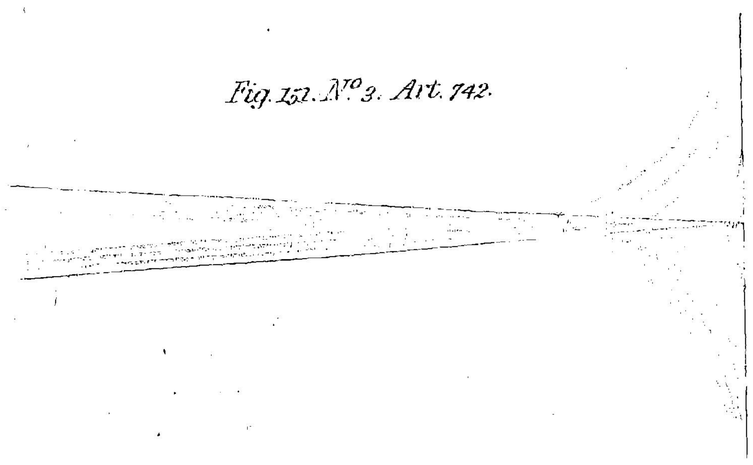


Fig. 158. Art. 771.

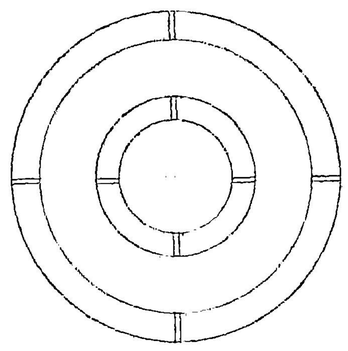


Fig. 159. Art. 771.

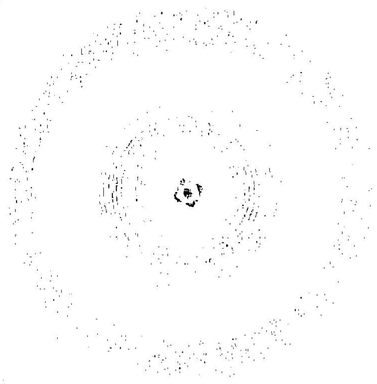


Fig. 160. Art. 772.

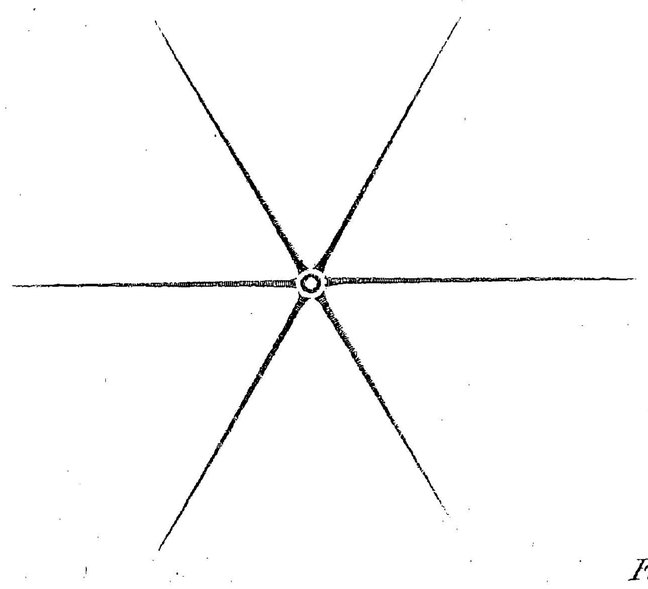


Fig. 161. Art. 773.

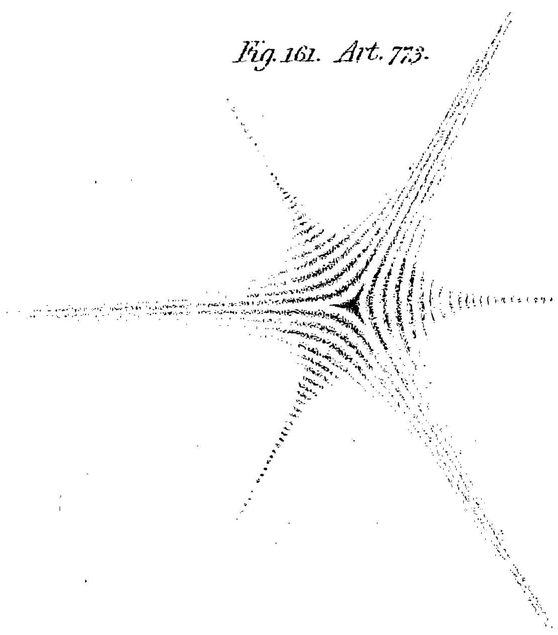


Fig. 162. Art. 775.

Fig. 163. Art. 775.

Fig. 165. Art. 776.



Fig. 164. Art. 775.

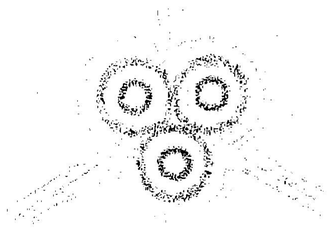




Fig. 166. Art. 777.

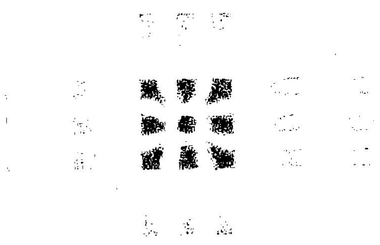


Fig. 167. Art. 778.

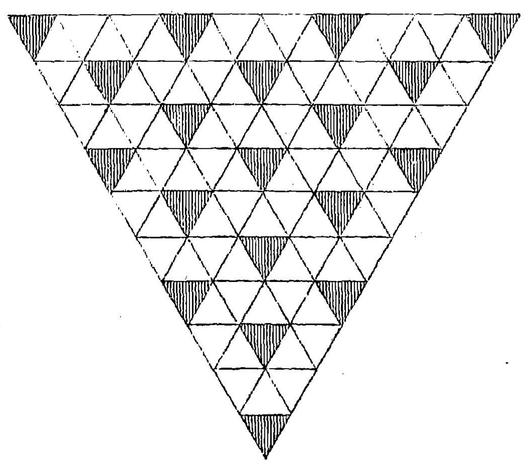


Fig. 168. Art. 778.

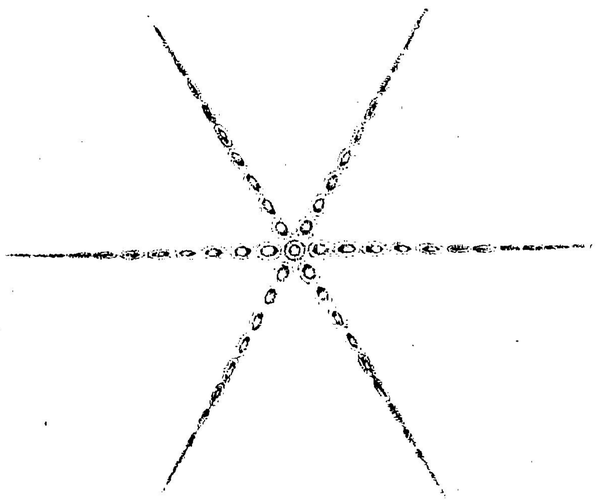


Fig. 170. Art. 836.

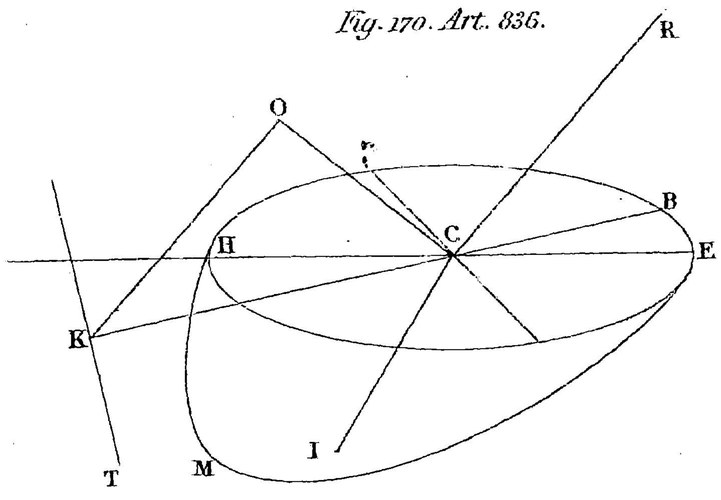
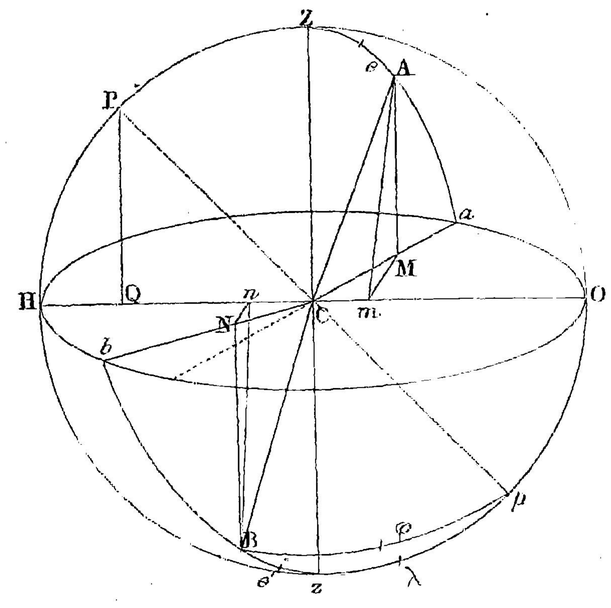


Fig. 169. Art. 790.



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Plate II.

Fig. 172. Art. 825.

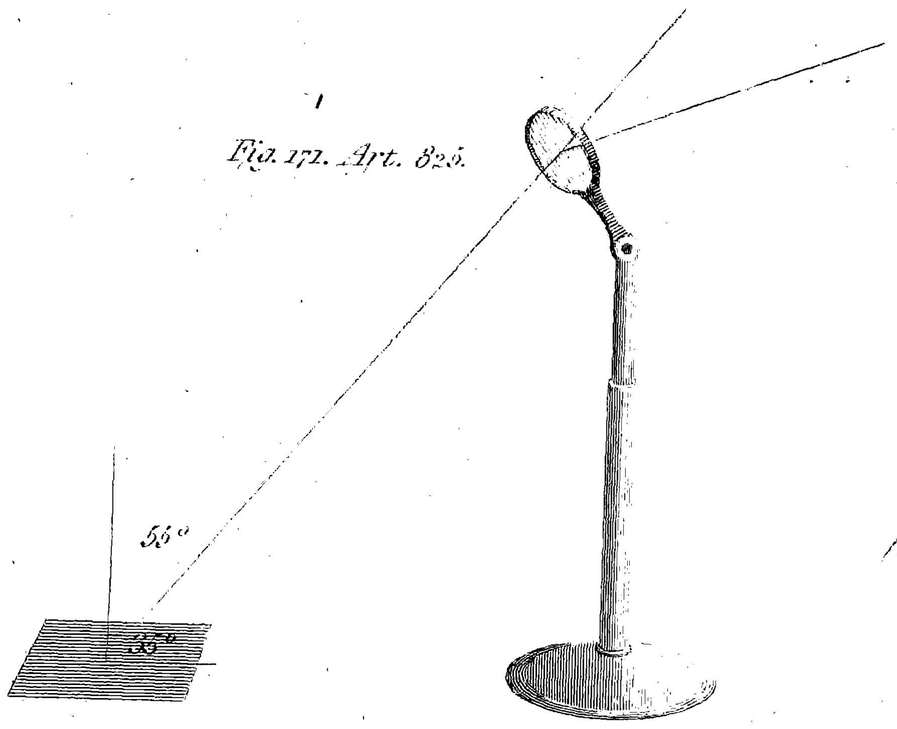


Fig. 172. Art. 826.

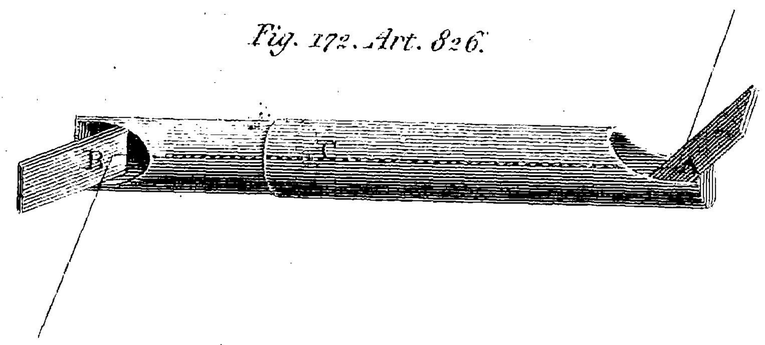


Fig. 173. Art. 879.

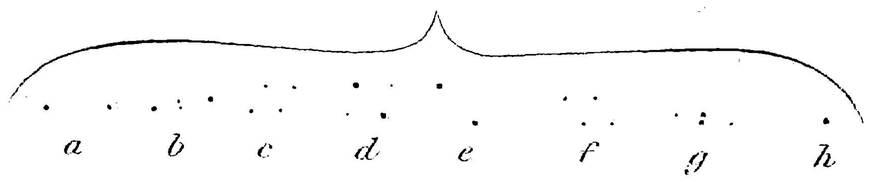


Fig. 174. Art. 881.

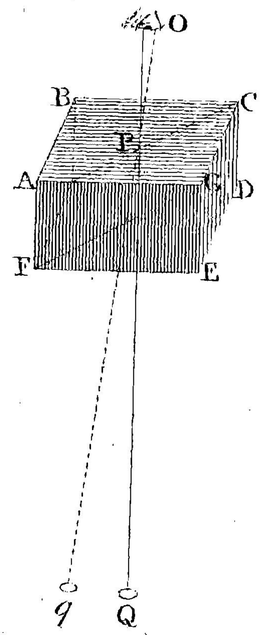
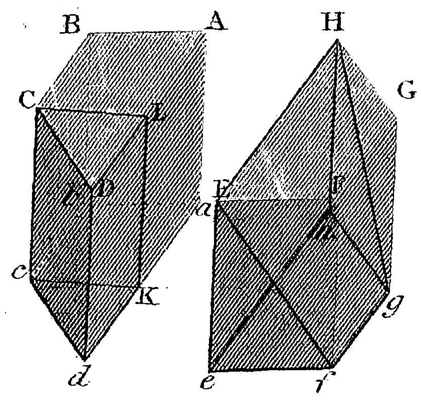


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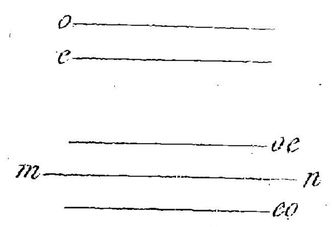


Fig. 176. Art. 892.

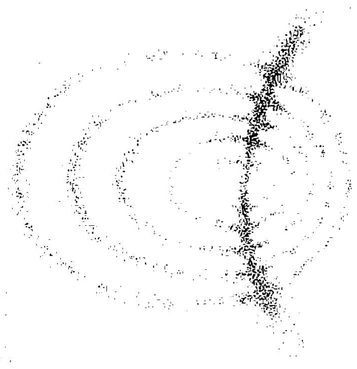


Fig. 177. Art. 893.

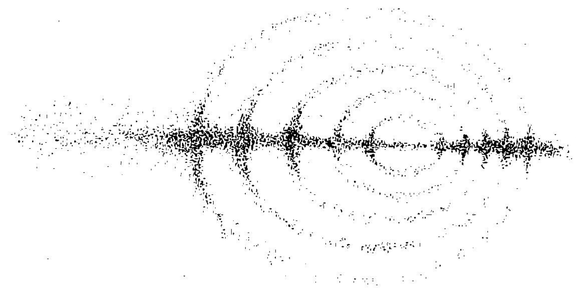


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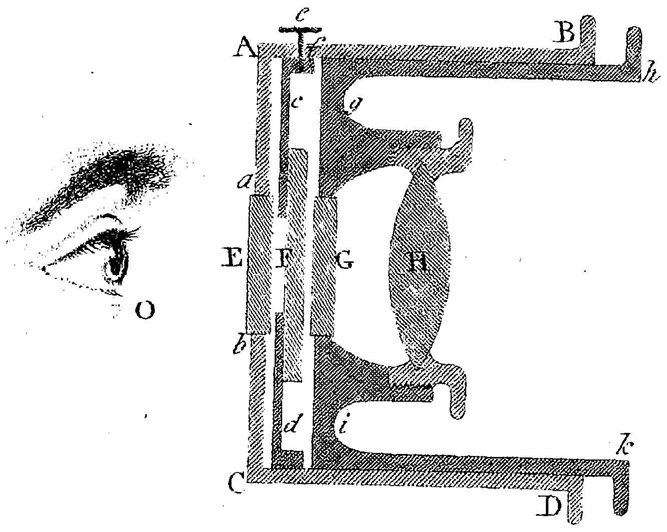


Fig. 179. Art. 900.

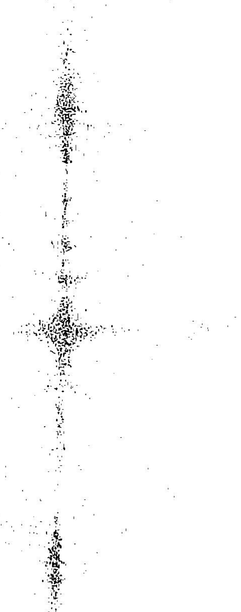


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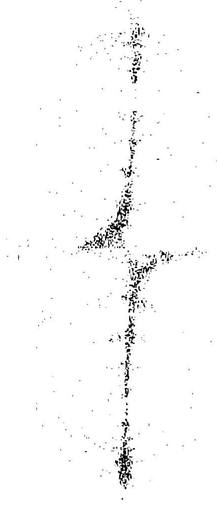


Fig. 181. Art. 900.

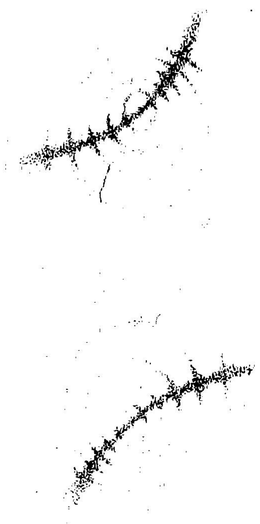


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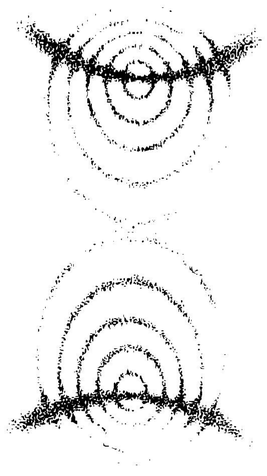


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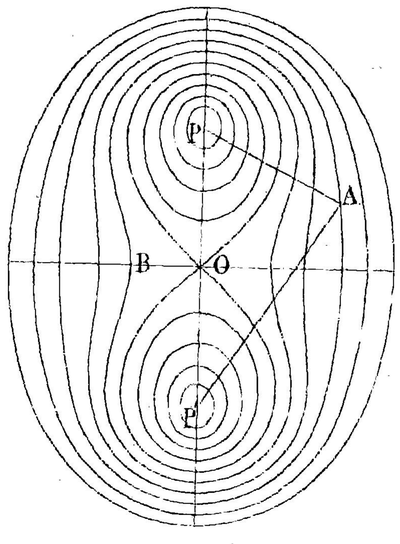
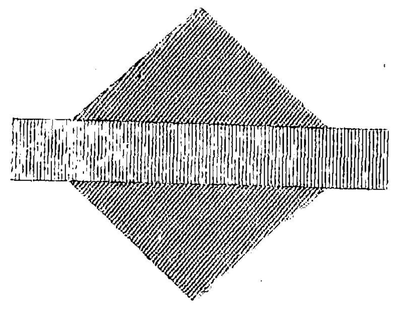


Fig. 184. Art. 910.



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Fig. 185. Art. 911.

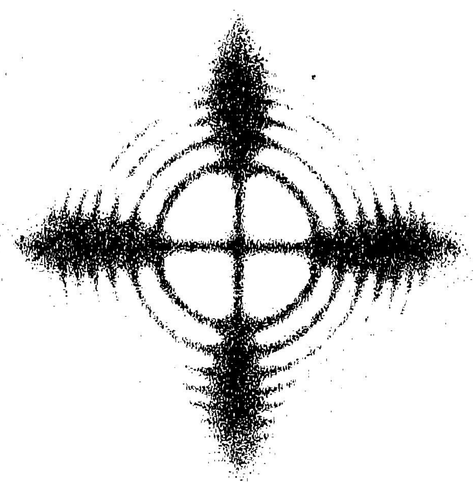


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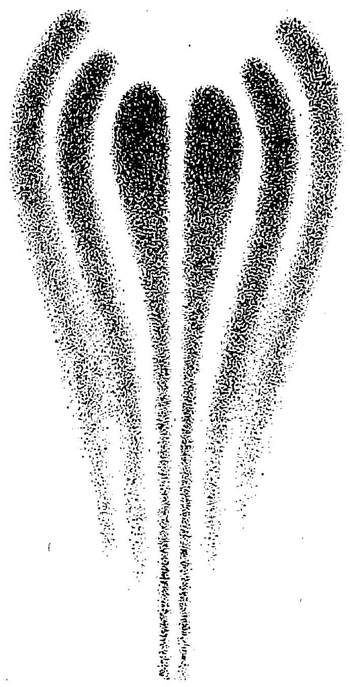


Fig. 187. Art. 922.

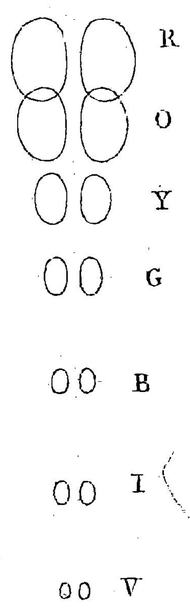


Fig. 188. Art. 926.

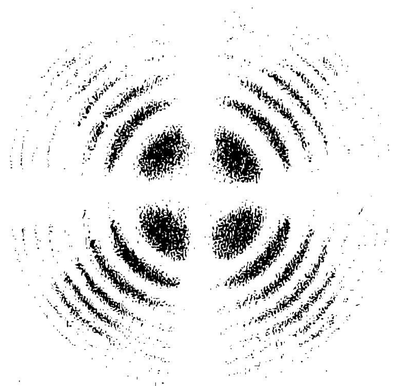


Fig. 189. Art. 929.

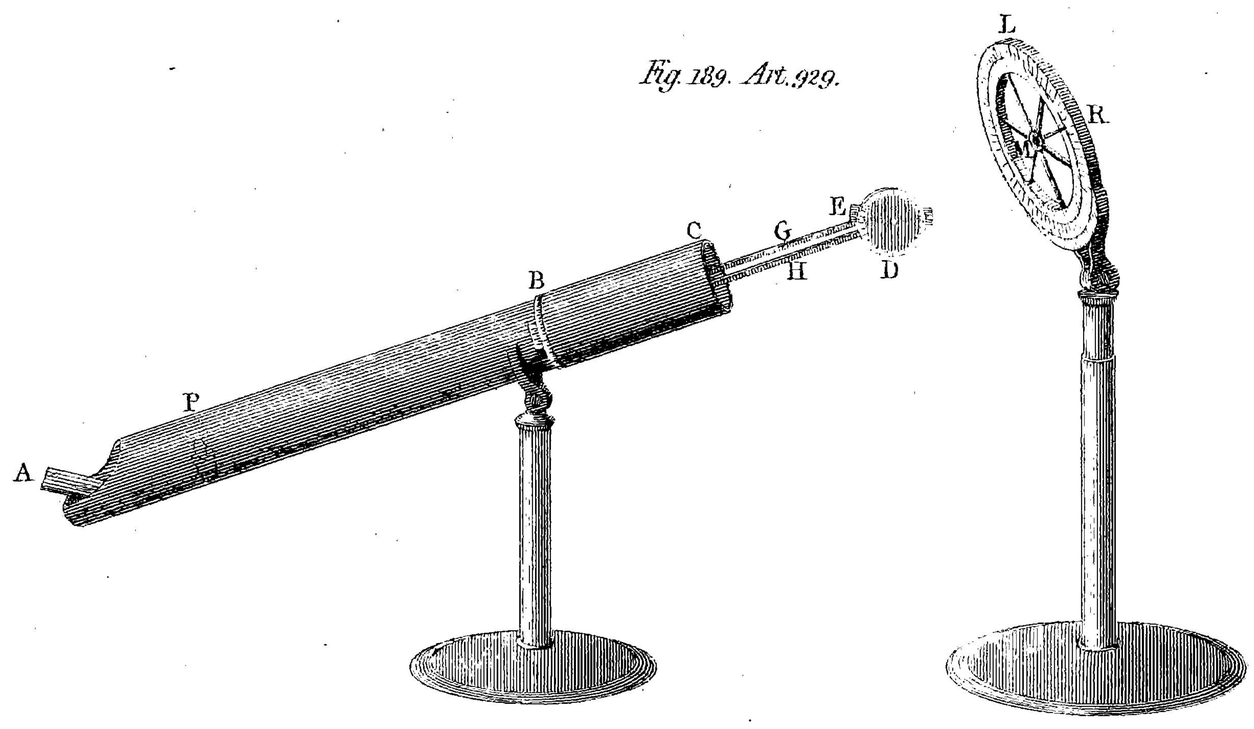


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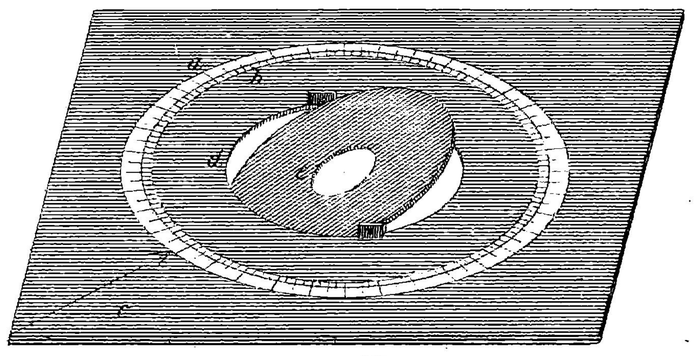


Fig. 191. Art. 935.

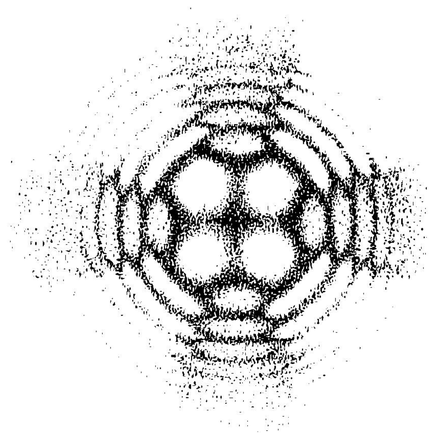


Fig. 192. Art. 939.

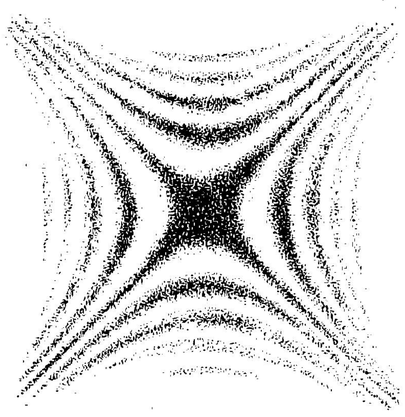


Fig. 193. Art. 939.

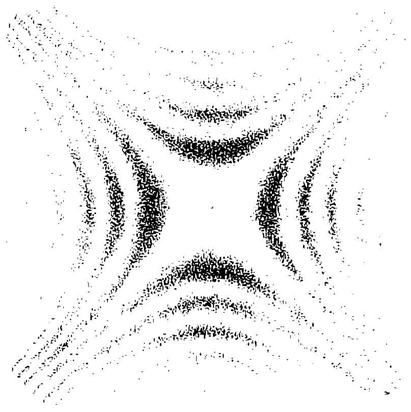


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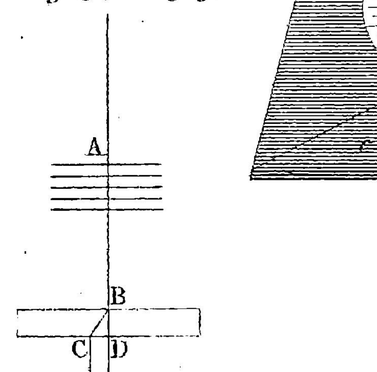
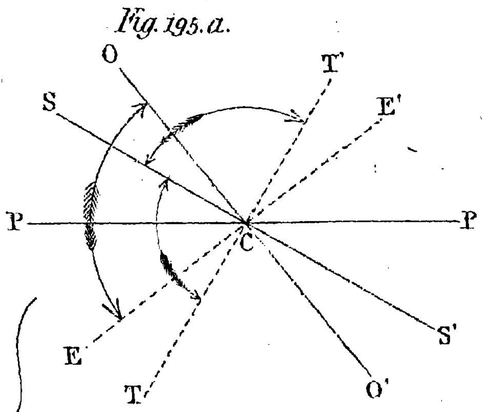


Fig. 195. a.



Art 972 and 983

Fig. 195. b.

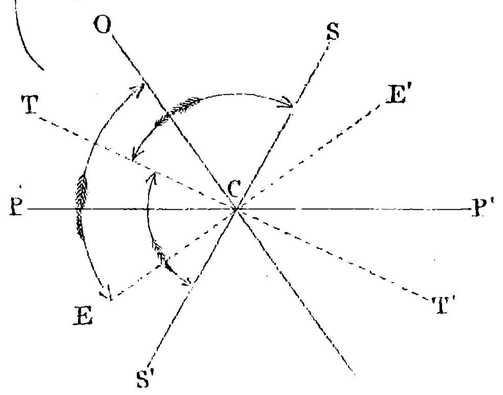


Fig. 196. Art. 986.

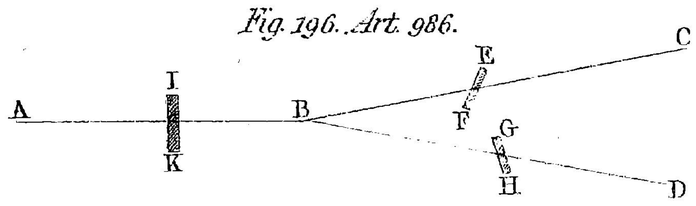


Fig. 197. Art. 993.

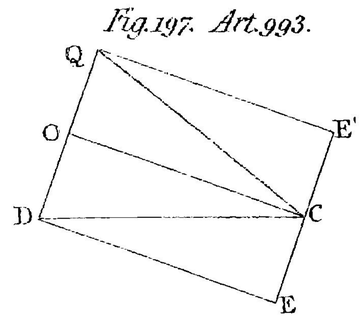
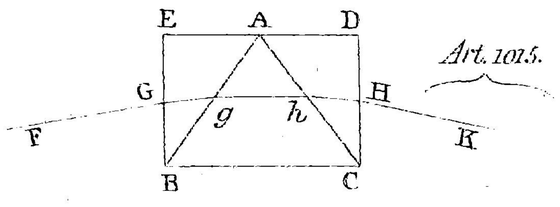
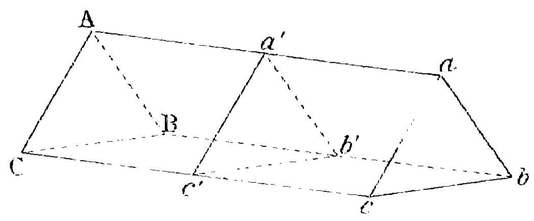


Fig. 198. N.o 1.



Art. 1015.

Fig. 198. N.o 2.



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LIGHT

Plate 13.

Fig. 199 Art. 1030

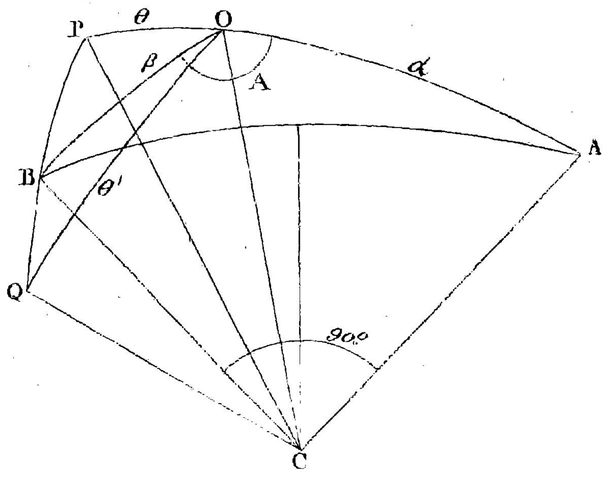


Fig. 200 Art. 1032

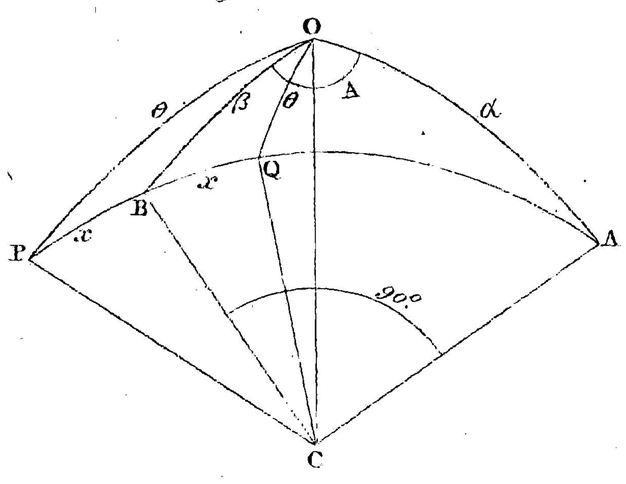


Fig. 201 Art. 1042

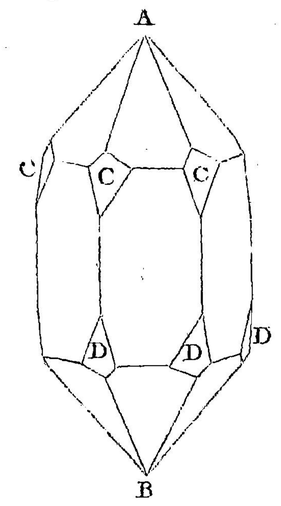


Fig. 202 Art. 1044

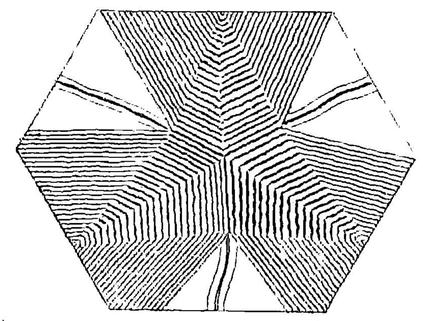


Fig. 203 Art. 1048

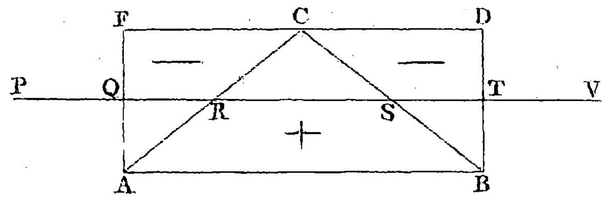


Fig. 204 Art. 1050

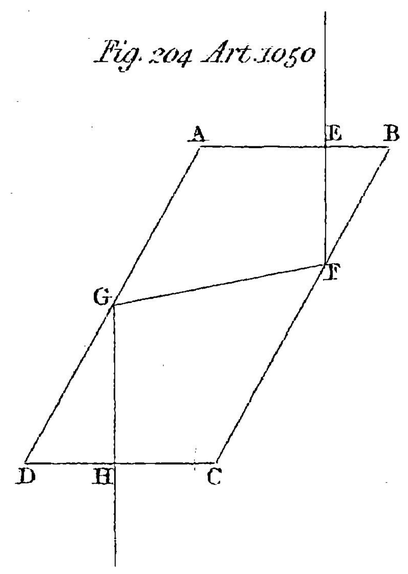


Fig. 205 Art. 1057

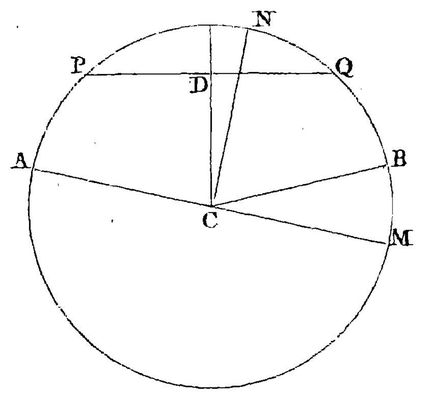


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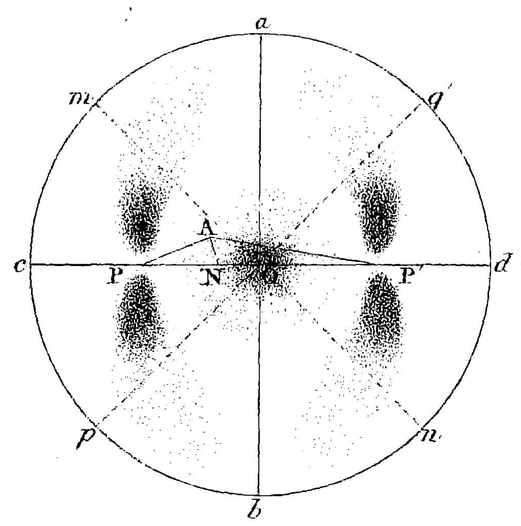


Fig. 207 Art. 1068.



Fig. 208 Art. 1071

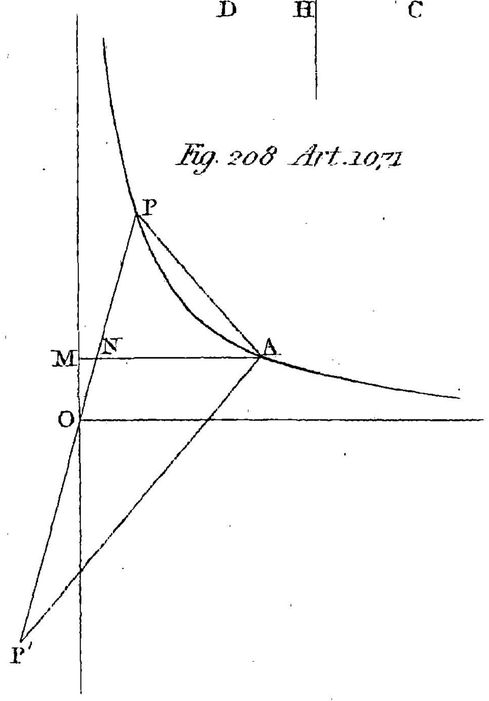


Fig. 209 Art. 1170

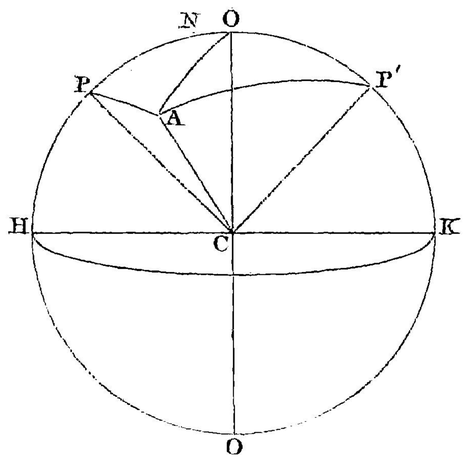


Fig. 210 Art. 1081 & 1079.

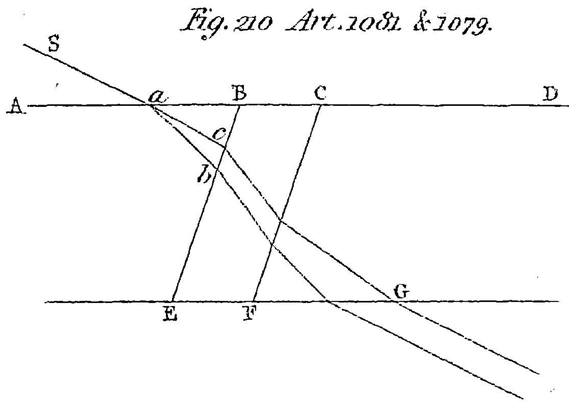


Fig. 211 Art. 1082

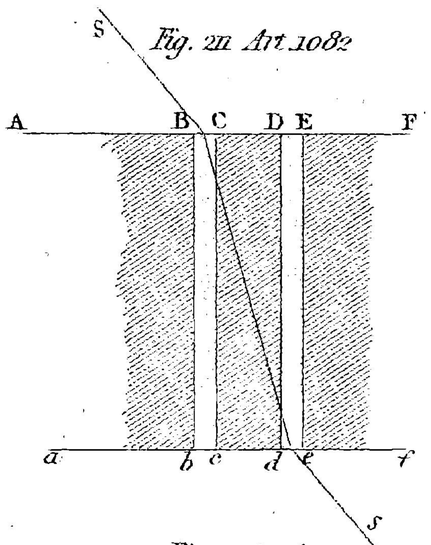


Fig. 212 Art. 1090

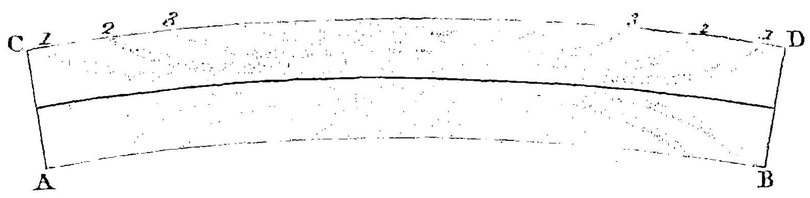


Fig. 213 Art. 1096

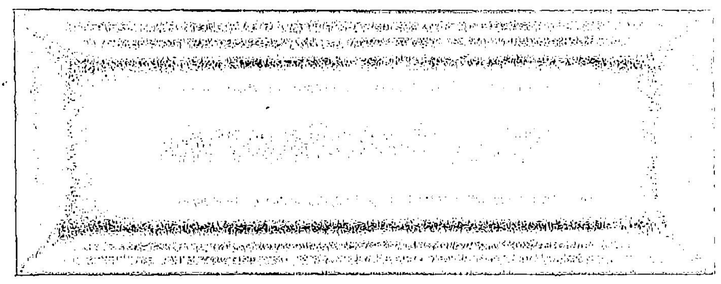


Fig. 214 Art. 1098

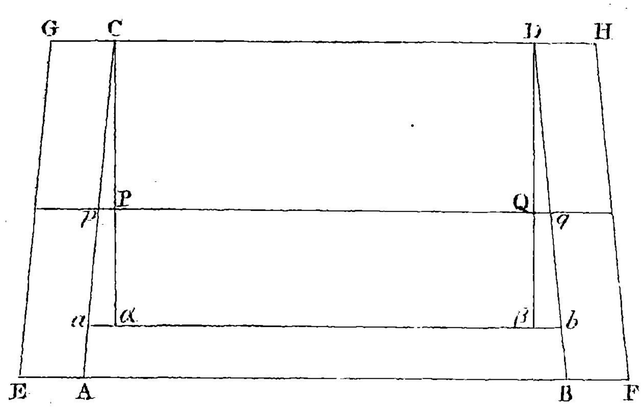


Fig. 215 Art. 1099

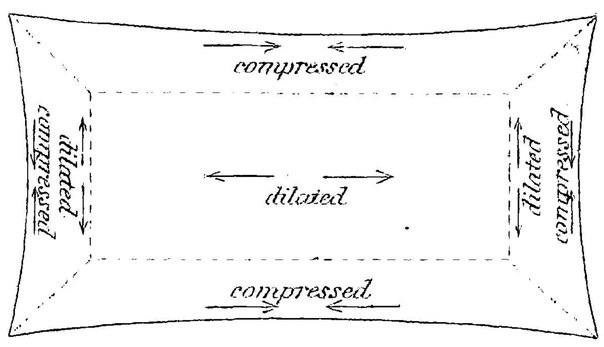
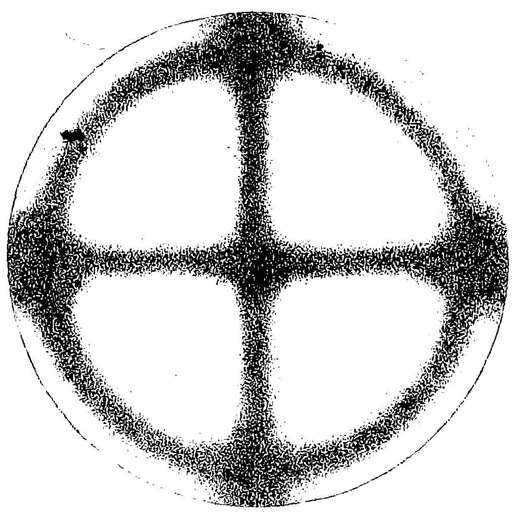


Fig. 216 Art. 1105



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Fig. 217 Art. 1105

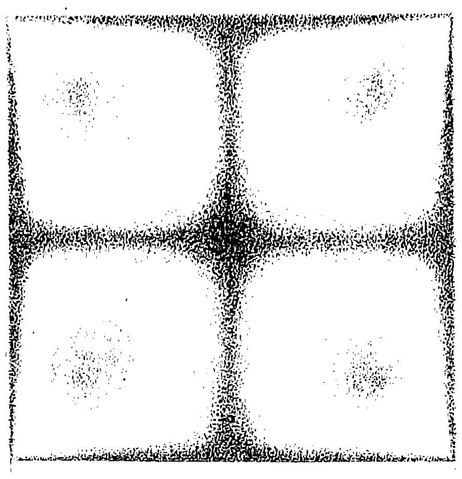


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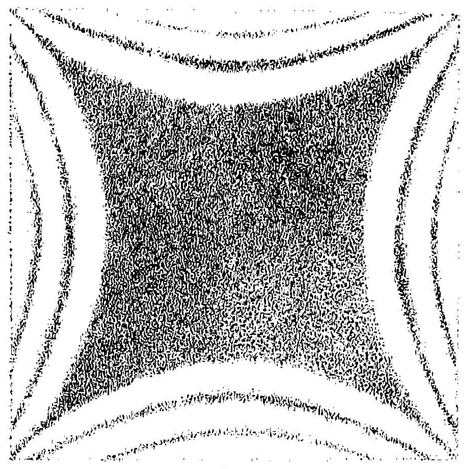


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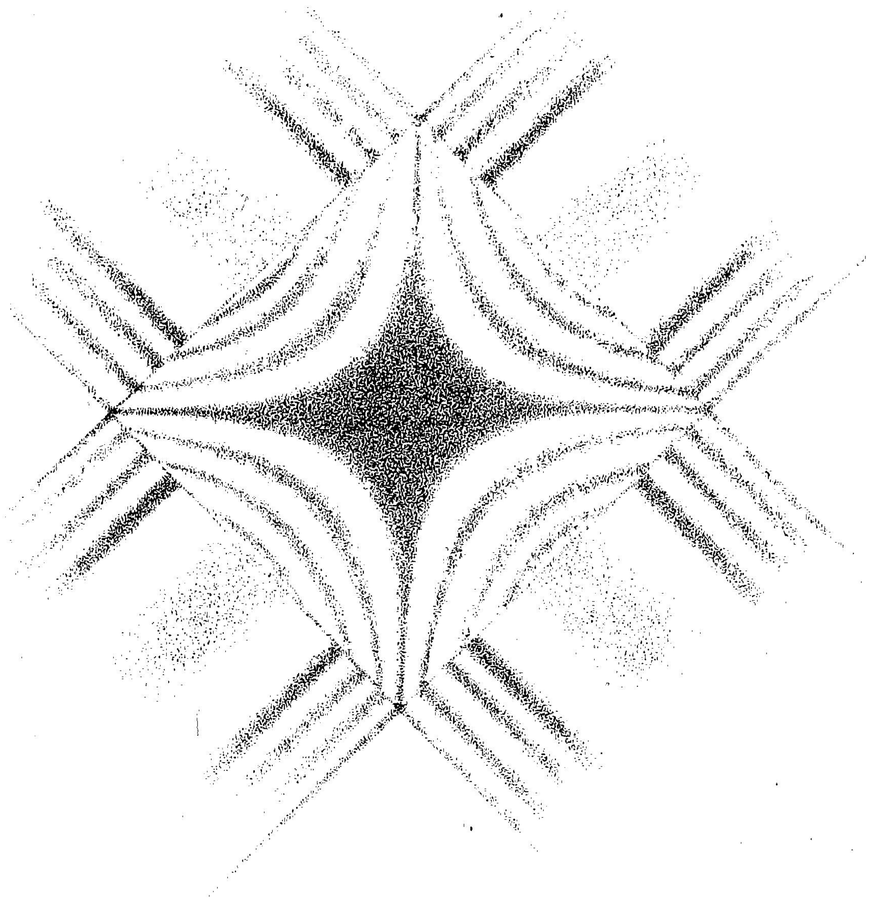


Fig. 218 Art. 1105

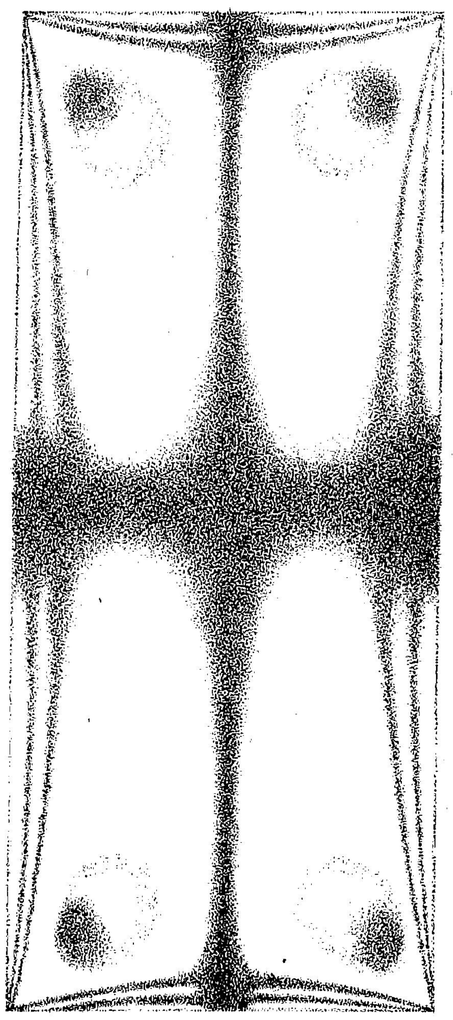


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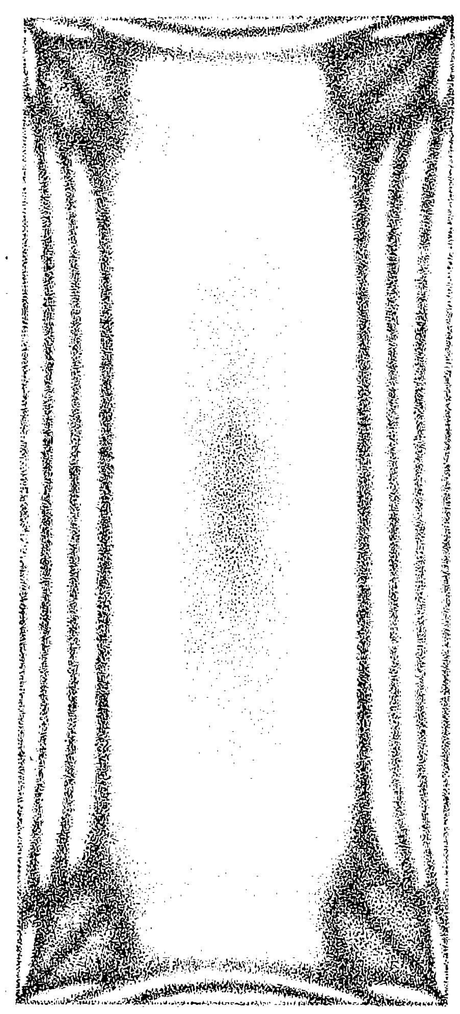


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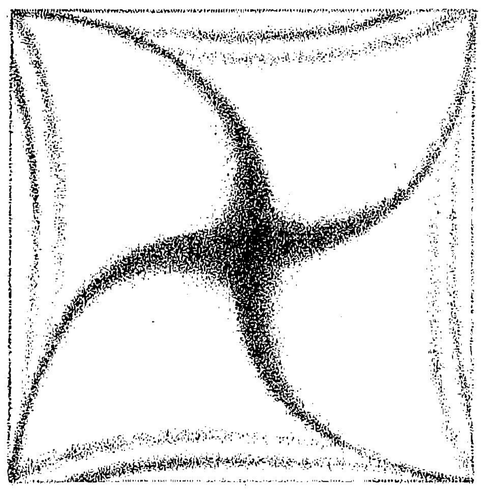


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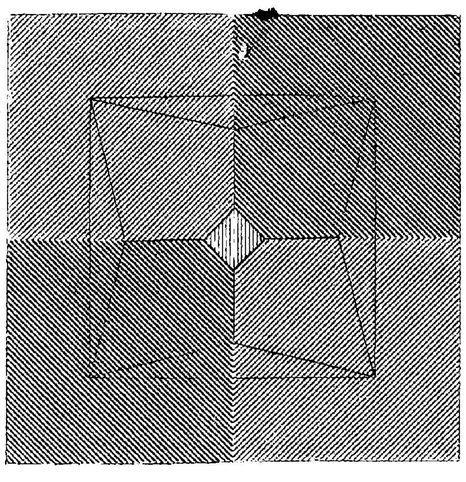
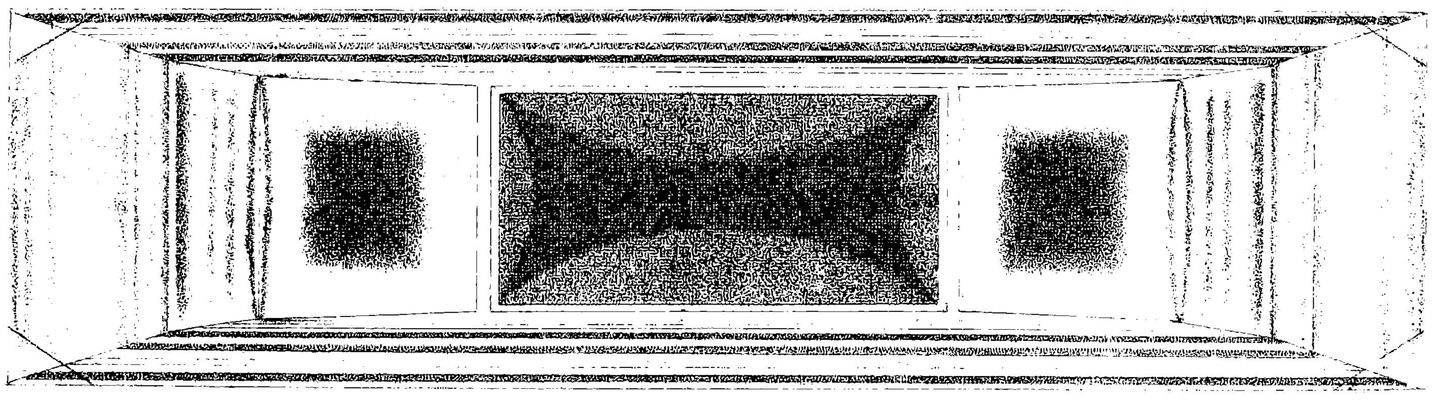


Fig. 224



J.W.Lowry sculp.

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Herschel, John F. W.

On the theory of light

S.l. 1828

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